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NON-LINEAR ELECTRODYNAMICS AND THE ABELIAN DYON IN GENERAL RELATIVITY

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The solutions for the electric and magnetic fields of a dyon are found using a specific non-linear electromagnetic Lagrangian. The metric is expanded near the origin and spatial infinity, the leading terms at spatial infinity agree with the solution for Maxwell's electrodynamics. For small r the metric displays a confining structure for particles moving in the field and also displays a phase structure which may suggest that abelian dyons may both confine orbiting particles gravitationally as well as generate a modified background phase to generate otherwise unexplainable bound state phenomena.

1. Introduction

The abelian magnetic monopole has been the subject of intense research during the last decade mainly because its existence would shed light on the fundamental nature of electric and magnetic charge. Dirac's quantization condition[1)]

$$eq = 2\pi n, \; e = \text{electric charge}, \; q = \text{magnetic charge}, \tag{1}$$

which follows from the laws of electric and magnetic fields of charges and poles and the fundamental quantization principle of quantum mechanics, intimates a fundamental connection between these two types of charges. This relation is sometimes contrasted with Schwinger's condition

$$e_1 q_2 - e_2 q_1 = 2\pi n, \quad n = \pm 1, \pm 2, \ldots, \tag{2}$$

which relates the electric and magnetic charges of two separate particles[2]. For a review of these quantization conditions the interested reader should refer to Witten's paper[3]. In this paper, Witten demonstrates that the electric charge need not be quantized due to CP violating interactions. Though his analysis deals with the charge degree of freedom of a t'Hooft Polykov magnetic monopole[4], the question of the quantization of electric charge takes on a new perspective. It is a specific interaction coupled to the principles of quantum mechanics that leads to the quantization condition for electric and magnetic charge, or the principles of quantum mechanics with certain specific symmetries that yield the quantization condition. It can be argued that the dual symmetry of Maxwell's equations can be traced to the four-dimensional character of the world we live in[5], but this does in itself not imply the quantization condition. What is needed is a principle following from the structure of a Lagrangian and its symmetries. The origin of the quantization condition as a restriction on a gauge transformation in the overlap regions between two sections that define the vector potential of a monopole does not really intimate the dynamical origin of the condition. It is obvious that the origin of the condition might very well tug at the very foundations of quantum mechanics.

The quantization of magnetic charge can be shown to follow purely from the topological properties of space-time and the primitive fields in the Lagrangian[6]. For a SO_3 gauge theory the magnetic charge resides at the zeros of the Higgs field. In the classical solution of Prasad Sommerfield[7], no restriction is put on the electric charge and it in fact can have an arbitrary value. All of these results refer to specific models and no general conclusions can be drawn. Perhaps the condition may follow from the semi-classical treatment of a field theory that has well defined classical soliton solutions[8].

Whatever the ultimate source of the quantization condition, it certainly questions the fundamental nature of space-time symmetries along with the dynamical structure of the theory of quantized fields.

In what follows we assume the presence of electrical and magnetic charges and simply calculate the electric, magnetic and gravitational field of a abelian dyon using a nonlinear Lagrangian. Our seemingly simple result exhibits rather profound properties of confinement and phase structure without the use of any contrived Higgs field. We feel, in this sense, the calculation is important because it lends value to an old nonlinear theory in a modern context.

2. *The field of an abelian dyon in general relativity*

Consider the classical field theory of the combined Lagrangian of electromagnetism and gravity[9]; G — gravitational constant,

$$\mathscr{L} = \frac{c^4}{16\pi G^R}\sqrt{-g} - \frac{b}{8\pi}\left(\sqrt{1+\frac{J}{b}}-1\right)\sqrt{-g} \tag{1}$$

b = empirical constant, $F_{\mu\nu}$ = E. M. field tensor, $J = F_{\mu\nu}F^{\mu\nu}$.

