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SUPERSYMMETRIC EXTENSION OF A CENTRAL POTENTIAL

MLADEN MARTINIS and VESNA MIKUTA-MARTINIS

Ruđer Bošković Institute, 41001 Zagreb, P. O. B. 1016, Croatia, Yugoslavia

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We extend the investigation of the supersymmetric hydrogen atom to the case of a central potential using the factorisation property of a three-dimensional one-particle Hamiltonian $H^{(0)}$. By introducing the »spin« operator into the space of fermions, we show that the total angular momentum operators, $J_k = L_k + S_k$ and $\vec{S}^2$, commute with the supersymmetric Hamiltonian and can be used to classify its eigenstates. We also show that the matrix Hamiltonians $H^{(1)}$ and $H^{(2)}$ can be written in the form which exhibits the structure corresponding to the motion of two spin $\frac{1}{2}$ particles in central and tensor potentials. The relative strength between the potentials is fixed by supersymmetry.

1. Introduction

The ideas of supersymmetry (SUSY) have been successfully applied not only as an extension of Lorentz symmetry in relativistic field theory[1], but also as a symmetry in nuclear physics[2] and in quantum-mechanical systems[3] to provide a testing ground for various mechanisms for breaking SUSY and to relate the spectra and wave-functions of different Hamiltonians.

The SUSY extension of quantum-mechanical systems in space dimensions ≥ 2 is most easily achieved by using the factorisation property of the Schrödinger operator[4,5] whose spectrum is bounded from below, i. e. possesses a normalised ground-state wavefunction.

In this paper we extend the investigation of paper I[6] and apply the factorisation method and its subsequent SUSY extension to the three-dimensional Schrödinger Hamiltonian with a central potential.

The plan of the paper is as follows. Following I[6], in Section 2 we give a brief presentation of the factorisation method for a three-dimensional quantum-mechanical system and its relation to SUSY. In Section 3 we apply the factorisation method to a three-dimensional central potential. We indicate the role of the spin and show that a tensor potential appears as a natural consequence of the SUSY extension. In Section 5 we give the concluding remarks.

2. *The factorisation method and SUSY*

For any three-dimensional time-independent Hamiltonian $H_0 = \frac{1}{2} p_k^2 + V$ whose energy spectrum is bounded from below there exist three generators[6,7]

$$Q_k = \frac{1}{\sqrt{2}}(\mathrm{i}p_k + \partial_k \chi) \qquad k = 1, 2, 3 \tag{2.1}$$

and their duals

$$\tilde{Q}_{ij} = \varepsilon_{ijk} Q_k \tag{2.1'}$$

in terms of which H_0 can be written in the factorised form[4,5]

$$H^{(0)} = H_0 - E_0 = Q_k^\dagger Q_k = \frac{1}{2} \mathrm{Tr}\,(\tilde{Q}^\dagger \cdot \tilde{Q}). \tag{2.2}$$

Here E_0 is the ground-state energy and $\chi = -\ln \psi_0$, with ψ_0 being the normalised ground-state wavefunction of H_0 satisfying

$$Q_k \psi_0 = 0. \tag{2.3}$$

By introducing a set of three fermion creation and annihilation operators f_k ($k = 1, 2, 3$), with the usual anticommutation relations

$$\{f_k^\dagger, f_l\} = \delta_{kl} \qquad \{f_k^\dagger, f_l^\dagger\} = 0 = \{f_k, f_l\} \tag{2.4}$$

we can construct a supersymmetric Hamiltonian H

$$\hat{H} = \{\hat{Q}^\dagger, \hat{Q}\} = \hat{Q}^\dagger \hat{Q} + \hat{Q}\hat{Q}^\dagger \tag{2.5}$$

