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ON THE HOPF BIFURCATIONS IN ELECTRON INTERACTION WITH PLASMA

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We have discussed the equation describing the interaction of an electron with plasma from the viewpoint of bifurcation theory, and have deduced the condition for stability.

1. Introduction

In recent years nonlinear systems have been classified in two basic category — integrable[1)] and chaotic[2)]. Systems which are integrable have been discussed from the viewpoint of inverse scattering theory, Lax pair, and symmetry[3)], while those belonging to the second class have been analysed with the help of bifurcation theory, chaos, and discrete mapping[4)]. In recent years a nonlinear equation describing the interaction between an electron and plasma has been deduced by Chain et al.[5)]. Here we have analysed this equation with the help of bifurcation theory.

2. Formulation

The equation under consideration reads[5)]

$$\frac{d^2u}{dt^2} + \frac{u}{\sqrt{1+u^2}} + \frac{\mu u - D}{\sqrt{1+(\mu u - D)^2}} = 0 \qquad (1)$$

where u is the electron velocity and μ, D are constants pertaining to the plasma. Let us define $v = u_t$ so that (1) becomes:

$$\begin{aligned} u_t &= v \\ v_t &= -f(u) \end{aligned} \tag{2}$$

with

$$f(u) = \frac{u}{\sqrt{1+u^2}} + \frac{\mu u - D}{\sqrt{1 + (\mu u - D)^2}}.$$

The fixed point (u^*, v^*) is given as the solution of

$$f(u^*) = 0; \qquad v^* = 0. \tag{3}$$

Equation (3) yields

$$u_+^* = -\frac{D}{1-\mu} \quad \text{and} \quad u_-^* = \frac{D}{1+\mu}. \tag{4}$$

(i) Linearizing about $\left(u_+^* = -\frac{D}{1-\mu},\ v^* = 0\right)$ we set:

$$\begin{aligned} u &= u_+^* + \varepsilon u' \\ v &= 0 + \varepsilon v'. \end{aligned} \tag{5}$$

Then the Jacobian at $(0, u_+^*)$ is

$$J = \begin{pmatrix} 0 & 1 \\ -f'(u_+^*) & 0 \end{pmatrix},$$

so that we have

$$\begin{pmatrix} u'_t \\ v'_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f'(u_+^*) & 0 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}. \tag{6}$$

To solve the linear equation (6) we require its eigenvalues which are:

$$\begin{aligned} \lambda_\pm &= \pm \mathrm{i} \sqrt{f'(u_+^*)} \\ f'(u_+^*) &= \frac{(1-\mu)^3 (1+\mu)}{[D^2 + (1-\mu)^2]^{3/2}}. \end{aligned} \tag{7}$$

Out of the two constants (μ and D), D is complex and let us write $D = D_1 + \mathrm{i} D_2$.

(ii) For $1 - \mu > 0$,

$$\lambda_{\pm} = \pm (KB + iKA),$$

where

$$K = \frac{(1+\mu)^{1/2} (1-\mu)^{3/2}}{[(D_1^2 - D_2^2 + (1-\mu)^2 + 4D_1^2 D_2^2]^{3/4}}$$

and

$$A = \exp\left[\frac{3}{8} \ln (a^2 + b^2)\right] \cos\left(\frac{3}{4} \tan^{-1} \frac{b}{a}\right)$$

$$B = \exp\left[\frac{3}{8} \ln (a^2 + b^2)\right] \sin\left(\frac{3}{4} \tan^{-1} \frac{b}{a}\right)$$

$$b = 2D_1 D_2$$

$$a = D_1^2 - D_2^2 + (1-\mu)^2. \tag{8}$$

For the case of Hopf-bifurcation we require

$$\text{Real } [\lambda_{\pm}] = 0$$

which gives

$$\frac{b}{a} = \tan (n (4\pi/3)) \qquad n \neq 0 \text{ or } 3m$$

$$m \text{ integer.}$$

$$= \varepsilon_1 \sqrt{3}; \quad \varepsilon_1 \text{ is } \pm 1. \tag{9}$$

Equation (9) yields the critical value $\mu = \mu_c$, and let us set:

$$\lambda_+ |_{\mu=\mu_c} = iKA|_{\mu=\mu_c} = i\, w_0 \text{ (say)}.$$

Then we make a transformation

$$\begin{pmatrix} u' \\ u' \end{pmatrix} = \begin{pmatrix} u_+^* \\ 0 \end{pmatrix} + P \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -w_0 \end{pmatrix}, \tag{10}$$

so that the linear part of the equation is of the form

$$\begin{pmatrix} 0 & -w_0 \\ w_0 & 0 \end{pmatrix}. \tag{11}$$

Performing the same on the nonlinear system we get

$$\left.\begin{aligned} \dot{y}_1 &= -w_0 y_2 = F' \\ \dot{y}_2 &= \frac{1}{w_0} f(u_+^* + y_1) = F^2 \end{aligned}\right\}. \tag{12}$$

