YU ISSN 0015-3206
FZKAAA 21 (1) 43 (1989)

POLARIZATION STATE OF THE LIGHT TRANSMITTED BY A SANDWICH OF ISOTROPIC DIELECTRIC LAYERS

JOSIP MOSER and LJILJANA JANIĆIJEVIĆ

Fizički Institut, Prirodno-matematički fakultet, Univerzitet »Kiril i Metodij« — Skopje, P. O. Box 162, Gazi Baba b. b., 91 000 Skopje, Yugoslavia

Received 28 April 1988

Revised manuscript received 9 September 1988

UDC 535.51

Original scientific paper

The analysis of the transmission coefficient of the plate with multilayer film (each layer is made of isotropic material of different thickness and refractive index), shows that the polarization state of the light, plane at incidence, turns into elliptical one, when the light is transmitted by the plate. General expressions for the ellipticity and the azimuth of the vibrating ellipse as functions of the incident light azimuth and angle of incidence are found. The results are applied to the five layer sandwich. Specialisations are done for plates of reduced number of layers, as well as to sandwiches of two successively repeating components. Azimuths of 0° and 90° of the incident light, as well as normal incidence, preserve the linear polarization of the transmitted light by the multilayer plate.

1. Introduction

When the light transmission by isotropic layers is studied, it is usualy the amplitude (intensity) distribution which occupies the main interest of the investigator. Much less attention has been paid to the state of polarization of the transmitted light.

In few recent papers[1–3] it has been shown that the plane polarized monochromatic wave of light changes its state of polarization when it is transmitted by an isotropic dielectric plate. The transmitted light is ellipticaly polarized and the values of the azimuth of the major axis of the vibrating ellipse and the ellipticity are highly sensitive to the changes of the incident angle values.

In this paper we give a theoretical proof that the change of the polarization state from linear at incidence, into elliptical one at the exit, occurs when the light is transmitted by a system of $(m - 1)$ plane parallel layers bounded by m boundary planes. The theory is based on the relation between the azimuth and ellipticity of the transmitted light and the real and imaginary parts of the inverse value of the transmission coefficient[3-4]. The obtained expressions concern the general case when the layers are made of isotropic dielectric material of different thickness and refractive indexes.

As an illustration of the general theory we have found the expressions for the characteristical inverse transmittance coefficients for the case of a five layer sandwich. Such choice was governed by the need of outlining the influence of the interchange of two neighbouring, two successive and two layers separated by more than one layer, to the final results.

By reducing the number of the layers, it is possible to get the known simplified expressions, found in the mentioned references.

The results of the general theory and those concerning plates with known number of layers, may be found interesting for investigations in the field of thin film optics.

2. *The general solution*

Let the incident light be monochromatic of wavelegth λ. It has a linear polarization and its vibrations go in a direction which makes an angle ψ_0 with the plane of incidence.

Since the amplitude of the incident light does not take part in the determination of its polarization, we shall take its value to be equal to one.

Indexing by p and n the parallel and normal to the plane of incidence the components of the electric vector of the incident light, we have

$$E_{0p} = \cos \psi_0$$
$$E_{0n} = \sin \psi_0. \tag{1}$$

The incident and the outgoing media are vacuum of refractive index $n_0 = 1$, while the incident angle is α. We shall use the dummy indexes from 1 to $(m - 1)$ attached to the quantities which characterize the layers media. The index 1 stands for the last layer, while the layer indexed by $(m - 1)$ is the one which is first illuminated by the incident light. The corresponding absolute refractive indexes are n_j $(j = 1, 2, 3, \dots (m - 1))$, while by β_j we denote the angles between the light propagation directions and the boundary planes normals. According to the Snells refraction law

$$\sin \beta_j = \frac{\sin \alpha}{n_j} \quad j = 1, 2, \dots (m - 1). \tag{2}$$

To express the transmittance of the light from one to another medium, we use the Fresnel formulae. In order to have the same mathematical representation of these formulae for both the parallel and normal components of the electric vector, we introduce notations:

$$g_{jp} = \frac{\cos \beta_j}{n_j} = \frac{1}{n_j^2} \sqrt{n^2 - \sin^2 \alpha} \qquad j = 1, 2, \dots (m-1) \tag{3}$$

for the parallel, and

$$g_{jn} = n_j \cos \beta_j = \sqrt{n_j^2 - \sin^2 \alpha} \qquad j = 1, 2, \dots (m-1) \tag{4}$$

for the normal components. Only for the vacuum

$$g_{0p} = g_{0n} = \cos \alpha. \tag{5}$$

The transmitted light components are proportional to the corresponding incident light components

$$E_{p,n} = t_{p,n} E_{0\,p,n} \tag{6}$$

where the transmission coefficients $t_{p,n}$ should be determined. If their values are complex, the transmitted light components $E_{p,n}$ are complex too, which means that the transmitted light is ellipticaly polarized.

