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RELATIVISTIC LIMITATION OF THE NAIVE QUARK MODEL

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The limitations of the nonrelativistic quark model are considered in order to build a unique quark model for mesons. By using the two-particle Dirac equation the basic properties of the energy spectrum of bound states are analyzed. For various forms of couplings the lower limits for masses of composite mesons are determined, and the character of the dynamics is clarified.

1. Introduction

After the discovery and study of meson families $\Psi(c\bar{c})$ and $Y(b\bar{b})$ it has become clear that naive quark model can be successfully applied to the systems of heavy quarks[1]. All such models are characterized by an interaction described by a Coulomb plus linearly confining potential, in conformity with QCD.

In the sector of light quarks (u, d, s) the nonrelativistic potential model becomes inappropriate, and it gives only a qualitative understanding of light quarks dynamics. In an attempt to understand completely the quark model such a situation leads to an investigation of limitations of the nonrelativistic treatment of quarks. This question is also important for a better understanding of other aspects of hadronic structure[2]. It turns out that for a successful and unified description of both old and new meson mass spectrum, it is essential to use relativistic quark model[1,3,4].

The question of the consistency of the nonrelativistic quark model has been considered in the literatuie. Horwitz[6] analyzed the case of quarks with equal masses and vector type interaction, and showed that the $J^P = O^-$ mesons have a lower mass limit which is given by $(2\sqrt{2}/3)\, 2m_q$. The characteristics of the spectrum of bound states of fermions has been studied in Ref. 7 by using the one-particle Dirac equation. When quarks have different masses, Koide[8] found that the mesonic mass cannot be higher than $(m_1 + m_2)/2$, independently of the type of interaction between quarks. It is found that quarks with equal masses and V-A interaction can produce very deep bound states[9].

In this paper we shall study the limitations of the quark model by using the wo-particle Dirac equation (2). Although noncovariant, this equation contains he relativistic kinematic. and spin[1,3,4]. We will try to find out the basic characteristics of the standard coupling (S, P, V, A, T) and their combinations (3) by assuming that the interaction is described by a spherical well potential. We shall also limit ourselves to the lowest $j = 0$ states.

2. *The energy spectrum of bound states*

As we mentioned above, the energy of the bound states of a quark (m_1) and an antiquark (m_2) is determined by the relativistic, two-particle Dirac equation

$$-\mathrm{i}\,[\vec{\alpha}, \vec{\nabla}\Phi(\vec{r})]_- + 1/2M\,[\beta, \Phi(\vec{r})]_- + 1/2m\,[\beta, \Phi(\vec{r})]_+ - E\Phi(\vec{r}) =$$

$$= -V(r)\,\Gamma^{(c)}\,\Phi(\vec{r}) \tag{1}$$

where $M = m_1 + m_2$, $m = m_1 - m_2$, and $\vec{\nabla}$ is the differential operator with respect to the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$.

For the standard interactions, which will be discussed here, the $\Gamma^{(c)}$ matrices are given by

$$\Gamma_S^{(c)} = -\beta^{(1)}\beta^{(2)} \qquad \Gamma_P^{(c)} = \div (\beta\gamma_5)^{(1)}(\beta\gamma_5)^{(2)}$$

$$\Gamma_V^{(c)} = 1^{(1)}\,1^{(2)} - \vec{\alpha}^{(1)}\,\vec{\alpha}^{(2)} \qquad \Gamma_A^{(c)} = -\vec{\Sigma}^{(1)}\,\vec{\Sigma}^{(2)} + \gamma_5^{(1)}\gamma_5^{(2)}$$

$$\Gamma_T^{(c)} = (\beta\,\vec{\Sigma})^{(1)}(\beta\,\vec{\Sigma})^{(2)} + (\beta\,\vec{\alpha})^{(1)}(\beta\,\vec{\alpha})^{(2)}. \tag{2}$$

The wave function $\Phi(\vec{r})$ is 16-component quantity, and for $j = 0$ it describes two different parity states,

$$\Phi^{(-)}({}^1S_0) = \begin{vmatrix} \mathrm{i}g_1(r)\,X_{11}^{00}\,\sigma_2^T & -\mathrm{i}f_1(r)\,X_{00}^{00}\,\sigma_2^T \\ f_2(r)\,X_{00}^{00}\,\sigma_2^T & -\mathrm{i}g_2(r)\,X_{11}^{00}\,\sigma_2^T \end{vmatrix} \tag{3a}$$

$$\Phi^{(+)}({}^3P_0) = \begin{vmatrix} \mathrm{i}f_1(r)\,X_{00}^{00}\,\sigma_2^T & -g_1(r)\,X_{11}^{00}\,\sigma_2^T \\ g_2(r)\,X_{11}^{00}\,\sigma_2^T & -\mathrm{i}f_2(r)\,X_{00}^{00}\,\sigma_2^T \end{vmatrix}. \tag{3b}$$

The angular momentum eigenstates $X_{1,}^{JM}$ can be obtained in a standard way, with the help of Clebsch-Gordon coefficients. Dirac's matrices are taken to be in the standard representation[6].

