YU ISSN 0015—3206
FZKAAA 21 (2) 167 (1989)

AN APPROXIMATE RIEMANN-HILBERT APPROACH TO THE SINGULAR SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATION

A. ROY CHOWDHURY and MRITYUNJOY DE

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta-700 032, India

Received 7 January 1988

UDC 530.182

Original scientific paper

An approximate Riemann-Hilbert technique is used to study the evolution of singular »seed« solution of nonlinear Schrödinger equation, which is seen to lead to a stationary regular approximate solution of NLSE.

1. Introduction

There have been different approaches to study the inverse problems of nonlinear equations, integrable with the help of a Lax pair. But the Riemann-Hilbert approach is perhaps the most suitable one in order to incorporate the different from of the starting »seed« solution$^{1)}$. Of late many new type of solutions for nonlinear Schrödinger equation have been constructed of which an important class is solutions with singularities in prescribed positions$^{2)}$. Furthermore it has been observed that in many situations approximate versions of inverse problems yields reasonable result$^{3)}$. Here we have shown that a new class of solutions can be constructed for nonlinear Schrödinger equation by starting from an exact singular solution.

2. Formulation

The nonlinear Schrödinger equation is written in the form:

$$iq_t + q_{xx} + |q|^2 q = 0 \tag{1}$$

for which the inverse problem is written as

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = \begin{pmatrix} m & n\\ p & r\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{2}$$

$$m = -\mathrm{i}\,(2\lambda^2 + |q|^2); \qquad r = \mathrm{i}\,(2\lambda^2 + |q|^2)$$

$$n = 2\lambda\, q + \mathrm{i}q_x; \qquad p = 2\lambda\, q^* - \mathrm{i}q_x^*$$

and

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = \begin{pmatrix} \mathrm{i}\lambda & q\\ q^* & -\mathrm{i}\lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{3}$$

Note that

$$q_0 = \frac{a}{x - x_j}$$

is an exact solution of (1) with x_j fixed, and a = constant. In the following we will assume that a is a small parameter. We shall start by assuming this to be the seed solution.

Following the usual procedure of Riemann-Hilbert approach, we set

$$\overline{\psi} = \chi_2 \psi_0 \tag{4}$$

where,

$$L_0 \psi_0 = \lambda \psi_0; \qquad L_0 = \begin{pmatrix} \mathrm{i}\partial_x & -\mathrm{i}q_0\\ \mathrm{i}q_0 & -\mathrm{i}\partial_x\end{pmatrix}$$

with the ansatz

$$\chi_2 = 1 + \frac{R}{\lambda - \mu_1}; \quad \chi_2^{-1} = \chi_1 = 1 + \frac{S}{\lambda - \lambda_1}. \tag{5}$$

It is not difficult now to prove that

$$\begin{pmatrix} -\mathrm{i}\lambda & \tilde{q}\\ \tilde{q}^* & \mathrm{i}\lambda\end{pmatrix} = \begin{pmatrix} -\mathrm{i}\lambda & q_0\\ q_0^* & \mathrm{i}\lambda\end{pmatrix} + \mathrm{i}\,[R, E] \tag{6}$$

where $\tilde{q}$ is a new solution and:

$$S = -R = -(\lambda_1 - \mu_1)\, P; \qquad P^2 = P; \qquad E = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{7}$$

The projection operator P is to be constructed out of the eigenfunctions ψ_0 corresponding to q_0.

3. The projection operator

At this point it is interesting to note that Eq. (3) with q_0 is equivalent to a Coulomb problem with a centrifugal force term, whose solution are known in terms of confluent hypergeometric functions[4)] $F(\alpha, \beta, z)$. So we can write the space part of the eigenfunction as:

$$\psi_0 = \begin{pmatrix} (x - x_j)^a \mathrm{e}^{-\mathrm{i}\lambda(x-x_j)} F \\ \frac{1}{a}(x - x_j)^{a+1} \mathrm{e}^{-\mathrm{i}\lambda(x-x_j)} \left(\frac{a}{x - x_j} F + F x\right) \end{pmatrix} \tag{8}$$

where

$$F = F(a, 2a + 1, 2\mathrm{i}\mu_1 (x - x_j)).$$

Up to the present stage of the calculation all our calculations are exact and no approximations are involved. But for the total information regarding the projection operator we need the temporal behaviour also. It is here we make use of the condition a is small. In the contrary situation, that is if we do not make this approximation the temporal equation is difficult to solve because of the x-dependence of the coefficients.