In Ref. 4 the transmission coefficient for a multilayer sandwich plate is defined. If the plate consists of $(m-1)$ layers bounded by m boundary planes, it is given by

$$t_{m\ p,n} = 2^m \prod_{j=0}^{(m-1)} g_{j\ p,n} / N_{m\ p,n}. \tag{7}$$

In the text which follows, instead of writing the three indexes, we shall keep only the first one, remembering that $t_m = t_{m,p}$ when $g_j = g_{j,p}$ and $t_m = t_{m,n}$ if $g_j = g_{j,n}$. The same stands for the quantity $N_{m\ p,n}$. It satisfies the following recursive formula:

$$N_{j+1} = [(g_{j+1} + g_j) N_j + (g_{j+1} - g_j) Z_j] \cos \varphi_j + \mathrm{i}\, [(g_{j+1} + g_j) N_j - - (g_{j+1} - g_j) Z_j] \sin \varphi_j \tag{8}$$

together with

$$Z_{j+1} = [(g_{j+1} - g_j) N_j + (g_{j+1} + g_j) Z_j] \cos \varphi_j - - \mathrm{i}\, [(g_{j+1} - g_j) N_j - (g_{j+1} + g_j) Z_j] \sin \varphi_j \tag{9}$$

and

$$\varphi_j = \frac{2\pi}{\lambda} d_j n_j \cos {}_j = \frac{2\pi}{\lambda} d_j \sqrt{n_j^2 - \sin^2 \alpha}. \tag{10}$$

$$\varphi_0 = 0.$$

In (8), (9) and (10) $j = 1, 2, 3, \dots (m-1)$. The starting values of the formulas (8) and (9) are

$$N_1 = g_1 + g_0 \qquad Z_1 = g_1 - g_0 \tag{11}$$

and by their help, for example, we get the well known Fresnel formulas[5] for the amplitude of the light refracted or reflected on the boundary between the plate 1 and the vacuum. With the meaning of our notations (3) and (4), for example

$$t_1 = \frac{2g_1}{g_1 + g_0} \tag{12}$$

for the transmitted, and

$$r_1 = \frac{(g_1 - g_0)}{(g_1 + g_0)} \tag{13}$$

for the reflected light.

As it is seen from the expressions (8), (9) and (11), except when $j = 0$, all expressions for N_{j+1} and Z_{j+1} are complex, i. e.

$$N_{j+1} = X_{j+1} + \mathrm{i} Y_{j+1} \text{ and } Z_{j+1} = U_{j+1} - \mathrm{i} V_{j+1} \; j = 1, 2, \dots (m-1). \tag{14}$$

Or, using notations (14), we have

$$X_1 = g_1 + g_0, \quad Y_1 = 0, \quad U_1 = g_1 - g_0, \quad V_1 = 0 \tag{15}$$

$$X_{j+1} = [(g_{j+1} + g_j) X_j + (g_{j+1} - g_j) U_j] \cos\varphi_j - [(g_{j+1} + g_j) Y_j + \\ + (g_{j+1} - g_j) V_j] \sin\varphi_j \qquad j = 1, 2, \dots (m-1) \tag{16}$$

$$Y_{j+1} = [(g_{j+1} + g_j) Y_j - (g_{j+1} - g_j) V_j] \cos\varphi_j + [(g_{j+1} + g_j) X_j - \\ - (g_{j+1} - g_j) U_j] \sin\varphi_j$$

$$U_{j+1} = [(g_{j+1} - g_j) X_j + (g_{j+1} + g_j) U_j] \cos\varphi_j + [(g_{j+1} - g_j) Y_j + \\ + (g_{j+1} + g_j) V_j] \sin\varphi_j \qquad j = 1, 2, \dots (m-1). \tag{17}$$

$$V_{j+1} = [(g_{j+1} - g_j) Y_j - (g_{j+1} + g_j) V_j] \cos\varphi_j - [(g_{j+1} - g_j) X_j - \\ - (g_{j+1} + g_j) U_j] \sin\varphi_j$$

Since N_m, appearing in the definition of the transmission coefficients (7), can be calculated by repeated application of the recursive formulas (15)—(17), starting with $j = 1$ and ending with $j = (m-1)$, it turns out, that the transmission coefficients $t_{m\,p,n}$ are complex too.

In other words

$$t_{m p,n}^{-1} = \frac{N_{m\ p,n}}{2^m \prod_{j=0}^{(m-1)} g_{j\ p,n}} = K_{m\ p,n} + \mathrm{i} L_{m\ p,n} \tag{18}$$

or

$$K_{m\ p,n} = \frac{X_{m\ p,n}}{2^m \prod_{j=0}^{(m-1)} g_{j\ p,n}} \tag{19}$$

$$L_{m\ p,n} = \frac{Y_{m\ p,n}}{2^m \prod_{j=0}^{(m-1)} g_{j\ p,n}}.$$

It is evident that both real and imaginary parts of the inverse value of the transmission coefficients (7) are different from zero and from each other. As we shall see in the next section, this supplies an elliptical polarization to the transmitted light.

3. Wolter transmission coefficients and characteristical parameters of the elliptical polarization

In one of our articles, here refered as Ref. 3, we have shown how the Wolter transmission coefficients are related to the main parameters which characterise the elliptical polarization of the light.