The radial part of the wave function for $^1S_0$ state satisfies the equation

$$2\mathrm{d}f_+/\mathrm{d}r + mg_- - Eg_+ = -V_0 g_+$$

$$mg_+ - Eg_- = -V_0 g_-$$

$$2\,(\mathrm{d}/\mathrm{d}r + 2/r)\,g_+ - Mf_- + Ef_+ = -V_0 f_+$$

$$-Mf_+ + Ef_- = -V_0 f_- \tag{4}$$

where $f_\pm = f_1 \pm f_2$, $g_\pm = g_1 \pm g_2$, and the *scalar* (*S*) interaction is represented by the spherical well potential,

$$V(r) = -V_0 \text{ for } r < r_0,\ V(r) = 0 \text{ for } r > r_0.$$

From Eq. (4) one obtaines the following relation for the energy of bound states[6]

$$\frac{E - m^2/E}{1 - \mathrm{i}\beta r_0} \quad \frac{E - V_0 - m^2/(E - V_0)}{1 - \alpha r_0 \operatorname{ctg} \alpha r_0}. \tag{5}$$

The parameters α and β are defined by

$$4\alpha^2 = \frac{[(E + V_0)^2 - M^2]\,[(E - V_0)^2 - m^2]}{(E - V_0)(E + V_0)}$$

$$4\beta^2 = \frac{(E^2 - M^2)(E^2 - m^2)}{E^2}. \tag{6}$$

For other types of interactions (P, V, A, T) the right-hand side of Eq. (4) gets changed, which leads to different value for α, and different equation for E, as summarized in Table 1.

The same kind of calculation can also be done with the wave function (3b) for $^3P_0$ state. These new equations can be obtained from (4) by the replacement $\vec{r} \to -\vec{r}$, $m \leftrightarrow M$, and also $V \to -V$ for S, P and T interactions. This means that a potential which is attractive in a certain state, becomes repulsive in the corresponding state of opposite parity. As a consequence, A and T types of interaction do not have bound states, as opposed to the P state (Table 2).

In Table 3 we have considered the cases S $\pm$ P and V $\pm$ A, S $\pm$ T + P, which often appear in the literature[9].

TABLE 1.

Type	Bound states	Parameter $4a^2$
S	$\frac{E}{1 - \mathrm{i}\beta r_0} = \frac{E - V}{1 - ar_0 \operatorname{ctg} ar_0}$	$\frac{E - V_0}{E + V_0} [(E + V_0)^2 - M^2]$
P	$\frac{E}{1 - \mathrm{i}\beta r_0} = \frac{E - V_0}{1 - ar_0 \operatorname{ctg} ar_0}$	$E^2 - V_0^2 - M^2$
V	$a \operatorname{ctg} ar_0 = \mathrm{i}\beta$	$\frac{E}{E - 2V_0} [(E + 4V_0)(E - 2V_0) - M^2]$
A	$a \operatorname{ctg} ar_0 = \mathrm{i}\beta$	$\frac{E}{E + 2V_0} [(E + 4V_0)(E + 2V_0) - M^2]$
T	$\frac{E}{1 - \mathrm{i}\beta r_0} = \frac{E + 2V_0}{1 - ar_0 \operatorname{ctg} ar_0}$	$\frac{E}{E + 2V_0} (E^2 + 6EV_0 - M^2)$

The form of the bound state equation and parameter a, for five basic types of potentials — $^1S_0$-state.

TABLE 2.

Type	Bound states	Parameter 4^2
S	$\frac{E - M^2/E}{1 - \mathrm{i}\beta r_0} = \frac{E + V_0 - M^2/(E + V_0)}{1 - ar_0 \operatorname{ctg} ar_0}$	$\frac{E - V_0}{E + V_0} [(E + V_0)^2 - M^2]$
V	$\frac{E - M^2/E}{1 - \mathrm{i}\beta r_0} = \frac{E - M^2/(E + 2V_0)}{1 - ar_0 \operatorname{ctg} ar_0}$	$\frac{E + 4V_0}{E + 2V_0} [E(E + 2V_0) - M^2]$

$^3P_0$-state.

3. *The case of equal quark masses*

We shall now analyze the dependence of the energy spectrum of bound states on the quark masses, the interaction radius (r_0) and strength (V_0). It is convenient to introduce dimensionless quantities

$$\tilde{E} = E/(m_1 + m_2), \quad \tilde{V}_0 = V_0/(m_1 + m_2), \quad (m_1 + m_2) r_0 = \lambda. \tag{7}$$

The energy of bound states satisfies the relation

$$m_1 - m_2 \leq E \leq m_1 + m_2 \tag{8a}$$

TABLE 3.

Type	Parameter $4\alpha^2$	$m = 0$	$m \neq 0$
S − P	$\frac{[E(E-2V_0)-m^2][E(E+2V_0)-M^2]}{E(E+2V_0)}$	$\tilde{E}_1 = -\tilde{V}_0 + \sqrt{1+\tilde{V}_0^2}$	$\tilde{E}_2 = \tilde{V}_0 + \sqrt{\tilde{V}_0^2 + \delta_2}$
S + P	$\frac{[E(E-2V_0)-m^2][E(E+2V_0)-M^2]}{E^2}$	$\tilde{E}_1 = -\tilde{V}_0^2 + \sqrt{1+\tilde{V}_0^2}$	$\tilde{E}_2 = \tilde{V}_0 + \sqrt{\tilde{V}_0^2 + \delta^2}$
V − A	$\frac{(E^2-m^2)(E^2+8EV_0-M^2)}{E^2}$	$\tilde{E}_1 = -4\tilde{V}_0 + \sqrt{1+16\tilde{V}_0^2}$	$\tilde{E}_2 = \delta$
V + A	$\frac{[E(E+2V_0)-m^2][E(E-4V_0)-M^2]}{E^2}$	$\tilde{E}_1 = 2\tilde{V}_0 + \sqrt{1+4\tilde{V}_0^2}$	
S + P ∓ T	$\frac{[(E\pm 2V_0)^2-m^2][E(E\pm 4V_0)-M^2]}{E(E\mp 2V_0)}$	$\tilde{E}_1 = \pm 2\tilde{V}_0 + \sqrt{1+4\tilde{V}_0^2}$	$\tilde{E}_2 = \mid \tilde{V}_0 + \sqrt{\tilde{V}^2 + \delta_0^2}$

