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ON THE SCATTERING AND TRANSVERSAL STABILITY OF SOLITONS IN A NEW NONLINEAR SYSTEM

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We have discussed the transversal stability and scattering of solitons in a newly discovered nonlinear partial differential equation obtained in studying the propagation of waves in a nonlinear elastic medium in relation to seismic theory.

1. Introduction

Recently Lund et al.[1] have deduced a new nonlinear partial differential equation for describing the propagation of elastic wave in a nonlinear medium. They were actually studying the properties of seismic waves inside the medium the earth crust. The nonlinear equation reads:

$$v_{tt} = C_1^2 v_{xx} + 3\alpha v_x^2 v_{xx} + C_1^2 \beta v_{xxxx} \tag{1}$$

which is similar in structure to the Boussinessque equation[2] but with a higher order nonlinearity. Unfortunately Eq. (1) is not exactly integrable and no Lax pair is known for it. So we thought it will be interesting to analyse the properties of solitons in such a non-integrable system by approximate methods. Two important aspects that we have touched are the scattering of solitons, which cannot be exactly treated due to lack of two-soliton solutions and the situation what will happen when the number of dimension is increased[3] by one in the transverse direction. In our study we have made extensive use of the variational principle.

2. Formulation (stability)

Eq. (1) can be obtained as the Euler-Lagrange equation from the Lagrangian

$$L = \int \mathrm{d}x\, \mathrm{d}t \left\{ \frac{1}{2} (v_t)^2 - \frac{C_2^2}{2} (v_x)^2 - \frac{\alpha}{4} (v_x)^4 + \frac{C_2^2 \beta}{2} (v_{xx})^2 \right\}. \tag{2}$$

Eq. (1) has got a single soliton solution given as

$$v(xt) = A \tan^{-1} \exp \{B (x - M(t))\} \tag{3}$$

$$A = \sqrt{\frac{8\beta C_2^2}{\alpha}}; \quad B = \left(\frac{v_2^2 - C_2^2}{C_2^2 \beta} \right)^{1/2}; \quad M = v_2 t.$$

We first formulate the problem of transversal stability by extending the Lagrangian by adding a y dependent term:

$$\bar{L} = \int \left\{ \frac{1}{2} (v_t)^2 - \frac{C_2^2}{2} (v_x)^2 - \frac{C_2^2}{2} (v_y)^2 - \frac{\alpha}{4} (v_x)^4 + \frac{C_2^2 \beta}{2} (v_{xx})^2 \right\} \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t \tag{4}$$

which has the effect that it adds a simple term v_{yy} on the right hand side of (1). We now consider a possible modulated form of the solitary wave due to the existence of the extra dimension y. For this let us imagine that we are moving with a coordinate from moving with the solitary wave. Then we can assume:

$$v(x, y, t) = A(y, t) \tan^{-1} \exp \{B(x, t) y\}. \tag{5}$$

Let us substitute (5) in (4) and integrate over x. Then we get the effective Lagrangian

$$\bar{L} = \frac{1}{2} \left(\frac{1}{2} A_t^2 - \frac{C_3^2}{2} A_y^2 \right) \frac{K}{B} - A A_t B_t \frac{K}{2B^2} + C_3^2 A A_y B_y \cdot \frac{K}{2B^2} -$$

$$- C_2^2 \frac{A^2 B}{4} - \frac{\alpha}{48} A^4 B^3 + \frac{C_2^2 \beta}{12} A^2 B^4 \left[2B - \frac{1}{B} \right] \tag{6}$$

where K stands for the integral

$$K = \int_{-\infty}^{\infty} B\, [\tan^{-1} \exp Bx]^2\, \mathrm{d}x. \tag{7}$$

We can now deduce the behaviour of the modulated coefficient A and B with respect to y and t if we deduce equations of motion from the effective Lagrangian (6).

Whence we get

$$A_{tt} - \frac{A_t B_t}{B} - \frac{AB_{tt}}{2B} + \frac{AB_t^2}{B^2} - C_3^2 A_{yy} + C_3^2 \frac{A_y B_y}{B} + C_3^2 \frac{AB_{yy}}{2B} -$$

$$- C_3^2 A \frac{B_y^2}{B^2} + \frac{C_2^2}{2K} AB^2 + \frac{\alpha A^3 B^4}{12K} - C_2^2 \beta \frac{AB^4}{6K} = 0 \tag{8}$$

$$\frac{1}{2} \frac{AA_{tt}}{B} - \frac{C_3^2 AA_{yy}}{2B} - \frac{C_2^2 A^2 B}{4K} - \frac{\alpha}{16} \frac{A^4 B^3}{K} + \frac{C_2^2 \beta A^2 B^3}{4K} = 0. \tag{9}$$