Namely, if the normal and parallel components of the light field vector are expressed by real amplitudes $A_{p,n}$ and phases $\delta_{p,n}$ such that $A_p \neq A_n$ and $\delta_p \neq \delta_n$, as

$$E_{p,n} = A_{p,n} \exp(\mathrm{i}\delta_{p,n}) = A_{p,n} \cos\delta_{p,n} + \mathrm{i} A_{p,n} \sin\delta_{p,n} \tag{20}$$

than, as it is shown in Ref. 6, the light is elliptically polarized. The azimuth ψ of the major semiaxis of the vibrating ellipse and the ellipticity ϑ are defined through $A_{p,n}$ and $\delta_{p,n}$ by

$$\mathrm{tg}\, 2\psi = \frac{2A_n A_p}{A_p^2 + A_n^2} (\cos\delta_p \cos\delta_n + \sin\delta_p \sin\delta_n) \tag{21}$$

$$\sin 2\vartheta = \frac{2A_p A_n}{A_n^2 + A_p^2} (\sin\delta_p \cos\delta_n - \cos\delta_p \sin\delta_n). \tag{22}$$

Let us now return to our definition of the transmitted light by the multilayer plate. According to (1), (6) and (18)

$$E_{p,n} = \frac{E_{0\ p,n}}{K_{p,n} + \mathrm{i} L_{p,n}}. \tag{23}$$

Thus

$$E_{mp} = \frac{K_{mp} \cos \psi_0}{K_{mp}^2 + L_{mp}^2} - \mathrm{i} \frac{L_{mp} \cos \psi_0}{K_{mp}^2 + L_{mp}^2} \tag{24}$$

$$E_{mn} = \frac{K_{mn} \sin \psi_0}{K_{mn}^2 + L_{mn}^2} - \mathrm{i} \frac{L_{mn} \sin \psi_0}{K_{mn}^2 + L_{mn}^2}. \tag{25}$$

The comparison of expression (20) with (24) and (25) gives

$$\begin{aligned} A_p \cos \delta_p &= \frac{K_{mp} \cos \psi_0}{K_{mp}^2 + L_{mp}^2} \quad A_p \sin \delta_p = - \frac{L_{mp} \cos \psi_0}{K_{mp}^2 + L_{mp}^2} \\ A_n \cos \delta_n &= \frac{K_{mn} \sin \psi_0}{K_{mn}^2 + L_{mn}^2} \quad A_n \sin \delta_n = - \frac{L_{mn} \sin \psi_0}{K_{mn}^2 + L_{mn}^2} \end{aligned} \tag{26}$$

which introduced in (21) and (22), defines the azimuth ψ and the ellipticity ϑ of the transmitted light as:

$$\mathrm{tg}\, 2\psi = \frac{2 (K_{mp} K_{mn} + L_{mp} L_{mn}) \,\mathrm{tg}\, \psi_0}{K_{mn}^2 + L_{mn}^2 - (K_{mp}^2 + L_{mp}^2) \,\mathrm{tg}^2\, \psi_0} \tag{27}$$

$$\sin 2\vartheta = \frac{2 (K_{mp} L_{mn} - K_{mn} L_{mp}) \,\mathrm{tg}\, \psi_0}{K_{mn}^2 + L_{mn}^2 + (K_{mp}^2 + L_{mp}^2) \,\mathrm{tg}^2\, \psi_0} \tag{28}$$

Therefore, for investigation of the polarization state of the transmitted light by any multilayer sandwich, it is important to determine, the coefficients $K_{m\ p,n}$ and $L_{m\ p,n}$.

4. *Application of the general theory to the case of a five layer sandwich*

Let us apply the general theory to the case of a five layer sandwich. Starting with $j = 1$, by repeated application of the recursive formulas (16) and (17) and ending with $j = 5$, we have calculated the values X_6 and Y_6. Than, using definitions (19), we have found that for the five layers bounded by six boundary planes the corresponding coefficients K_6 and L_6 are:

$$\begin{aligned} K_6 = {} & \cos \varphi_1 \cos \varphi_2 \cos \varphi_3 \cos \varphi_4 \cos \varphi_5 - (1/2) (g_1/g_2 + g_2/g_1) \sin \varphi_1 \cdot \\ & \cdot \sin \varphi_2 \cos \varphi_3 \cos \varphi_4 \cos \varphi_5 - (1/2) (g_1/g_3 + g_3/g_1) \sin \varphi_1 \cos \varphi_2 \sin \varphi_3 \cdot \\ & \cdot \cos \varphi_4 \cos \varphi_5 - (1/2) (g_1/g_4 + g_4/g_1) \sin \varphi_1 \cos \varphi_2 \cos \varphi_3 \sin \varphi_4 \cos \varphi_5 - \\ & - (1/2) (g_1/g_5 + g_5/g_1) \sin \varphi_1 \cos \varphi_2 \cos \varphi_3 \cos \varphi_4 \sin \varphi_5 - (1/2) (g_2/g_3 + \end{aligned}$$