The lower limit of the spectra at $^1S_0$ state for some combinations of the potentials, $\delta = m/M$.

which steems, for all five interactions, from the finiteness of the wave function at the origin and infinity. Besides, as Eq. (1) mixes positive and negative energy states, the physical solutions should satisfy

$$\lim_{V_0 \to 0} E_1(V_0) = m_1 + m_2 \tag{8b}$$

where E_1 is the solution of the equation $\alpha^2 = 0$. Therefore, the energy of the boune states must lie above ($\alpha^2 > 0$) the curve $E_1(V_0)$. These limiting curves are shown in Fig. 1, giving the basic insight into the spectrum of the $^1S_0$ states. These minima can be reached asimptotically when quark masses (or the interaction radius) tend to infinity.

For the scalar interaction and sufficiently heavy quarks, $m_q \approx 10$ GeV/c^2 ($r_0 \sim 0.1$—1 fm), there exist arbitrarily deep bound states for $V_0 \approx m_q$, but they disappear for $V_0 \approx 2m_q$.

In the case of the vector interaction (Figs. 2 and 3) the bound state appears on a certain depth of the potential V_0; when V_0 is increased it goes through the minimum, and then disappears. The absolute minimum $E = 0.942M$ appears for $V_0 = 0.117M$, and it represents the lower limit of the (O^-) meson mass. Since, here, the binding energy is small, the vector interaction cannot produce bound states for $\lambda \lesssim 4$, i. e. for light quarks.

For A and T interactions, which depend on the spin of quarks, one can find, asymptotically, arbitrarily deep bound states, Fig. 4. If the quark masses are suffi-

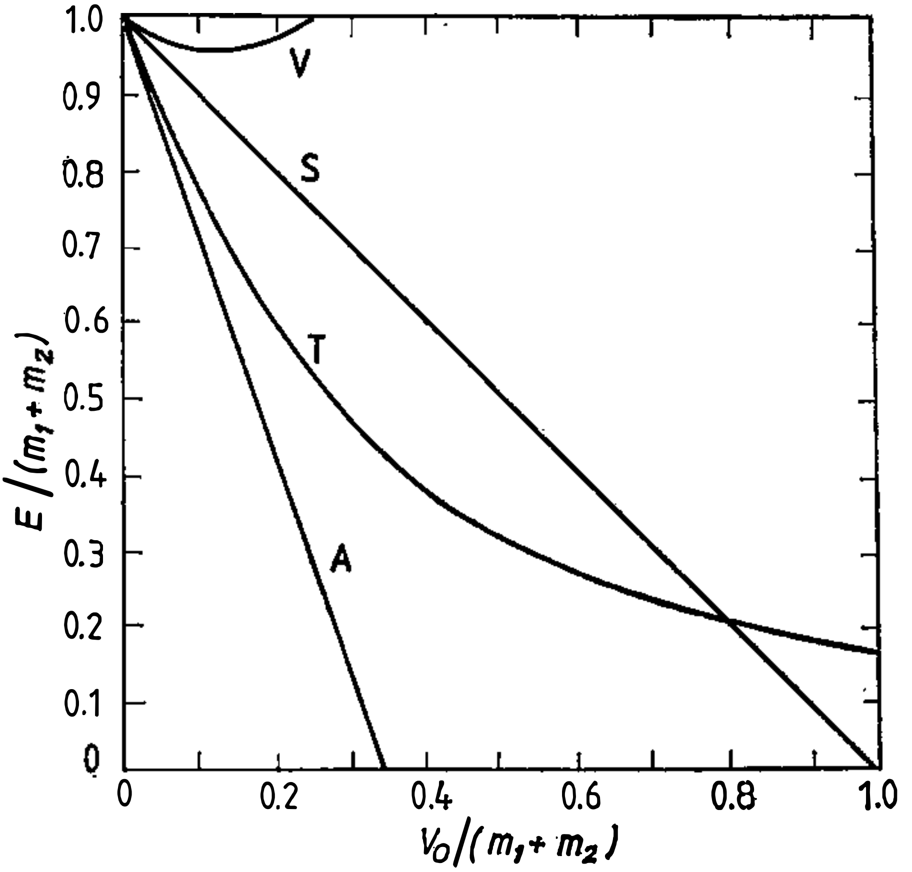

Fig. 1. The lower limits of the energy in the case of five basic potentials.

ciently high, the axial interaction develops an absolute minimum ($E = 0$) for finite value of the potential depth $V_0 = 0.353M$. A — T (and P) interaction is characterized by the absence of $^3P_0$ state. For the T interaction, the bound states of light quarks appear at somewhat higher values of V_0, as compared to the A interaction. When quarks are heavier, the bound state energy decreases more slowly with V_0. The tensor binding has the least sensitivity on the change of the parameter $\lambda = m_q r_0$.