Actually on elimination the time evolution of ψ_1 is given as

$$\psi_{1tt} = (pn - mr)\,\psi_1 = (-4\lambda^4 - |q|^4)\,\psi_1. \tag{9}$$

The second term involves terms of order a^4 which if a is assumed to be small can be neglected in a region away from $x \approx x_j$. Hence we obtain

$$\begin{aligned} \psi_1(t) &= \frac{1}{2}\mathrm{e}^{2\mathrm{i}\lambda^2 t} + \frac{K^2}{8\lambda^4}\mathrm{e}^{-2\mathrm{i}\lambda^2 t} \\ \psi_2(t) &= \frac{1}{2}\mathrm{e}^{2\mathrm{i}\lambda^2 t} + \frac{K'^2}{8\lambda^4}\mathrm{e}^{-2\mathrm{i}\lambda^2 t}. \end{aligned} \tag{10}$$

So we finally get

$$\psi_0 = \begin{pmatrix} (x - x_j)^a \mathrm{e}^{-\mathrm{i}\lambda_1(x-x_j)} \left(\frac{1}{2}\mathrm{e}^{2\mathrm{i}\lambda_1^2} + \frac{K^2}{8\lambda_1^4}\mathrm{e}^{-2\mathrm{i}\lambda_1^2 t}\right)_F \\ \frac{(x - x_j)^{a+1}}{a} \mathrm{e}^{-\mathrm{i}\lambda_1(x-x_j)} \left(\frac{1}{2}\mathrm{e}^{2\mathrm{i}\lambda_1^2} + \frac{K'^2}{8\lambda_1^4}\mathrm{e}^{-2\mathrm{i}\lambda_1^2 t}\right)\left(\frac{a}{x - x_j} F_1 + F_1 x\right) \end{pmatrix}. \tag{11}$$

Now we supplement with the usual degeneracy argument of the projection operator R or P. Therefore

$$(R)_{ij} = r_i t_j, \qquad (S)_{ij} = p_i q_j$$

whence

$$\begin{aligned} r_t &= \psi_0(\mu_1)\, a_t \\ q_t &= \bar{a}_t \bar{\psi}_0(\lambda_1) \end{aligned} \tag{12}$$

where a_t is an arbitrary vector.

Writing out explicitly we obtain from Eq. (6)

$$\tilde{q} = q_0 - 2\mathrm{i}(\lambda_1 - \mu_1)\frac{a_1}{a_2} a \cdot \frac{F_2 F_{1x}^*}{a(F_1^* F_{2x} + F_{1x}^* F_2) + (x - x_j) F_{1x}^* F_{2x}}$$

$$\frac{\mathrm{e}^{-2\mathrm{i}\sigma_- t} + \dfrac{K^2 K'^2}{16\lambda_1^4 \mu_1^4}\mathrm{e}^{2\mathrm{i}\sigma_- t} + \dfrac{K^2}{4\mu_1^4}\mathrm{e}^{-2\mathrm{i}\sigma_+ t} + \dfrac{K'^2}{4\lambda_1^4}\mathrm{e}^{-2\mathrm{i}\sigma_+ t}}{\mathrm{e}^{-2\mathrm{i}\sigma_- t} + \dfrac{K'^2}{16\lambda_1^4 \mu_1^4}\mathrm{e}^{2\mathrm{i}\sigma_- t} + \dfrac{K'^2}{4\lambda_1^4}\mathrm{e}^{2\mathrm{i}\sigma_+ t} + \dfrac{K'^2}{4\mu_1^4}\mathrm{e}^{-2\mathrm{i}\sigma_+ t}}. \tag{13}$$

So we have generated now approximate solution of NLSE, which is not propagating type, by starting from a singular solution.

4. *Discussions*

It is now interesting to observe that for small $(x - x_j)$ that is near the singularity the hypergeometric function $F(\alpha, \beta, z)$ behaves as z^{-1} for $z = (x - x_j)$ and the rational function is constant. Hence the solution generated is no more singular but regular in the neighbourhood of the point $x = x_j$.

References

1) V. E. Zakharov, S. V. Manakov, S. P. Novikov and L. P. Pitaevski, *Theory of Solitons*, C. B., New York (1980);
2) A. K. Pogrebov and M. K. Potiavanov, Teor. Mat. Phys. **53** (1982) 163; **54** (1983) 54;
3) V. Chudnovsky, Springer Lecture in Phys. **120**;
4) S. Flüge, *Practical Quantum Mechanics*, Springer Verlag.

PRIBLIŽAN RIEMANN-HILBERTOV PRISTUP SINGULARNIM RJEŠENJIMA NELINEARNE SCHRÖDINGEROVE JEDNADŽBE

A. ROY CHOWDHURY i MRITYUNJOY DE

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta-700 032, India

UDK 530.182

Originalni znanstveni rad

Koristi se približna Riemann-Hilbertova tehnika za proučavanje evolucije singularnog »zrno«-rješenja nelinearne Schrödingerove jednadžbe, za koju se opaža da vodi na stacionarno regularno približno rješenje NLSE.