$$
\begin{aligned}
&+ g_3/g_2) \cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \cos\varphi_5 - (1/2)(g_2/g_4 + g_4/g_2) \cos\varphi_1 \cdot \\
&\cdot \sin\varphi_2 \cos\varphi_3 \sin\varphi_4 \cos\varphi_5 - (1/2)(g_2/g_5 + g_5/g_2) \cos\varphi_1 \sin\varphi_2 \cos\varphi_3 \cdot \\
&\cdot \cos\varphi_4 \sin\varphi_5 - (1/2)(g_3/g_4 + g_4/g_3) \cos\varphi_1 \cos\varphi_2 \sin\varphi_3 \sin\varphi_4 \cos\varphi_5 - \\
&- (1/2)(g_3/g_5 + g_5/g_3) \cos\varphi_1 \cos\varphi_2 \sin\varphi_3 \cos\varphi_4 \sin\varphi_5 - (1/2)(g_4/g_5 + \\
&+ g_5/g_4) \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 \sin\varphi_4 \sin\varphi_5 + (1/2)(g_1g_3/g_2g_4 + g_2g_4/g_1g_3) \cdot \\
&\cdot \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \cos\varphi_5 + (1/2)(g_1g_3/g_2g_5 + g_2g_5/g_1g_3) \sin\varphi_1 \cdot \\
&\cdot \sin\varphi_2 \sin\varphi_3 \sin\varphi_5 \cos\varphi_4 + (1/2)(g_1g_4/g_2g_5 + g_2g_5/g_1g_4) \sin\varphi_1 \sin\varphi_2 \cdot \\
&\cdot \cos\varphi_3 \sin\varphi_4 \sin\varphi_5 + (1/2)(g_1g_4/g_3g_5 + g_3g_5/g_1g_4) \sin\varphi_1 \cos\varphi_2 \sin\varphi_3 \cdot \\
&\cdot \sin\varphi_4 \sin\varphi_5 + (1/2)(g_2g_4/g_3g_5 + g_3g_5/g_2g_4) \cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \cdot \\
&\cdot \sin\varphi_4 \sin\varphi_5
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
L_6 = {} & (1/2)(g_0/g_1 + g_1/g_0) \sin\varphi_1 \cos\varphi_2 \cos\varphi_3 \cos\varphi_4 \cos\varphi_5 + (1/2)(g_0/g_2 + g_2/g_0) \cdot \\
&\cdot \cos\varphi_1 \sin\varphi_2 \cos\varphi_3 \cos\varphi_4 \cos\varphi_5 + (1/2)(g_0/g_3 + g_3/g_0) \cos\varphi_1 \cos\varphi_2 \cdot \\
&\cdot \sin\varphi_3 \cos\varphi_4 \cos\varphi_5 + (1/2)(g_0/g_4 + g_4/g_0) \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 \sin\varphi_4 \cdot \\
&\cdot \cos\varphi_5 + (1/2)(g_0/g_5 + g_5/g_0) \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 \cos\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_2/g_1g_3 + g_1g_3/g_0g_2) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \cos\varphi_5 - \\
&- (1/2)(g_0g_2/g_1g_4 + g_1g_4/g_0g_2) \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \sin\varphi_4 \cos\varphi_5 - \\
&- (1/2)(g_0g_2/g_1g_5 + g_1g_5/g_0g_2) \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \cos\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_3/g_1g_4 + g_1g_4/g_0g_3) \sin\varphi_1 \cos\varphi_2 \sin\varphi_3 \sin\varphi_4 \cos\varphi_5 - \\
&- (1/2)(g_0g_3/g_1g_5 + g_1g_5/g_0g_3) \sin\varphi_1 \cos\varphi_2 \sin\varphi_3 \cos\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_3/g_2g_4 + g_2g_4/g_0g_3) \cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \cos\varphi_5 - \\
&- (1/2)(g_0g_3/g_2g_5 + g_2g_5/g_0g_3) \cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_4/g_1g_5 + g_1g_5/g_0g_4) \sin\varphi_1 \cos\varphi_2 \cos\varphi_3 \sin\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_4/g_2g_5 + g_2g_5/g_0g_4) \cos\varphi_1 \sin\varphi_2 \cos\varphi_3 \sin\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_4/g_3g_5 + g_3g_5/g_0g_4) \cos\varphi_1 \cos\varphi_2 \sin\varphi_3 \sin\varphi_4 \sin\varphi_5 - \\
&- (1/2)(g_0g_2g_4/g_1g_3g_5 + g_1g_3g_5/g_0g_2g_4) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \sin\varphi_4 \cdot \\
&\cdot \sin\varphi_5 .
\end{aligned}
\tag{30}
$$

In both expressions (29) and (30) it is understood that $K_6 = K_{6p}$ and $L_6 = L_{6p}$ if all g_j ($j = 1, 2, \ldots, 5$) appearing in the expressions are defined by (3) i. e $g_j = g_{jp}$, while $K_6 = K_{6n}$ and $L_6 = L_{6n}$ when each $g_j = g_{jn}$.

Next step in solving the five layer problem is to use the relations (27) and (28) and determine the azimuth ψ and ellipticity ϑ as parameters of the elliptical polarization.

As it is seen from expressions (29) and (30), both K_6 and L_6 consists of 16 terms. Their number can be reduced if some of the layers are the same.

4.1. *The light is transmitted by two equal systems of thin films*

Of all possible combinations we shall pay a special attention to the case when on the transparent carrier the layers keep repeating periodically. If the carrier is assigned by index 1, the next two different layers have indexes 2 and 3, and than instead of the layers indexed by 4 and 5, we have again 2 and 3 as indexes of the layers. By doing so in the expressions (29) and (30) we get