To study the stability of the solitary wave in the y direction we set,

$$\begin{aligned} A &= A_0 + \delta A \exp \mathrm{i} (K_1 y - \omega_1 t) \\ B &= B_0 + \delta B \exp \mathrm{i} (K_1 y - \omega_1 t) \end{aligned} \tag{10}$$

where A_0, B_0 are the constant values for the soliton in the one dimension. Substituting (10) in (8) and (9) and keeping only first order terms in δA, δB, we get equations (11) and (12). It is to be noted that for the study of the stability problem we always keep the perturbation small compared to the original solution. Hence it is sufficient to keep only first order terms in δA and δB

$$\left(\omega_1^2 \frac{K}{B_0} - C_3^2 K_1^2 \frac{K}{B_0} - \frac{C_2^2}{2} B_0 - \frac{\alpha}{4} A_0^2 B_0^3 + \frac{C_2 \beta}{6} B_0^3\right) \delta A +$$

$$+ \left(- \frac{1}{2} A_0 \omega_1^2 \frac{K}{B_0^2} + \frac{1}{2} C_3^2 A_0 \frac{K}{B_0^2} K_1 - \frac{C_2^2}{2} A_0 - \frac{\alpha}{4} B_0 A_0^2 + \frac{C_2^2 \beta}{2} A_0 B_0^2\right) \delta B = 0 \tag{11}$$

$$\left(- \frac{1}{2} \omega_1^2 A_0 \frac{K}{B_0^2} + \frac{1}{2} C_3^2 A_0 K_1^2 \frac{K}{B_0^2} - \frac{C_2^2}{2} A_0 - \frac{\alpha}{4} + A_0^3 B_0^2 + \right.$$

$$\left. + \frac{C_2^2 \beta}{2} B_0^2 A_0 \right) \delta A + \left(- \frac{\alpha}{B} A_0^4 B_0 + \frac{C_2^2 \beta}{2} A_0^2 B_0\right) \delta B = 0. \tag{12}$$

For nontrivial solution of δA, δB we should have the determinant of the coefficients to vanish leading to a quadratic equation for ω^2 having the solution:

$$\omega_1^2 = \frac{1}{2a^2} \{- (2ab\, K_1^2 + 2aC - pA) \pm \sqrt{E^2 - GH}\} \tag{13}$$

$$E = 2abK_1^2 + 2aC - PA$$

$$GH = 4a^2 (b^2 K_1^4 + C^2 + 2bC\, K_1^2 - PB\, K_1^2 - PD)$$

where,

$$a = -\frac{A_0 K}{2B_0^2}$$

$$b = C_3^2 \frac{A_0 K}{2B_0^2}$$

$$C = -\frac{C_3^2}{2} A_0 - \frac{\alpha}{4} A_0^2 B_0^2 + \frac{C_0^2 \beta}{2} A_0 B_0^2$$

$$P = \frac{\alpha}{8} A_0^4 B_0 + \frac{C_2^2 \beta}{2} A_0^2 B_0$$

$$A = K/B_0; \qquad B = -C_3^2 K/B_0;$$

$$D = -\frac{C_2^2}{2} B_0 - \frac{\alpha}{4} A_0^2 B_0^3 + \frac{C_2^2}{6} B_0^3 \beta.$$

After some simplification Eq. (13) reduces to:

$$\omega_1^2 = \frac{2B_0^2}{A_0 K} \left(\frac{1}{2} C_3^2 A_0 \frac{K}{B_0^2} K_1^2 - \frac{3}{8} A_0^3 B_0^2 + C_2^2 \beta A_0 B_0^2 - \right.$$

$$\left. - \frac{C_2^2}{2} A_0 \right) \pm \left(- \frac{\alpha}{8} A_0^4 B_0 + \frac{C_2^2 \beta}{2} A_0^2 B_0 \right) \left(- \frac{7}{8} A_0^2 B_0^3 + \frac{5}{3} C_2^2 \beta B_0^3 - \right.$$

$$\left. - \frac{3}{2} C_0^2 B_0 \right)^{1/2} = \frac{2B_0^2}{A_0 K} [\tilde{\alpha} K_1^2 - \tilde{\gamma}],$$

so that

$$\omega_1 = \left(\frac{2B_0}{A_0 K} \right)^{1/2} [\tilde{\alpha} K_1^2 - \tilde{\gamma}]^{1/2}.$$

Now if $K_1^2 > \frac{\tilde{\gamma}}{\tilde{\alpha}}$, ω_1 real, the perturbation is $A\, e^{i(K_1 y - \omega_1 t)}$.