3. *Stability of the system*

The stability of the present nonlinear system could be easily ascertained if the real part of $\lambda_\pm$ was not equal to zero. But near the bifurcation point as we deviate from the critical value $\mu = \mu_c$, at which the real $\lambda_\pm = 0$, we follow Hassard et al.[6] and compute τ_2, μ_2 and β_2 to estimate the period, direction and the Floquet exponent associated with the bifurcation. To calculate these exponents we consider the coefficients g_{11}, g_{02}, g_{20}, G_{21}, as defined in Chapter 2, of Ref. 6. We then get

$$g_{11} = \left.\frac{\partial^2 F^2}{\partial y_1^2}\right|_{y_1 = 0} = K_1 (A'' + \mathrm{i}B'') \tag{13}$$

$$g_{21} = \left.\frac{|\partial^3 F^2|}{\partial y_1^3}\right|_{y_1 = 0} = K_2 (M' + \mathrm{i}N'), \tag{14}$$

where we have set

$$K_1 = \frac{3}{w_0} \frac{(1 + \mu^2)(1 - \mu)^4}{\{[D_1^2 - D_2^2 + (1 - \mu)^2]^3 + 4D_1^2 D_2^2\}^{5/2}}$$

$$A'' = D_1 A' + D_2 B' \tag{15}$$

$$B'' = D_2 A' - D_1 B'$$

$$\left.\begin{matrix} A' \\ B' \end{matrix}\right\} = \exp\left(\frac{5}{2} \ln\sqrt{a^2 + b^2}\right) \begin{matrix} \cos \\ \sin \end{matrix} \left(\frac{5}{2} \tan^{-1} \frac{b}{a}\right)$$

$$a = D_1^2 - D_2^2 + (1 - \mu_c)^2$$

$$b = 2D_1 D_2$$

and
$$\frac{b}{a} = \varepsilon_1 \sqrt{3} \quad \text{when} \quad \mu = \mu_c.$$

$$K_2 = -\frac{3}{w_0} \frac{(1 - \mu)^5 (1 + \mu^3)}{\{[D_1^2 - D_2^2 + (1 - \mu)^2]^2 + 4D_1^2 D_2^2\}^{7/2}} \tag{16}$$

$$M' = L A_2 + 8D_1D_2B_2$$

$$N' = 8D_1D_2A_2 - L B_2$$

$$L = 4(D_1^2 - D_2^2) - (1 - \mu)^2$$

$$\left.\begin{matrix} A_2 \\ B_2 \end{matrix}\right\} = \exp\left(\frac{7}{2}\ln\sqrt{a^2 + b^2}\right) \begin{matrix}\cos \\ \sin\end{matrix} \left(\frac{7}{2}\tan^{-1}\frac{b}{a}\right).$$

All these are to be evaluated at $\mu = \mu_c$. Finally,

$$\mathrm{Re}\, C_1(0) = \frac{1}{w_0}\frac{K_1^2}{16} A'' B'' - \frac{K_2}{16} N' \tag{17}$$

$$\mathrm{Im}\, C_1(0) = \frac{1}{2w_0} \cdot \frac{K_1^2}{16}\left(\frac{10}{3} A''^2 + \frac{4}{3} B''^2\right) + \frac{K^2}{16} M' \tag{18}$$

which immediately leads to

$$\mu_2 = -\frac{\mathrm{Re}\, C_1(0)}{\alpha'(0)} = -\frac{\mathrm{Re}\, C_1(0)}{\mathrm{Re}\, \lambda_1'(\mu_c)}, \quad \lambda_1'(\mu_c) = \left.\frac{\partial \lambda_1}{\partial \mu}\right|_{\mu = \mu_c}. \tag{19}$$

where λ_1 is given by formulae (8).

An explicit evaluation leads to

$$\cos n\pi \cdot \mu_2 = -\frac{3(1 + \mu_c^2)(1 - \mu_c)^{11/2}}{16\varepsilon_1 \sqrt{3} w_0^3 (2a)^{13/4}} \left[-\frac{D_1 D_2 \sqrt{3}}{2 \quad 4}(D_1^2 - D_2^2)\right] -$$

$$-\frac{(1 + \mu_c^2)(1 - \mu_c)^{5/2}}{\varepsilon_1 16 \sqrt{3} w_0^2 (2a)^{7/4}} [4D_1 D_2 - 2\sqrt{3}(D_1^2 - D_2^2) + \frac{3}{2}(1 - \mu_c)^2]. \tag{20}$$

On the other hand,

$$\beta_2 = 2\,\mathrm{Re}\, C_1(0) = \frac{K_1^2}{8w_0} A'' B'' - \frac{K_2}{8} N', \tag{21}$$

and

$$\tau_2 = -(\mathrm{Im}\, C_1(0) + \mu_2 w'(0))/w_0. \tag{22}$$

4. Conclusion

Since none of the parameters τ_2, μ_2 and β_2 are zero, our system do possess a definite trend in the stability zone. It only depends upon the relative magnitude of the parameter values (D_1, D_2) of which all the τ_2, β_2 and μ_2 are composed. So the period of the motion is given as $T = \frac{2\pi}{w_0}(1 + \tau_2 \varepsilon^2)$. A similar analysis can be developed for the fixed point $u_+^* = + \frac{D}{1+\mu} v^* = 0$ and also for the case $1 - \mu < 0$.

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O HOPFOVIM BIFURKACIJAMA U INTERAKCIJI ELEKTRONA SA PLAZMOM

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U ovom radu diskutirane su i opisane interakcije elektrona sa plazmom s točke gledišta teorije bifurkacija, i nađeni su uvjeti stabilnosti.

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