$$\begin{aligned} K_6 = {} & \cos\varphi_1 (\cos^2\varphi_2 \cos^2\varphi_3 - \cos^2\varphi_2 \sin^2\varphi_3 - \sin^2\varphi_2 \cos^2\varphi_3) - \\ & - (1/2)(g_1/g_2 + g_2/g_1) \sin\varphi_1 \cos\varphi_2 \sin\varphi_2 (2\cos^2\varphi_3 - \sin^2\varphi_3) - \\ & - (1/2)(g_1/g_3 + g_1/g_3) \sin\varphi_1 \cos\varphi_3 \sin\varphi_3 (2\cos^2\varphi_2 - \sin^2\varphi_2) - \\ & - 2(g_2/g_3 + g_3/g_2) \cos\varphi_1 \cos\varphi_2 \sin\varphi_2 \cos\varphi_3 \sin\varphi_3 + \\ & + (1/2)(g_1 g_2/g_3^2 + g_3^2/g_1 g_2) \sin\varphi_1 \cos\varphi_2 \sin\varphi_2 \sin^2\varphi_3 + (1/2)(g_1 g_3/g_2^2 + \\ & + g_2^2/g_1 g_3) \sin\varphi_1 \sin^2\varphi_2 \cos\varphi_3 \sin\varphi_3 + (1/2)(g_2^2/g_3^2 + g_3^2/g_2^2) \cos\varphi_1 \cdot \\ & \cdot \sin^2\varphi_2 \sin^2\varphi_3 \end{aligned} \tag{31}$$

$$\begin{aligned} L_6 = {} & (1/2)(g_0/g_1 + g_1/g_0) \sin\varphi_1 (\cos^2\varphi_2 \cos^2\varphi_2 - \sin^2\varphi_2 \cos^2\varphi_3 - \\ & - \cos^2\varphi_2 \sin^2\varphi_3) + (1/2)(g_0/g_2 + g_2/g_0) \cos\varphi_1 \cos\varphi_2 \sin\varphi_2 (2\cos^2\varphi_3 - \\ & - \sin^2\varphi_3) + (1/2)(g_0/g_3 + g_3/g_0) \cos\varphi_1 \cos\varphi_3 \sin\varphi_3 (2\cos^2\varphi_2 - \\ & - \sin^2\varphi_2) - [(3/2)(g_0 g_2/g_1 g_3 + g_1 g_3/g_0 g_2) + (1/2)(g_0 g_3/g_1 g_2 + \\ & + g_1 g_2/g_0 g_3)] \sin\varphi_1 \cos\varphi_2 \sin\varphi_2 \cos\varphi_3 \sin\varphi_3 - (1/2)(g_0 g_2/g_3^2 + \\ & + g_3^2/g_0 g_2) \cos\varphi_1 \cos\varphi_2 \sin\varphi_2 \sin^2\varphi_3 - (1/2)(g_0 g_3/g_2^2 + g_2^2/g_0 g_3) \cdot \\ & \cdot \cos\varphi_1 \sin^2\varphi_2 \cos\varphi_3 \sin\varphi_3 + (1/2)(g_0 g_2^2/g_1 g_3^2 + g_1 g_3^2/g_0 g_2^2) \sin\varphi_1 \cdot \\ & \cdot \sin^2\varphi_2 \sin^2\varphi_3. \end{aligned} \tag{32}$$

Their further specialization for the parallel and normal component and insertion in expressions (27) and (28), determines the polarization state of the light transmitted by this type of layer.

When the optical characteristics of the thin films are studied, it is accustomed to eliminate the influence of the carrier by taking $d_1 = 0$ which implies $\varphi_1 = 0$. So from (31) and (32) it follows that

$$\begin{aligned} K = {} & \cos^2\varphi_2 \cos^2\varphi_3 - \cos^2\varphi_2 \sin^2\varphi_3 - \sin^2\varphi_2 \cos^2\varphi_3 - 2(g_2 g_3 + \\ & + g_3/g_2) \cos\varphi_2 \sin\varphi_2 \cos\varphi_3 \sin\varphi_3 + (1/2)(g_2^2/g_3^2 + g_3^2/g_2^2) \sin^2\varphi_2 \cdot \\ & \cdot \sin^2\varphi_3 \end{aligned} \tag{33}$$

$$L = (1/2)(g_0/g_2 + g_2/g_0)\cos\varphi_2 \sin\varphi_2 (2\cos^2\varphi_3 - \sin^2\varphi_3) + \\ + (1/2)(g_0/g_3 + g_3/g_0)\cos\varphi_3 \sin\varphi_3 (2\cos^2\varphi_2 - \sin^2\varphi_2) - \\ - (1/2)(g_0 g_2/g_3^2 + g_3^2/g_0 g_2)\cos\varphi_2 \sin\varphi_2 \sin^2\varphi_3 - (1/2)(g_0 g_3/g_2^2 + \\ + g_2^2/g_0 g_3)\sin^2\varphi_2 \cos\varphi_3 \sin\varphi_3. \qquad (34)$$

It is interesting to notice that there exist angles of incidence which give the same polarization to the light propagating through the carrier and the thin film, as it was passing through the thin film only. These angles are defined by

$$\varphi_1 = 2\pi N \qquad N = 1, 2, 3, \ldots \qquad (35)$$

In this case $\sin\varphi_1 \neq 0$ and $\cos\varphi_1 = 1$, and it becomes clear that expressions (33) and (34) are obtained from (31) and (32). Specialising expression (10) we have

$$\sin\alpha = \sqrt{n_1^2 - (\lambda N/d_1)^2}. \qquad (36)$$

If in addition we take into account that $0 < \sin\alpha < 1$, it follows that the integer N satisfies the unequalities

$$d_1 n_1/\lambda > N > (d_1/\lambda)\sqrt{n_1^2 - 1} \qquad N = 1, 2, 3, \ldots \qquad (37)$$

This indicates that these special angles of incidence depend on the carrier characteristics only.