The above analysis shows that, independently of the Lorentz structure of the interaction, arbitrarily deep bound states cannot be obtained for finite values of V_0. For S and V interactions, the bound states disappear with increasing V_0. Only for A and T interactions we can expect asimptotically light meson states if quarks are sufficiently heavy.

Besides the five basic interactions we have also studied some of their linear combinations (Table 3). The results are shown in Fig. 5, where the lower limit of the energy spectrum is given as a function of V_0. One can see that V + A and

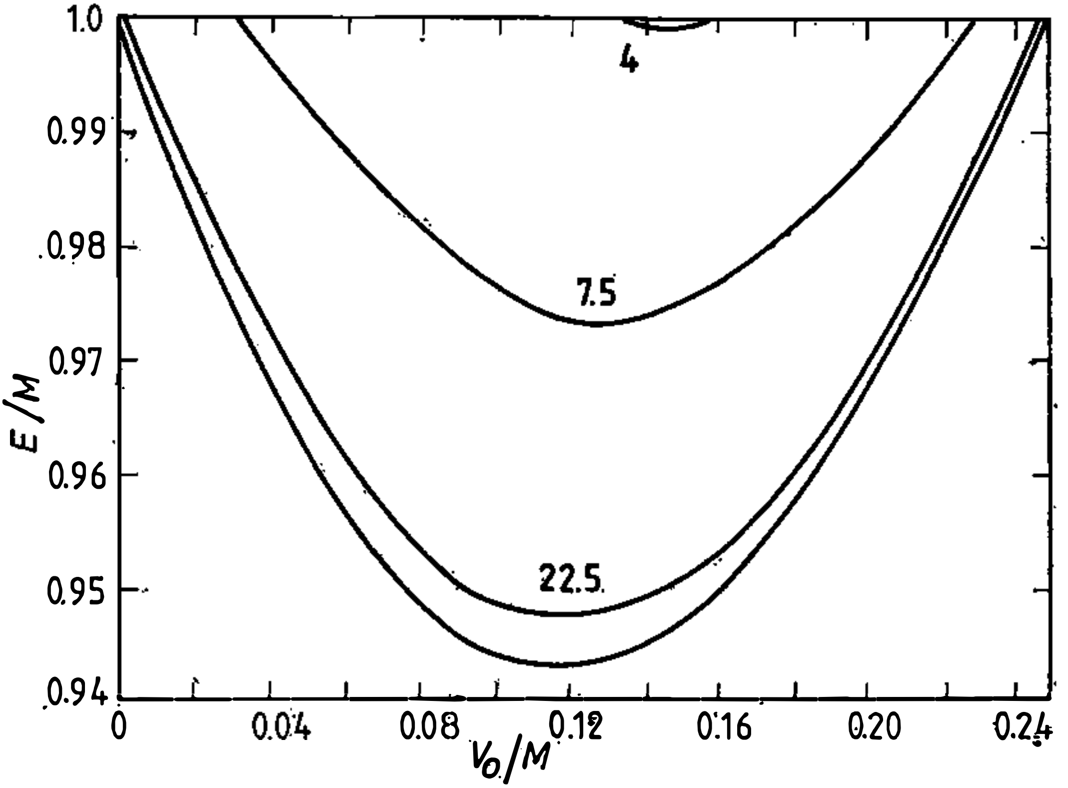

Fig. 2. Energy levels of the bound state at vector's potential for different values of the parameter $\lambda = m_q r_0$.

S + P + T interactions do not lead to $^1S_0$ states. Among the other interactions, only V − A and S + P − T can give deep bound states, in the asymptotic region ($V_0 \to \infty$).

4. *The one-particle limit*

Horwitz[6)] found that the existence of very small quark mass differences can completely change the nature of the solutions of the transcendental equation, so that many informations about the energy spectrum are practically lost in the case of equal quark masses. It is therefore, interesting to study the change of the energy of the lowest bound state with the change of the mass of one of two quarks. Let as fix the mass of one quark at $m_2 = 0.2$ GeV/c^2 (light quark) and choose the potential as