But if $K_1^2 < \tilde{\gamma}/\tilde{\alpha}$ we can write

$$\omega_1 = i\nu \text{ which gives } \delta A\, e^{i(K_1 y - i\nu t)} = A\, e^{iK_1 y + \nu t}.$$

In the limit $t \to \infty$ this is stable for $\nu < 0$ and unstable for $\nu > 0$.

3. *Scattering of solitons*[4]

Since we do not have an IST mechanism, we must take recourse to the similarity between the particle and soliton to discuss the scattering of solitons. Let us go back to Eq. (3) which represents a simple single Kink solution. Assume that we do have too such extended objects nearly which influence each other due to the force acting between them, so that the coefficient A and B become time dependent and we can write the solution as

$$v\,(xt) = A\,(t) \tan^{-1} \exp B\,(t)\,[x - M\,(t)].$$

Now inserting this in the Lagrangian (2) and integrating over x we get

$$\bar{L} = -\frac{1}{8}\bar{\Lambda}^2 M A_t^2 + \frac{1}{4} A^2 B_t^2 \frac{M^2}{B} + \frac{1}{4} A^2 B M_t^2 - \frac{C_2^2}{4} A^2 B - \frac{\alpha}{48} A^4 B^3 + \frac{C_2^2 \beta}{12} A^2 B^3 - A A_t B_t \frac{\bar{\Lambda}^2 M}{16 B} - A A_t M_t \frac{\bar{\Lambda}^2}{8}. \qquad (14)$$

Now utilising the variational principle with respect to (A, B, M) we can now generate three equations governing the evolution of (A, B, M) with respect time. These are, respectively:

for B

$$A^2 B_{tt} \frac{M^2}{2B} + A A_t B_t \frac{M^2}{B} + A^2 \frac{B_t M_t M}{B} - \frac{3}{4} A^2 B_t^2 \frac{M^2}{B^2} - A_t^2 \frac{\bar{\Lambda}^2 M}{16B} - A A_{tt} \frac{\bar{\Lambda}^2 B}{16B} - A A_t \frac{\bar{\Lambda}^2 M_t}{16B} - \frac{1}{4} A^2 M_t^2 + \frac{C_2^2 A^2}{4} + \frac{\alpha}{16} A^4 B^2 - C_2^2 \frac{\beta}{4} A^2 B^2 = 0 \qquad (15)$$

for A

$$2A_{tt} \frac{\bar{\Lambda}^2 M}{4} + A B_{tt} \frac{\bar{\Lambda}^2 M}{16B} - A B_t \frac{\bar{\Lambda}^2 M B_t}{16B^2} + A B_t \frac{\bar{\Lambda}^2 M_t}{16B} + A M_{tt} \frac{\bar{\Lambda}^2}{8} + A B_t^2 \frac{M^2}{2B} + A B \frac{M_t^2}{2} - \frac{C_2^2}{2} A B - \frac{\alpha}{12} A^3 B^3 + C_2^2 \beta A B^3 + A_t M_t \frac{\bar{\Lambda}^2}{4} = 0 \qquad (16)$$

for M

$$AA_t B\, M_t + \frac{1}{2} A^2 B_t M_t + \frac{1}{2} A^2 B\, M_{tt} - A_t^2 \frac{\overline{\Lambda}^2}{8} + AA_{tt} \frac{\overline{\Lambda}^2}{8} -$$

$$- \frac{1}{2} A^2 B_t^2 \frac{M}{B} + AA_t B_t \frac{\overline{\Lambda}^2}{16B} + A_t^2 \frac{\overline{\Lambda}^2}{8} = 0. \tag{17}$$

Let us now search for the physical parameters of the soliton (3) which are:

$$\text{Mass} = - \int_{-\infty}^{\infty} v_t \, \mathrm{d}x. \tag{18}$$

$$\text{Momentum} = \widetilde{P} = - \int_{-\infty}^{\infty} v_t \, v_x \, \mathrm{d}x. \tag{19}$$

$$\text{Energy} = \widetilde{E} = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\partial L}{\partial v_t} \cdot v - L \right). \tag{20}$$

Evaluating these equations with the explicit solution (3) we get expressions for $\widetilde{P}$ and $\widetilde{E}$.

If we set $A\,(t)$, $B\,(t)$ to the values for which (3) was an exact solution then,

$$\widetilde{P} = \frac{1}{2} A^2 B\, M_t \tag{21}$$

$$\widetilde{E} = \widetilde{H} = \frac{1}{4} A^2 B\, M_t^2 + \frac{C_2^2}{4} A^2 B + \frac{\alpha}{48} A^4 B^3 - \frac{C_2^2 \beta}{12} A^2 B^3. \tag{22}$$

Now it is pertinent to observe that

$$M_t \cdot \widetilde{P} - \widetilde{H} = \widetilde{L}$$

so that the soliton behaves like a particle following the same relation between Lagrangian, Hamiltonian (H) and momentum.