4.2 *Three equal layers separated by other two equal layers*

Let the five layer film be constructed as a sandwich of three equal components (made of same material and having the same thickness) split up by other two equal layers, whose substance and thickness differ from those of the three layers. Since in this case the first, third and the fifth layer are the same, the indexes 3 and 5 should be replaced by 1, while for the second and the fourth layer we use the index 2. Such changes introduced in (29) and (30) yield

$$K = \cos^3\varphi_1 (\cos^2\varphi_2 - \sin^2\varphi_2) - \cos\varphi_1 \sin\varphi_1 (3\cos^2\varphi_2 - \sin^2\varphi_2) - \\ - (1/2)(g_1/g_2 + g_2/g_1)\sin\varphi_1 \cos\varphi_2 \sin\varphi_2 (3\cos^2\varphi_1 + \sin^2\varphi_1) + \\ + (g_1^2/g_2^2 + g_2^2/g_1^2)\sin^2\varphi_1 \cos\varphi_1 \sin^2\varphi_2 \qquad (38)$$

$$L = (1/2)(g_0/g_1 + g_1/g_0)(3\cos^2\varphi_1 \sin\varphi_1 \cos^2\varphi_2 - 2\cos^2\varphi_1 \sin\varphi_1 \cdot \\ \cdot \sin^2\varphi_2 - \sin^3\varphi_1 \cos^2\varphi_2) + (1/2)(g_0/g_2 + g_2/g_0)(2\cos^3\varphi_1 \cos\varphi_2 \cdot \\ \cdot \sin\varphi_2 - 2\cos\varphi_1 \sin^2\varphi_1 \cos\varphi_2 \sin\varphi_2) + (1/2)(g_0 g_1/g_2^2 + \\ + g_2^2/g_0 g_1)\cos^2\varphi_1 \sin\varphi_1 \sin^2\varphi_2 - 2(g_0 g_2^2/g_1^2 + g_1^2/g_0 g_2)\cos\varphi_1 \sin^2\varphi_1 \cdot \\ \cdot \cos\varphi_2 \sin\varphi_2 + (1/2)(g_0 g_2^2/g_1^3 + g_1^3/g_0 g_2^2)\sin^3\varphi_1 \sin^2\varphi_2. \qquad (39)$$

These values calculated for a particular problem, and introduced in (27) and (28), determine an elliptical polarization of the transmitted light.

In all of the discussed cases it is possible for one of the layers to be empty. In such case the layer is »made« of vacuum, and, according to (10), we put

$$\varphi_j = (2\pi/\lambda)\, d_j \cos \alpha \tag{40}$$

while everything else is the same as we had a substancial layer instead.

5. *Change of the order of the layers*

In the preceding discussion we considered that the light goes through the layers ordered by indexes 5, 4, 3, 2, 1. The opposite order of the layers can be arranged either by rotating the plate for 180° or by replacing the light source on the opposite side of the plate. For this case we must replace the indexes $1 \leftrightarrow 5$ and $2 \leftrightarrow 4$ in expressions (29) and (30). It is not hard to see that this type of change of the propagation direction does not affect the result and therefore does not influence the state of the transmitted light polarization.

On the other hand, if we try to interchange the places of two neighbouring, two succesive, two third or fourth neighbour layers, expressions (29) and (30) will give different values for K and L. This means that when such layer interchange occurs, the ellipticity (27) and the azimuth (28) of the transmitted light possess different values.

6. *Specialisation of the results for systems of lower number of layers*

The solutions (29) and (30) can be used to get the corresponding expressions for the systems of lower number of layers.

Let us first consider a four layer plate. The fifth layer does not exist i. e. its thickness $d_5 = 0$. From the expression (10) it follows that $\varphi_5 = 0$, which reduces expressions (29) and (30) to:

$$\begin{aligned} K_5 = {} & \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 \cos\varphi_4 - (1/2)(g_1/g_2 + g_2/g_1) \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \cdot \\ & \cdot \cos\varphi_4 - (1/2)(g_1/g_3 + g_3/g_1) \sin\varphi_1 \cos\varphi_2 \sin\varphi_3 \cos\varphi_4 - \\ & - (1/2)(g_1/g_4 + g_4/g_1) \sin\varphi_1 \cos\varphi_2 \cos\varphi_3 \sin\varphi_4 - (1/2)(g_2/g_3 + \\ & + g_3/g_2) \cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 - (1/2)(g_2/g_4 + g_4/g_2) \cos\varphi_1 \cdot \\ & \cdot \sin\varphi_2 \cos\varphi_3 \sin\varphi_4 - (1/2)(g_3/g_4 + g_4/g_3) \cos\varphi_1 \cos\varphi_2 \sin\varphi_3 \sin\varphi_4 + \\ & + (1/2)(g_1 g_3/g_2 g_4 + g_2 g_4/g_1 g_3) \sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \sin\varphi_4. \end{aligned} \tag{41}$$

$$\begin{aligned} L_5 = {} & (1/2)(g_0/g_1 + g_1/g_0) \sin\varphi_1 \cos\varphi_2 \cos\varphi_3 \cos\varphi_4 + (1/2)(g_0/g_2 + \\ & + g_2/g_0) \cos\varphi_1 \sin\varphi_2 \cos\varphi_3 \cos\varphi_4 + (1/2)(g_0/g_3 + g_3/g_0) \cos\varphi_1 \cdot \end{aligned}$$

$$\cdot \cos\varphi_2 \sin\varphi_3 \cos\varphi_4 + (1/2)(g_0/g_4 + g_4/g_0)\cos\varphi_1 \cos\varphi_2 \cos\varphi_3 \sin\varphi_4 - \\ - (1/2)(g_0 g_2/g_1 g_3 + g_1 g_3/g_0 g_2)\sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \cos\varphi_4 - (1/2)(g_0 g_2/ \\ /g_1 g_4 + g_1 g_4/g_0 g_2)\sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \sin\varphi_4 - (1/2)(g_0 g_3/g_1 g_4 + \\ + g_1 g_4/g_0 g_3)\sin\varphi_1 \cos\varphi_2 \sin\varphi_3 \sin\varphi_4 - (1/2)(g_0 g_3/g_2 g_4 + \\ + g_2 g_4/g_0 g_3)\cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \sin\varphi_4. \quad (42)$$