$$r_0 = 2 \times 200/(100 + m_1) = 0.7465 \text{ fm} \tag{9}$$

$$\widetilde{V}_0 = V_0/(100 + m_1). \tag{10}$$

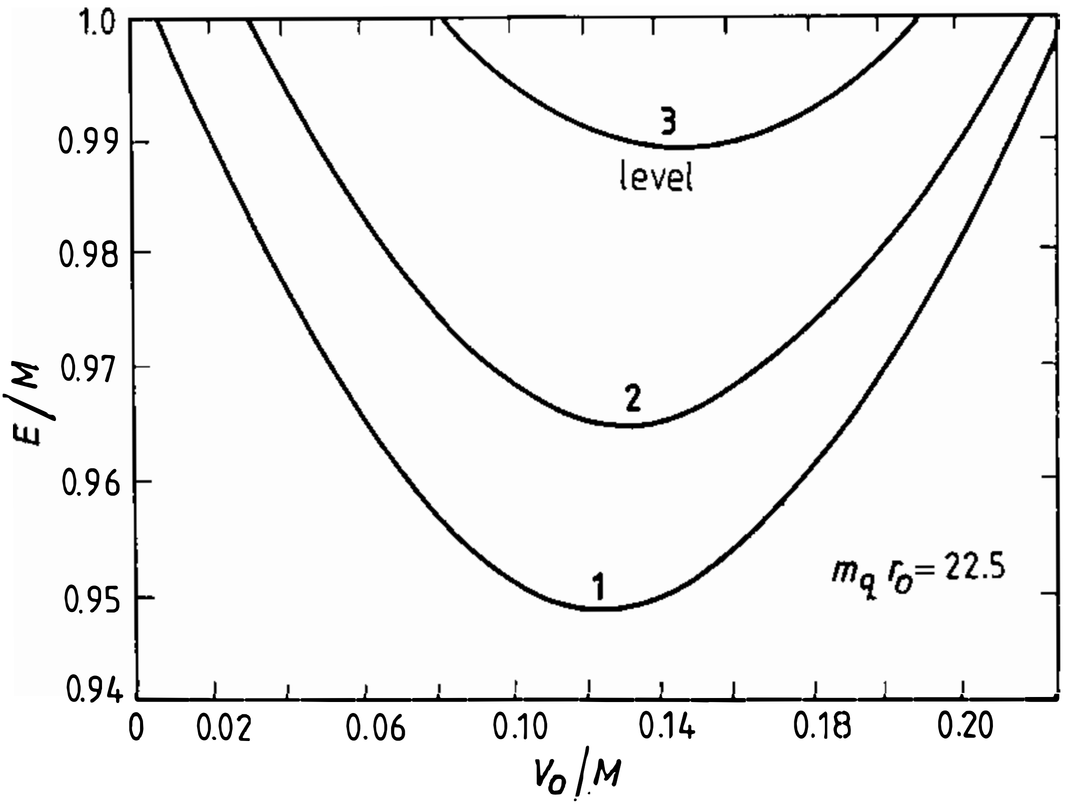

Fig. 3. The first three radial's levels as the functions of the depth of the potential for the vector coupling.

We also introduce dimensionless energy parameter

$$\widetilde{E} = (E - m_1)/m_2 = M/m_1 \times (E/M - m_2/M) \tag{11}$$

lying in the interval [−1, 1].

It should be noted that there are more limitations on the physical solutions here than in the case with equal quark masses. So, the scalar potential must lie above the curve $E_1 = -V_0 + M$, and above the line $E_s = V_0$ where the poles of the parameter $\alpha (E, V_0)$ are situated. Taking also into account $E > E_2 = V_0 + m$ we see that the minimal value of the energy of the bound states increases with the quark mass difference (Table 4).

In the case of the vector coupling, the singularity line $E_s = 2V_0$ lies below the curve $\widetilde{E}_1 = -\widetilde{V}_0 + \sqrt{1 + 9\widetilde{V}_0^2}$, so that it does not have any influence on the energy spectrum. For the axial coupling, however, the singularity line intersects the curve $\widetilde{E}_1 = -3\widetilde{V}_0 + \sqrt{1 + \widetilde{V}_0^2}$, which significantly changes the dependence

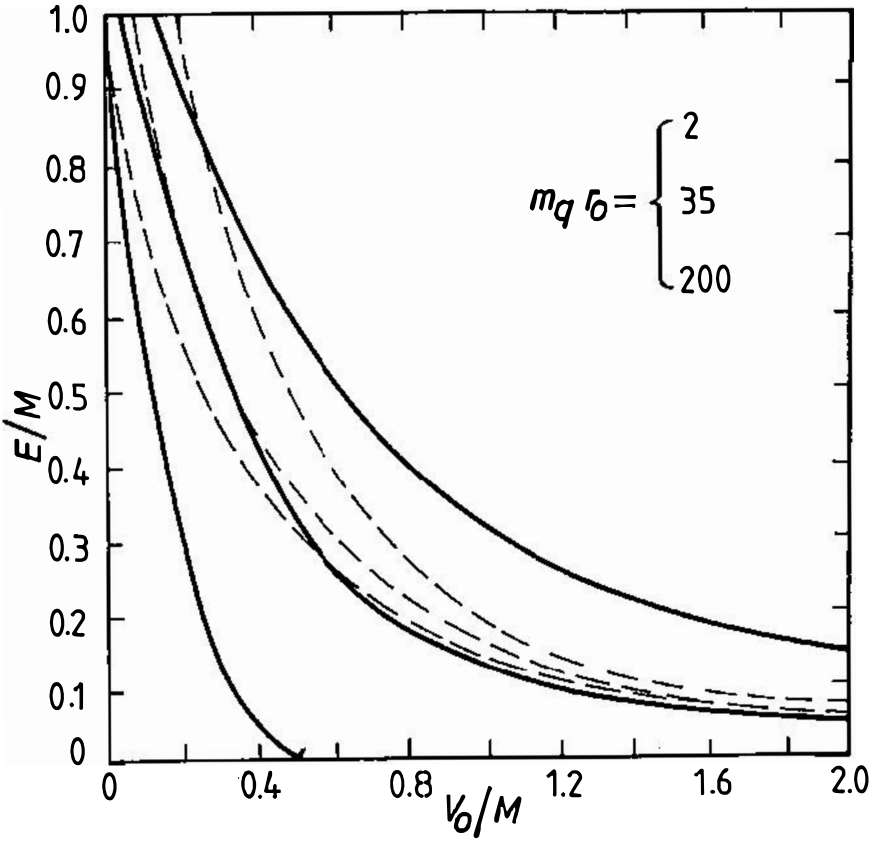

Fig. 4. A and T type of interactions.

TABLE 4.