whose supercharges $\hat{Q}$ and $\hat{Q}^\dagger$ are

$$\hat{Q} = Q_l f_l^\dagger \qquad \hat{Q}^\dagger = Q_l^\dagger f_l \tag{2.6}$$

and satisfy the algebra

$$\hat{Q}^2 = 0 \qquad \hat{Q}^{\dagger 2} = 0 \tag{2.7}$$

$$[\hat{H}, \hat{Q}] = 0 \qquad [\hat{H}, \hat{Q}^{+}] = 0.$$

Since the Hamiltonian (2.5) also commutes with the fermion-number operator $N_F = f_l^\dagger f_l$, it can be written as a direct sum of Hamiltonians $H^{(n)}$ corresponding to a fixed number of fermions, $n = 0, 1, 2$ and 3:

$$\hat{H} = \bigoplus_{n=0}^{3} H^{(n)}. \tag{2.8}$$

Note that $\hat{H}$ also has a more convenient form

$$\hat{H} = H^{(0)} + [Q_r Q_s^\dagger] f_r^\dagger f_s \tag{2.9}$$

which shows how the SUSY extension of $H^{(0)}$ is achieved.

Since the action of the supercharge $\hat{Q}$ changes the fermion number of the state by one,

$$N_F \hat{Q} = \hat{Q}\,(N_F + 1) \tag{2.10}$$

we can connect Hamiltonians differing in the number of fermions by one. They are

$$Q_l H^{(0)} = H_k^{(1)} Q_k$$

$$\hat{Q} \cdot H^{(1)} = H^{(2)} \cdot \hat{Q} \tag{2.11}$$

(matrix multiplication is implied)

$$Q_l H_{lk}^{(2)} = H^{(3)} Q_k.$$

Each energy level of an intermediate Hamiltonian $H^{(n)}$ in the chain (2.11) is doubly degenerate except the ground-state level of $H^{(0)}$, which is singly degenerate because of (2.3).

3. *The central potential*

For the central potential the operators Q_k become

$$Q_k = \frac{1}{\sqrt{2}} \left(\mathrm{i} p_k + \frac{x_k}{r} \chi' \right) \qquad k = 1, 2, 3 \tag{3.1}$$

where

$$\chi' = \frac{\mathrm{d}}{\mathrm{d}r} \chi(r).$$

It is easy to see that the SUSY Hamiltonian $\hat{H}$ does not commute with the components of the orbital angular momenta L_k. In fact, we find that

$$[L_k, \hat{Q}] = \mathrm{i} \varepsilon_{klj} f_l^\dagger Q_j \tag{3.2}$$

showing that the notion of spin is necessary.

In the space of fermions we may define components of a spin operator as

$$\mathrm{i} S_k = \varepsilon_{klj} f_l^\dagger f_j. \tag{3.3}$$

They satisfy the usual spin commutation relations

$$[S_k, S_l] = \mathrm{i} \varepsilon_{klm} S_m$$

with

$$\vec{S}^2 = N_F (3 - N_F). \tag{3.4}$$

Their commutation relations with the supercharge $\hat{Q}$ are

$$[S_k, \hat{Q}] = -[L_k, \hat{Q}] \quad \text{and} \quad [\vec{S}^2, \hat{Q}] = 0. \tag{3.5}$$

We conclude that the components of the total angular momenta $J_k = L_k + S_k$ and the total spin operator $\vec{S}^2$ commute with the SUSY Hamiltonian $\hat{H}$, i. e.

$$[\hat{H}, J_k] = 0 \qquad [\hat{H}, \vec{S}^2] = 0 \tag{3.6}$$

and are therefore conserved. The eigenstates of $\hat{H}$ can be classified according to the number of fermions which determines the total spin through (3.4) and the value of the total angular momentum J.