As it is seen, both K_5 and L_5 are expressed by eight terms.

For the three layer plate $d_4 = 0$ and $\varphi_4 = 0$, so that the further reduction leads to the following coefficients:

$$K_4 = \cos\varphi_1 \cos\varphi_2 \cos\varphi_3 - (1/2)(g_1/g_2 + g_2/g_1)\sin\varphi_1 \sin\varphi_2 \cos\varphi_3 - \\ - (1/2)(g_1/g_3 + g_3/g_1)\sin\varphi_1 \cos\varphi_2 \sin\varphi_3 - \\ - (1/2)(g_2/g_3 + g_3/g_2)\cos\varphi_1 \sin\varphi_2 \sin\varphi_3 \quad (43)$$

and

$$L_4 = (1/2)(g_0/g_1 + g_1/g_0)\sin\varphi_1 \cos\varphi_2 \cos\varphi_3 + (1/2)(g_0/g_2 + \\ + g_2/g_0)\cos\varphi_1 \sin\varphi_2 \cos\varphi_3 + (1/2)(g_0/g_3 + g_3/g_0)\cos\varphi_1 \cos\varphi_2 \sin\varphi_3 - \\ - (1/2)(g_0 g_2/g_1 g_3 + g_1 g_3/g_0 g_2)\sin\varphi_1 \sin\varphi_2 \sin\varphi_3 \quad (44)$$

K_4 and L_4 have four terms each.

Much simpler is the two layer plate. Than $d_3 = 0$ and $\varphi_3 = 0$ so that (43) and (44) reduce to:

$$K_3 = \cos\varphi_1 \cos\varphi_2 - (1/2)(g_1/g_2 + g_2/g_1)\sin\varphi_1 \sin\varphi_2 \\ L_3 = (1/2)(g_0/g_1 + g_1/g_0)\sin\varphi_1 \cos\varphi_2 + (1/2)(g_0/g_2 + \\ + g_2/g_0)\cos\varphi_1 \sin\varphi_2. \quad (45)$$

The same expressions were obtained in Ref. 3 by direct calculation.

And finaly, when the light goes through one layer only, $d_2 = 0$ and $\varphi_2 = 0$, so that

$$K_2 = \cos\varphi_1 \\ L_2 = (1/2)(g_0/g_1 + g_1/g_0)\sin\varphi_1, \quad (46)$$

i. e. we obtain the known formulae for this case (Ref. 2).

These specialisations suggest that each additional layer on the plate, doubles the number of the terms in the expresions for the coefficients $K_{m\ p,n}$ and $L_{m\ p,n}$. For example, if instead of five we had a six layer system, the corresponding coefficients for the transmission on the seven boundary planes $K_{7\ p,n}$ and $L_{7\ p,n}$ would contain 32 terms each. In other words, for the $(m - 1)$ layer system bounded by m planes, the coefficients $K_{m\ p,n}$ and $L_{m\ p,n}$ consist of 2^{m-2} terms each.

7. The linear polarization of the transmitted light

The expressions (27) and (28) show that the two parameters which characterize the elliptical polarization of the transmitted light, depend on the azimuth of polarization of the incident light and on the angle of incidence. This last dependence is rather complicated and is implicit, since the coefficients $K_{m\ p,n}$ and $L_{m\ p,n}$ are functions of α through $g_{j\ p,n}$ and φ_j ($j = 1, 2, 3, \ldots (m-1)$).

The transmitted light will be linearly polarized if its ellipticity $\vartheta = \text{arc tg}\,(b/a) = 0$ (a and b are the major and minor semiaxes of the vibrating ellipse of the light field vector). As it is seen from expression (28) this occurs when

$$K_{mp}(\alpha)\, L_{mn}(\alpha) - K_{mn}(\alpha)\, L_{mp}(\alpha) = 0. \tag{46'}$$

The difference between the coefficients K and L for the parallel and norma[1] component is dictated by the differences in the definitions (3) and (4) of the g_j-parameters. It is not hard to verify, by using these definitions, that

$$g_{jp}/g_{kp} = g_{kn}/g_{jn} \quad j, k = 0, 1, 2, \ldots (m-1) \tag{47}$$

$$\text{if} \quad \alpha = 0 \tag{48}$$

Thus, the sums of the two inverse quotients of the g_j-parameters, which multiply the products of the sine and cosine functions in expressions (29) and (30), (31) and (32), (33) and (34), (38) and (39), (41) and (42), (43) and (44), (45) and (46) will remain unchanged if they are specialised for the parallel and normal component when the incidence is normal, i. e.

$$K_{mp}(0) = K_{mn}(0) \quad \text{and} \quad L_{mp}(0) = L_{mn}(0) \tag{49}$$

and equation (46) is satisfied.