Type	$m = 0$	$m \neq 0$
S	0.5, 0.5	$(1 + \delta)/2$, $(1 - \delta)/2$
V	0.942, 0.117	0.942, 0.117
A	0, 0.353	0.408, 0.204
T	0, ∞	δ, $(1 - \delta^2)/6\delta$

The absolute minimum ($E_{min.}$, V_0), $\delta = (m_1 - m_2)/(m_1 + m_2)$.

of the binding energy on the potential depth. The absolute minimum of energy is now $M/\sqrt{6}$, whereas in the case of equal quark masses it was asymptotically vanishing. The tensor coupling does not have any new limitations, so that one can find deeply bound states.

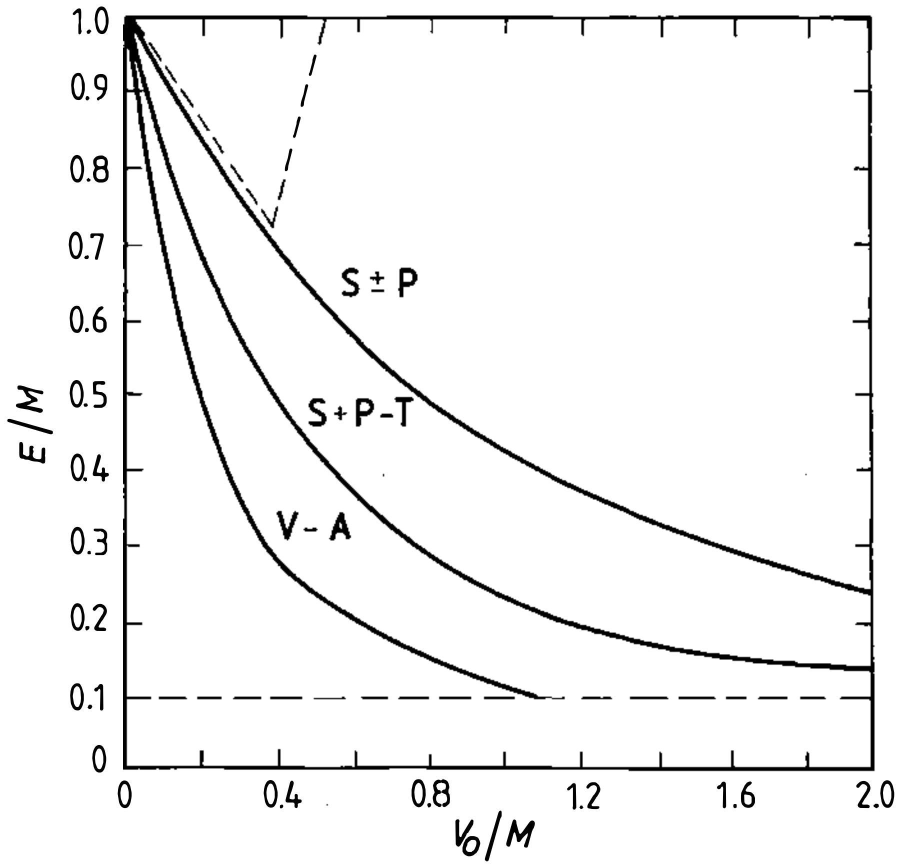

Fig. 5. Some combinations of the basic potentials.

The dependence of the energy of the lowest bound state on the depth of the potential is shown in Fig. 6 (for various values of m_1, $m_1 > m_2$). One can see that for a given value m_2 there exists a critical value m_1,

$$m_1^{(C)} = 6.94 \text{ GeV}/c^2.$$

For $m_1 < m_1^{(C)}$ the energy spectrum behaves in a way similar to equal mass case (the minimum of the energy spectrum decreases with increasing m_1). However, for $m_1 > m_1^{(C)}$ the spectrum intersects the limit for the lower continuum ($\widetilde{E} = -1$), i. e. the bound states disappear at $V_0 = V_{01}$, and appear again at $V_0 = V_{02} > V_{01}$. Further increase of m_1 leads to a wider energy gap $[V_{01}, V_{02}]$.

It has been shown in Ref. 7 that in the case of vector coupling, and for $m_1 \gg m_2$, one obtained results which are very close to the ones obtained from the one-particle Dirac equation. This fact justifies the use of an external field approximation, i. e. the relativistic model of quasi-independent quarks.