For the central potential we can verify that the Hamiltonians $H^{(n)}$, where $n = 0, 1, 2, 3$, have the form

$$H^{(n)} = \frac{1}{2} p_k^2 + V_C^{(n)}(r) + V_T^{(n)}(r)\, S_{12} - E_0 \tag{3.7}$$

where

$$V_C^{(n)}(r) = V(r) + \frac{n}{3} \frac{1}{r} (r\chi)'' \tag{3.8}$$

and

$$V_T^{(n)}(r) = (-)^n \frac{n(3-n)}{12} r \left(\frac{\chi'}{r}\right)'.$$

Here S_{12} is the usual »tensor operator«

$$S_{12} = 2\left[3\frac{(\vec{S}\cdot\vec{r})^2}{r^2} - 2\right] \tag{3.9}$$

with

$$(S_i)_{jk} = -i\varepsilon_{ijk}.$$

Only the Hamiltonians $H^{(1)}$ and $H^{(2)}$ contain the tensor potential whose strength is related to the original central potential V by

$$2(V - E_0) = (\chi')^2 - \frac{1}{r}(r\chi)''. \tag{3.10}$$

The solution of (3.10) for χ is subjected to the boundary conditions

$$\chi(0) = 0 \qquad \chi(\infty) = \infty. \tag{3.11}$$

These conditions are requirements that the wavefunction $\psi_0 = \exp(-\chi)$ be a normalisable ground-state wavefunction.

We also note that the tensor potential disappears if $\chi \sim r^2$. This happens when the central potential is harmonic.

4. *Conclusion*

We have shown that the three-dimensional one-particle quantum-mechanical problem in the central potential can be extended to complete SUSY algebra. By introducing the »spin« operator into the space of fermions, we find that the complete set of commuting operators is H, Q (or $\bar{Q}$), $\vec{J}^2$, J_3 and $\vec{S}^2 = N_F(3 - N_F)$. The eigenfunctions of $\hat{H}$ may be sought among the common eigenfunctions of $\vec{J}^2$, J_3 and N_F.

We have shown that the appearance of the tensor potential in some of SUSY Hamiltonians with a definite number of fermions is a natural consequence of the SUSY extension of the central potential.

To conclude, we have constructed a model of SUSY quantum mechanics which describes the motion of two spin $\frac{1}{2}$ particles in the central and tensor potentials, both of which are determined by a single central potential $V(r)$.

It is interesting to note that in a similar way it is also possible to generate a SUSY Hamiltonian which describes the motion of spin $-\frac{1}{2}$-particles in central and spin-orbit potentials[7,8].

The SUSY charges are then given in the following matrix form:

$$\hat{Q} = \begin{bmatrix} 0 & 0 \\ \vec{\sigma} \cdot \vec{Q}^{\dagger} & 0 \end{bmatrix} \quad \text{and} \quad \hat{Q}^{\dagger} = \begin{bmatrix} 0 & \vec{\sigma} \cdot \vec{Q} \\ 0 & 0 \end{bmatrix}$$

where $\vec{\sigma}$ are the usual Pauli matrices.

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SUPERSIMETRIČNO PROŠIRENJE CENTRALNOG POTENCIJALA

MLADEN MARTINIS i VESNA MIKUTA-MARTINIS

Institut »Ruđer Bošković«, 41001 Zagreb, p. p. 1016

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Prošireno je istraživanje supersimetričnog vodikovog atoma na slučaj centralnog potencijala koristeći svojstvo faktorizacije trodimenzionalnog jednočestičnog hamiltonijana $H^{(0)}$. Uvođenjem operatora spina u prostor fermiona pokazano je da operatori ukupnog angularnog momenta, $J_k = L_k + S_k$ i $\vec{S}^2$, komutiraju sa supersimetričnim hamiltonijanom i mogu se upotrijebiti za klasificiranje vlastitih stanja. Također je pokazano da se matrični hamiltonijani $H^{(1)}$ i $H^{(2)}$ mogu napisati u obliku koji pokazuje strukturu koja odgovara gibanju dviju čestica spina 1/2 u centralnom i tenzorskom potencijalu. Relativna jakost između ovih potencijala određena je supersimetrijom.