The field equations for the gravitational field are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = - \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2}$$

$R_{\mu\nu}$ = Ricci tensor, R = curvature scaler, $g_{\mu\nu}$ = metric, $-g$ in $\sqrt{-g}$ denotes the square root of determinant, where

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} \left[\frac{b}{8\pi}\left(\sqrt{1 + \frac{J}{b}} - 1\right)\right]. \tag{3}$$

The equations for the electromagnetic field are

$$\frac{\partial}{\partial x^\nu} (4NF^{\mu\nu}\sqrt{-g}) = 0, \tag{4}$$

where $N = \frac{1}{16\pi\sqrt{1 + J/b}}$, $J = F^{\mu\nu} F_{\mu\nu}$. The dual to the electromagnetic field is $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, where $\varepsilon^{4123} = 1$, -1 for any odd permutation, and 0 for any two indicies equal. The equation

$$\frac{\partial}{\partial x^\nu} (\tilde{F}^{\mu\nu}) = 0 \tag{5}$$

is identically true when a potential A_μ exists. If we are in the region outside of a monopole, equation (5) is still valid but the potential involves a line or string singularity. For a monopole we need only have an A_φ component of A_μ and hence $F_{23} = r^2 \sin\Theta\, B_r$. Equation (5) then gives

$$\begin{aligned} &\frac{\partial}{\partial r} (\varepsilon^{4123} F_{23}) = 0, \\ &\frac{\partial}{\partial r} (r^2 \sin\Theta\, B_r) = 0 \end{aligned} \tag{6}$$

yielding $B_r = \frac{q}{r^2}$, where q = magnetic charge.

Here the field B_r is independent of the line singularity in A_μ and since we need only the field $F_{\mu\nu}$ in the calculation of the metric, we needn't dicuss the potential A_μ.

Now employing the metric

$$(\mathrm{d}s)^2 = \mathrm{e}^\nu (\mathrm{d}x^4)^2 - \mathrm{e}^\lambda (\mathrm{d}r)^2 - r^2 (\mathrm{d}\Theta)^2 - r^2 \sin^2\Theta\, (\mathrm{d}\varphi)^2$$

where $x^4 = ct$, $x^1 = r$, $x^2 = \Theta$, $x^3 = \varphi$, we have

$$J = 2F^{14}F_{14} + 2F^{23}F_{23}$$

$$J = 2B_r^2 - 2E_r^2.$$

From Eq. (3) we have

$$T^{\mu\nu} = 4NE^{\mu\nu} + g^{\mu\nu}(L - JN), \tag{7}$$

$$L = \frac{b}{8\pi}\left(\sqrt{1 + \frac{J}{b}} - 1\right), \quad N = \frac{\partial L}{\partial J}, \quad E^{\mu\nu} = F^{\mu\alpha} F^{\nu}_{\alpha} + \frac{g^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta}. \tag{8}$$

Now from Eqs. (7) and (8) we have $T_4^4 = T_1^1$ and hence $\lambda + \nu = 0$ from the spherical symmetry of the Einstein equations. Also using Eq. (4) for $F^{41} = -F^{14} = E$, we have

$$\frac{\partial}{\partial r}\left(\frac{4r^2 \sin\Theta\, E}{16\pi\sqrt{1 + 2B_r^2/b - 2E_r^2/b}}\right) = 0.$$

Integrating and identifying the electric charge e, we have

$$E^2 = e^2 \frac{\left(1 + \frac{2g^2}{br^4}\right)}{r^4 + \frac{2e^2}{b}}, \tag{9}$$

where q = magnetic charge, e = electric charge. Now the 4, 4 component of the energy momentum tensor can be calculated from Eqs. (7), (8) and (9)

$$T_4^4 = \frac{e^2}{4\pi r^4}\left(\frac{r^4 + \frac{2q^2}{b}}{r^4 + \frac{2e^2}{b}}\right)^{1/2} + \frac{b}{8\pi}\left[\frac{\left(r^4 + \frac{2q^2}{b}\right)^{1/2}}{\left(r^4 + \frac{2e^2}{b}\right)^{1/2}} - 1\right]. \tag{10}$$