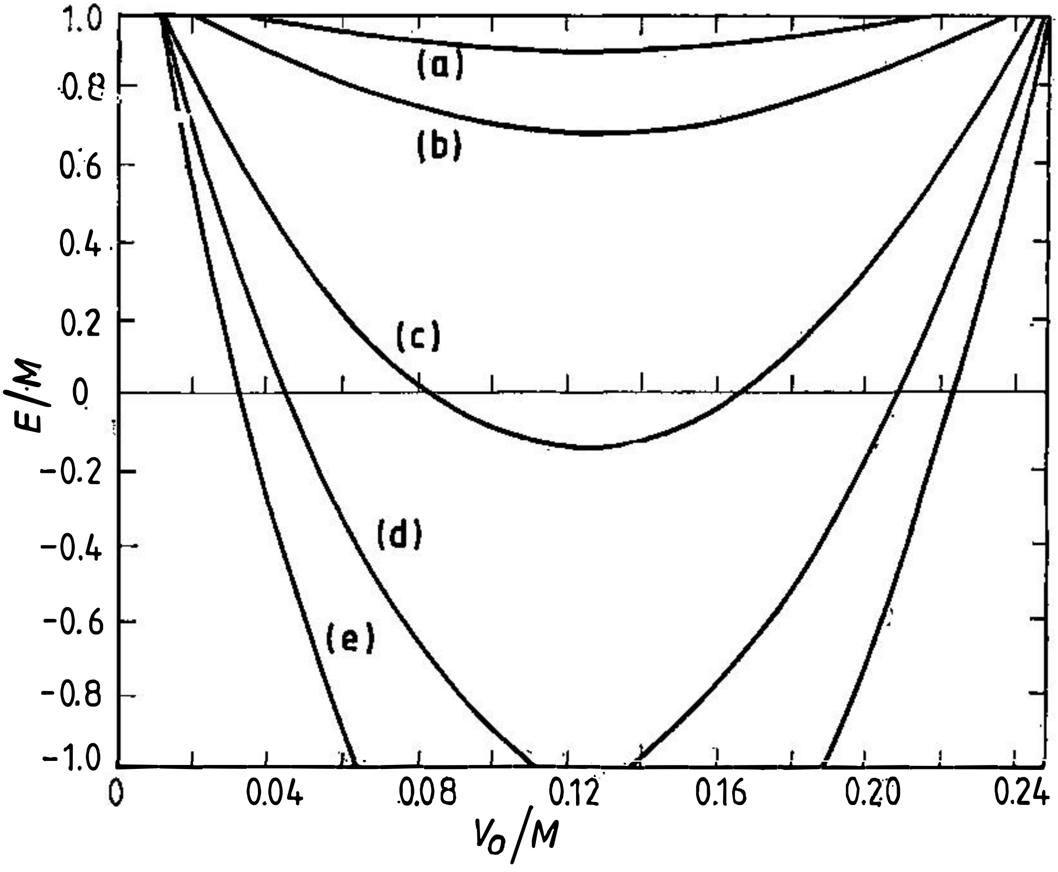

Fig. 6. The dependence of the lowest bound state on the depth of the potential for various values of m_1, $m_1 > m_2$.

5. *Discussion*

One of the basic characteristics of the non-relativistic (N. R.) quark model is that one can obtain a bound state independently of quark masses, by increasing the strength of interaction. In the semi-relativistic (S. R.) quark model this is not so for S- and V- type interactions. Second important property of the N. R. quark model is that the whole energy of the system, including the rest-masses, vanishes for a finite strength of the potential. The relativistic case is different. The axial and tensor coupling give deep bound states only asymptotically, when quarks are heavy. The similar situation occurs for V — A and S + P — T couplings.

In order to compare the N. R. and S. R. treatments more clearly we displayed in Figs. 7 and 8 the functions $E(V_0)$ for A and T coupling together with the corresponding N. R. results. For light quarks, $m_q \sim 0.1$—1 GeV/c^2, and $r_0 \cong$

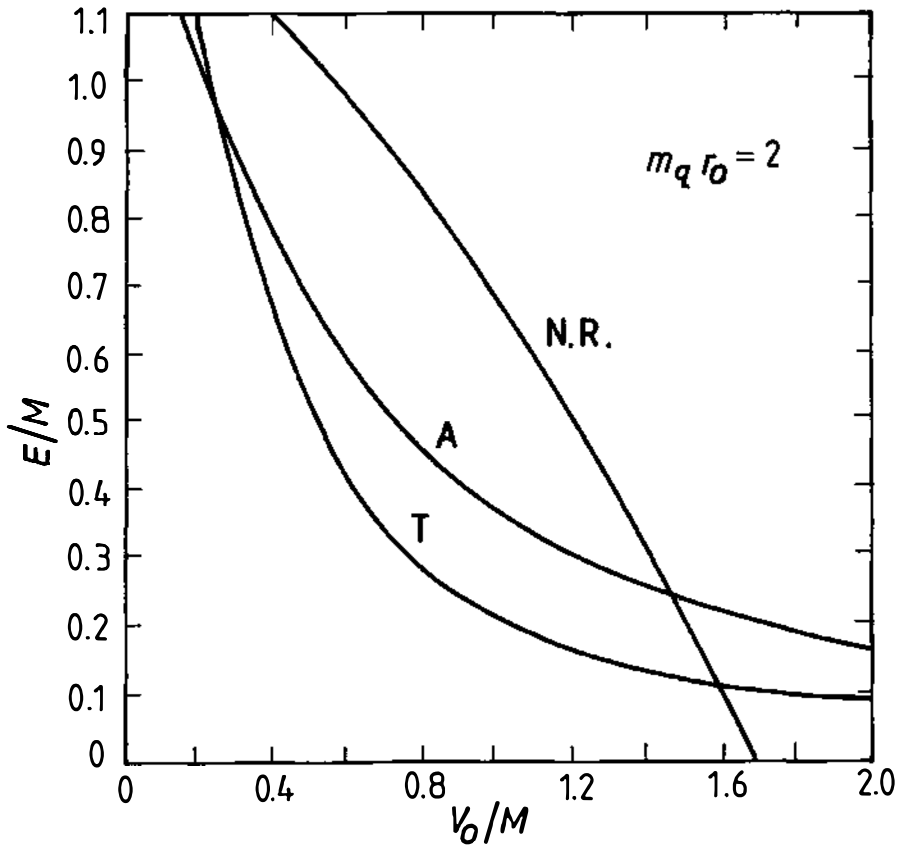

Fig. 7. The functions $E(V_0)$ for A and T couplings together with the corresponding N. R. result for $m_q r_0 = 2$.