Substituting conditions (49) in expression (27), we find that than

$$\text{tg}\, 2\psi = 2\, \text{tg}\, \psi_0 / (1 - \text{tg}^2\, \psi_0) = \text{tg}\, 2\psi_0. \tag{50}$$

or

$$\psi = \psi_0.$$

It means that at normal incidence, the linear polarization of the incident light at transmission by five, four, three, two and one layer sandwich, remains unchanged. The azimuth of the transmitted light is the same as that of the incident light.

For other angles of incidence, the transcendental equation (46) should be solved for a particular type of sandwich plate.

Regardless to the value of the angle of incidence, to the physical properties and to the number of the layers in the sandwich, there exist two other initial conditions which preserve the linear polarization of the light transmitted by the sandwich.

When the vibrations of the incident light occur in the incident plane, their azimuth of polarization is

$$\psi_0 = 0. \tag{51}$$

Than, it is not hard to see from (27) and (28) that

$$\vartheta = 0 \quad \text{and} \quad \psi = 0 \tag{52}$$

the vibrations of the outgoing light also occur in the incident plane.

If we put $\operatorname{tg} \psi_0 = 1/\operatorname{ctg} \psi_0$ in expressions (27) and (28), they can be written as

$$\operatorname{tg} 2\psi = \frac{(K_{mp} K_{mn} + L_{mp} L_{mn}) \operatorname{ctg} \psi_0}{(K_{mp}^2 + L_{mp}^2) \operatorname{ctg}^2 \psi_0 - (K_{mn}^2 + L_{mn}^2)} \tag{53}$$

$$\sin 2\vartheta = \frac{(K_{mp} L_{mn} - K_{mn} L_{mp}) \operatorname{ctg} \psi_0}{(K_{mp}^2 + L_{mp}^2) \operatorname{ctg}^2 \psi_0 + (K_{mn}^2 + L_{mn}^2)}. \tag{54}$$

Now it becomes clear that when $\psi_0 = \pi/2$ i. e. when the vibrations of the incident light are orthogonal to the incident plane,

$$\vartheta = 0 \quad \text{and} \quad \psi = \pi/2 \tag{55}$$

the transmitted light is linear with an azimuth equal as that of the incident light.

For all other azimuth values the transmitted light is ellipticaly polarized.

8. Conclusion

Calculated by the Wolter formula, the complex transmission coefficient of a multilayer film indicates that the transmitted light, having linear polarization at incidence, is ellipticaly polarized. The ellipticity and the azimuth of the transmitted light are connected with $K_{m\,p,n}$ and $L_{mp,n}$ — the coefficients which represent the real and imaginary parts of the inverse transmission coefficient values. These coefficients can be calculated for the parallel and normal field components, from the found recursive formulas (16), (17) and (19) for any sandwich consisting of m boundary planes between the $(m - 1)$ layers. From the calculated expressions for a five, four, three, two and one layer plate, it can be concluded that the transmitted light is ellipticaly polarized, except at normal incidence and when the incident light azimuth is 0 or $\pi/2$. Under this incident conditions, the transmitted light preserves its linear polarization and the azimuth of the vector field vibrations.

The azimuth and ellipticity of the transmitted light do not change when the propagation through the plate is taken in the inverse direction. These quantities are sensitive upon the interchange of the order of any two layers. The results obtained for the five layer plate can be specialised for films of lower number of layers, as well as to films with periodically changing layers.

References

1) K. Zander, J. Mozer and H. Melle, Optik **70** (1985) 6;
2) J. Moser, Godišen zbornik-fizika **35** (1985) 69;
3) J. Moser, Lj. Janićijević and M. Jonoska, Optik **76** (1987) 27;
4) H. Wolter, Hdb. der Physik, Bd. XXIV, Springer (1956) 472;
5) M. Born and E. Wolf, *Principles of Optics*, Fourth ed., Pergamon Press (1970), p. 27.

STANJE POLARIZACIJE SVJETLOSTI NAKON TRANSMISIJE KROZ SENDVIČ OD VIŠE DIELEKTRIČNIH IZOTROPNIH SLOJEVA

JOSIP MOZER i LJILJANA JANIĆIJEVIĆ

Fizički Institut, Prirodno-matematički fakultet, Univerzitet »Kiril i Metodij«-Skopje, p. p. 162, Gazi Baba b. b., 91 000 Skopje

UDK 535.51

Originalni znanstveni rad

Analiza transmisionog koeficijenta pločice sa višeslojnim filmom (svaki je sloj od izotropnog materijala sa različitom debljinom i indeksom prelamanja) pokazuje da se stanje polarizacije ravno polarizirane upadne svjetlosti menja u eliptično, nakon prolaza svjetlosti kroz pločicu. Nađeni su opšti izrazi za eliptičnost i azimut titranja kao funkcije upadnog ugla i azimuta upadne svjetlosti. Rezultati su primenjeni na slučaj petoslojnog filma. Ovi se rezultati lako dadu specijalizovati za pločice sa četiri, tri, dva i jednim slojem, kao i za slučaj sendviča sa periodično ponovljivim slojevima. Kod normalnog upadanja ili kad je azimut upadne svjetlosti 0° ili 90°, propuštena je svjetlost sa linearnom polarizacijom ili sa azimutom titranja koji je isti kao kod upadne svjetlosti.