For small r, Eq. (10) can be expanded as

$$T_4^4 = \frac{eq}{4\pi r^4} + \frac{b}{16\pi}\left[\frac{q^2 + e^2}{eq}\right] - \frac{b}{8\pi}, \tag{11}$$

while for large r

$$T_4^4 = \frac{e^2}{8\pi r^4} + \frac{q^2}{8\pi r^4}. \tag{12}$$

The time, time component of Einstein's equation reads

$$\frac{\mathrm{d}}{\mathrm{d}r}(re^{-\lambda}) = 1 - \frac{8\pi G}{c^4} T_4^4, \tag{13}$$

yielding

$$(e^{-\lambda})_{r\to 0} = 1 + \frac{G}{c^4}\left(\frac{2eq}{r^2}\right) - \frac{Gb}{6c^4}\left(\frac{q^2+e^2}{eg}\right) r^2 + \frac{Gbr^2}{3c^4} - \frac{2GM}{rc^2} \tag{14}$$

$$(e^{-\lambda})_{r\to\infty} = 1 + \frac{Ge^2}{r^2c^4} + \frac{Gq^2}{r^2c^4} - \frac{2GM}{rc^2} \tag{15}$$

where the constant of integration is identified with the total mass of the dyon.

Notice that in Eq. (14) the metric near the origin is dominated by the factor $2eq$ with coefficient of $1/r^2$. There is also a symmetry between $e \to q$, $q \to e$. A particle moving in the field of such a dyon would be subject to an interaction near the origin which varies as the product of the electric and magnetic charge of the dyon. Note also, as r increases the nature of the coupling changes. If $q \ll e$, then the coupling would vary as bq/e for increasing r. Thus we see a transition from the weak coupling (eq) to strong coupling (bq/e) as r increases. In a sense, this is a form of infrared slavery applied to a particle moving in the dyon field where the strength of the gravitational attraction increases for increasing r. We also note the rather curious anomalous behaviour of the metric if the electric and magnetic charge are equal; $e = q$ gives for $r \to 0$

$$e^{-\lambda} = 1 + \frac{2Ge^2}{r^2c^4} - \frac{Gbr^2}{3c^4} + \frac{Gbr^2}{3c^4}$$

$$e^{-\lambda} = 1 + \frac{2Ge^2}{r^2c^4} \text{ for } r \to 0.$$

This implies that the metric for small r is independent of the constant b and the gravitational effects of the dyon are independent of the electromagnetic scale set by b and depend only on the electric and magnetic charge $e = q$. Thus, both the near region, $r = 0$, and the far region, $r \to 0$, produce a gravitational field independent of the electromagnetic scale b. Allow me to compare this situation which refers to gravitational effects with the apparent alteration of the vacuum structure in Q. E. D. conjectured to accommodate the 1.7—1.8 MeV e^+e^- peak in heavy ion collisions wherein Caldi and Chodos have suggested that a bound state of e^+e^- pair forms in a modified vacuum structure of Q. E. D. in somewhat the same fashion that the chiral phase transition occurs in Q. C. D. with the acquisition of quark masses with the onset of the confining phase[10]. In other words, chiral symmetry breaking and confinement are closely linked. This is the conjectured analogy with the Q. E. D. case for the formation of a modified vacuum structure through strong electric and magnetic fields formed in heavy ion collisions. In the above problem of a dyon, we see that b measures this modified background for the gravitational field of the dyon wherein the physics for small r and large r is independent of b, but for intermediate r, the gravitational field of a dyon depends on b. If we borrow from the thinking of Salam, in his theory of strong gravity[11], we find that in his picture, strong gravity represents the strong interaction and is influential in confinement of quarks through the strong gravitational field. In fact, he has discussed the confinement of a scalar particle in the strong gravitational field of a point mass by analogizing this to quark confinement[12]. Somewhat related R. Magnani[13] has discussed the analogy between quarks and monopoles within the framework of strong gravity with a Yukawa like solution being identified with a monopole solution for the case of vanishing gluon mass.

In closing, we gently suggest that a purely classical theory with a non-linear abelian gauge field has both confinement properties of the associated gravitational field of the dyon solution and a phase structure dependent on the length scale studied. For large r, the gravitational field is identical to the Maxwell case.

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NELINEARNA ELEKTRODINAMIKA I ABELOV DION U OPĆOJ TEORIJI RELATIVNOSTI

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Koristeći nespecifični nelinearni elektromagnetski lagranžijan nađena su rješenja za električno i magnetsko polje diona. Metrika je razvijena u oklolišu ishodišta i u prostornoj beskonačnosti. Vodeći članovi u prostornoj beskonačnosti slažu se s rješenjima Maxwellove elektrodinamike.