For simulating the soliton collision we set:

$$v = v_1 + v_2$$

$$v_1 \cong A_1\,(t) \tan^{-1} \exp B_1\,(t)\,[x - M_1\,(t)]$$

$$v_2 \cong A_2\,(t) \tan^{-1} \exp B_2\,(t)\,[x - M_2\,(t)].$$

We again substitute v in L and integrate whence we obtain:

$$\overline{L} = L_1 + L_2 + L_{12} \tag{23}$$

where L_1 and L_2 are the Lagrangians for the 1st and 2nd solitons and L_{12} gives an interaction Lagrangian between them, with each L_i, satisfying,

$$L_i = P_i M_{it} - H_i \tag{24}$$

$$P_i = \frac{1}{2} A^2 B M_{it}. \tag{25}$$

Now if we write the Euler equation with (23) then we get

$$M_{1t} - \frac{\delta}{\delta P_1}(H_1) = \frac{\delta}{\delta t}\left(\frac{\delta L_{12}}{\delta P_{1t}}\right) - \frac{\delta L_{12}}{\delta P_1} \tag{26}$$

whence,

$$\Delta M_1 = \int_{-\infty}^{\infty} \mathrm{d}t \left(M_{1t} - \frac{\delta H}{\delta P_1}\right) = - \int_{-\infty}^{\infty} \mathrm{d}t \frac{\delta L_{12}}{\delta P_1}. \tag{27}$$

So Eq. (27) immediately leads to an expression for the phase-shift of the soliton due to scattering via the interaction Lagrangian.

The interaction Lagrangian is given as:

$$\int L_{12}\,\mathrm{d}x = A^2B^2M_{1t}M_{2t}\left(\frac{M_1 - M_2}{2}\right) \operatorname{cosech} \varkappa + \frac{3}{8}\,\alpha A^4B^4 \sigma \operatorname{cosech}^2 \varkappa -$$

$$- C_2^2A^2B^2 \frac{\sigma}{2} \operatorname{cosech} \varkappa + \operatorname{cosech}^2 \varkappa \left\{\frac{3\alpha A^4B^3}{8} - \alpha A^4B^4 \frac{\sigma}{4} + \right.$$

$$\left. + C_2^2 \beta A^2B^4 \left(\frac{\varkappa}{2} + \frac{1}{B}\right)\right\} + \frac{3\alpha A^4B^4}{8} \cosh^3 \varkappa\, \mathrm{e}^{\varkappa} + \operatorname{cosech} \varkappa \left(\frac{\alpha}{8} A^4B^3 + \right.$$

$$\left. + C_2^2 \beta A^2B^4 \frac{\sigma}{2}\right) + \frac{\alpha A^4B^4}{8} \sigma \operatorname{cosech}^3 \varkappa\, \mathrm{e}^{2\varkappa} - \frac{\alpha A^4B^3}{16} \operatorname{cosech}^2 \varkappa\, \mathrm{e}^{\varkappa} +$$

$$+ \frac{\alpha A^4B^3}{8} \operatorname{cosech} \varkappa\, \mathrm{e}^{2\varkappa} - C_2^2 \beta A^2B^4 \sigma \operatorname{cosech}^3 \varkappa -$$

$$- C_2^2 \beta \frac{A^2B^4}{4} \sigma \operatorname{cosech}^2 \varkappa \cdot \mathrm{e}^{\varkappa}$$

where

$$\varkappa = B(M_1 - M_2); \qquad \sigma = M_1 - M_2.$$

Using now formulae (24), (25), (26) and (27) we can compute ΔM_1 or ΔM_2 explicitly. But one should keep in mind that it is impossible to know $A_l(t)$ and $B_l(t)$ exactly. So as an approximation we have set these to their constant values. Such an evaluation leads to:

$$\Delta M_1 = \frac{B}{2(v_1 - v_2) v_1} (3A^2 + 8C_2^2 \beta).$$

4. Conclusion

In our above analysis we have formulated the stability and scattering of solitons in a non-integrable system. The methodology rests on the use of Lagrangian formalism and similar behaviour of a classical particle and solitary wave.

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O RASPRŠENJU I TRANSVERZALNOJ STABILNOSTI SOLITONA U NOVOM NELINEARNOM SISTEMU

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Diskutirana je transverzalna stabilnost i raspršenje solitona u novootkrivenoj nelinearnoj parcijalnoj diferencijalnoj jednadžbi, koja je dobivena u okviru seizmičke teorije pri studiju širenja valova u nelinearnom elastičnom mediju.