$\cong$ 1 fm, i. e. $\lambda = 1-10$, these energy spectra are significantly different for $V_0 \approx \approx M$. That means that the use of the naive, N. R. quark model is not well sounded when the quarks are light. However, with increasing λ this difference decreases, so that for $m_1 \cong 10$ GeV/c^2 and $r_0 \cong$ a few fm, deep bound states can be considered as non-relativistic, specifically in the case of axial interaction with equal quark masses. If the quark masses are different, this interaction does not give bound states below $E = 0.408M$ (Table 4). In that case, the strongest binding can be realized by the V — A coupling[9].

6. Conclusion

We studied in this work the limitations on the naive quark model by using the two-particle Dirac equation. It is shown that arbitrary deep bound states can not be obtained for finite potential depths, independently of the type of interaction.

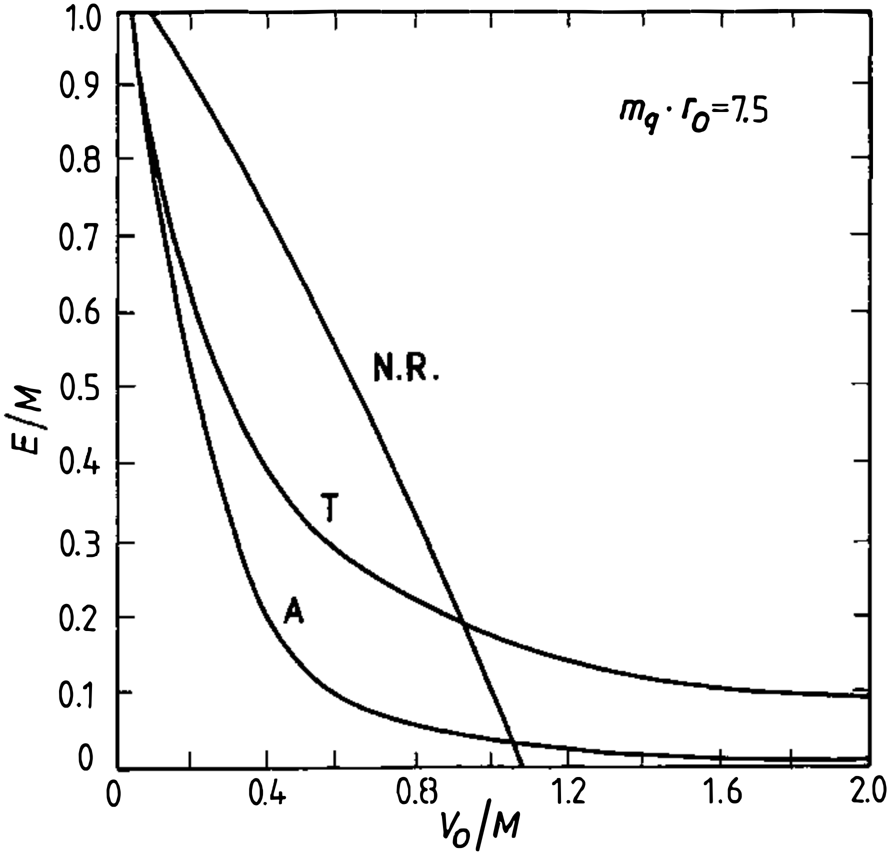

Fig. 8. The functions $E(V_0)$ for A and T couplings together with the corresponding N. R. result for $m_q r_0 = 7.5$.

For some interactions (S, A, T, V — A) there are solutions whose energy vanishes asymptotically. However, with unequal quark masses, the energy of the bound state must be higher than the quark mass difference.

We also investigated the dependence of the energy spectrum on the quark masses. It is shown that for $m_q \cong 10$ GeV/c^2 and $r_0 \cong$ a few fm, the deep bound states may be considered non-relativistic. This fact explains the success of the naive quark model in the spectroscopy of heavy meson resonances. However, the Schrödinger equation cannot be used for a unique treatment of old and new mesons. One of the practical possibility for building a unique quark model is the instant interaction of quark, with a dynamics described by the Bethe-Salpeter equation.

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References

1) Stephen Godfrey and Nathan Isqur, University of Toronto, Preprint MSS1A7 (1984);
2) D. Lalović, M. Blagojević and V. Zlatarov, Lettere al Nuovo Cimento **21** (1978) 73;
3) P. F. Smith and J. D. Lewin, Il Nuovo Cimento **64** (1981) 421;
4) Zhang Jianzu, Il Nouvo Cimento **78A** (1983) 435;
5) O. W. Greenberg, Phys. Rev. **447** (1966);
6) P. Horwitz, Phys. Rev. **461** (1967) 1415;
7) M. Bowin and J. P. Lavine, Il Nuovo Cimento **37B** (1977) 45;
8) Yoshio Koide, Prog. Theor. Phys. **39** (1968) 817;
9) M. Nagasaki, Prog. Theor. Phys. **37** (1967) 437;
K. Schilcher and H. J. W. Muller, Nuovo Cimento **4A** (1971) 243.

RELATIVISTIČKA OGRANIČENJA NAIVNOG KVARK MODELA

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U cilju izgradnje jedinstvenog kvark modela za mezone, razmotrena su ograničenja nerelativističkog potencijalnog modela. Na bazi dvočestične Dirac-ove jednačine, analizirana su osnovna svojstva spektra energija vezanih stanja. Za razne oblike kuplovanja, određene su donje granice masa kompozitnih mezona i ispitan karakter dinamike kvarkova.