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RELATIVISTIC NONLINEARLY-COUPLED ELECTROMAGNETIC AND LANGMUIR WAVES

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In a plasma acted on by an intense laser field, one encounters the problem of relativistic nonlinearly-coupled electromagnetic and Langmuir waves. Previous works on this problem consider one of the two coupled waves as a driven wave and consequently consider a highly specialized subclass of solutions. In this paper, we use a generalized perturbation theory to exhibit general nonlinearly-coupled wave solutions that exist and provide a unified framework for previously known specialized solutions of this problem.

1. Introduction

An example of nonlinearly-coupled plasma waves arises with the nonlinear propagation in a plasma of an electromagnetic wave which is strong enough to drive the electrons to relativistic speeds. Two nonlinear effects which arise in this context are:

(i) Relativistic variation of the electron mass;

(ii) Excitation of space-charged fields by strong $\mathbf{v} \times \mathbf{B}$ forces driving electrons along the direction of propagation of the electromagnetic wave.

The first effect leads to a propagation of the electromagnetic wave a normally overdense plasma (recall that according to linear theory of propagation of electromagnetic waves in a plasma, an electromagnetic wave with a frequency less than the plasma frequency cannot propagate in the plasma). The second effect leads to a coupling of the electromagnetic wave to the Langmuir wave. This problem has been considered by Akhiezer and Polovin[1], Lunow[2], Kaw and Dawson[3], Max and Perkins[4], Chian and Clemmow[5] and Decoster[6].

However, in most of the previous works dealing with the latter problem, one of the waves is treated as a driven wave (i. e. of constant amplitude) and as a consequence, one picks out for investigation a highly specialized subclass of solutions. In this paper, we use a generalized perturbation theory to exhibit general nonlinearly-coupled wave solutions that exist and thus put older specialized solutions in the proper perspective. We will show in particular that the known results for a nonlinear relativistic longitudinal wave (Akhiezer and Polovin[1]) and a nonlinear relativistic transverse wave (Sluijter and Montgomery[7]) are recovered as special cases of the general nonlinearly-coupled wave solution.

2. *Relativistically-coupled electromagnetic and Langmuir waves*

Most of the previous works on this problem are concerned with obtaining exact analytical solution for some special cases like purely transverse waves and purely longitudinal waves. DeCoster[6] considered the general case, for which he gave some approximate solutions. However, DeCoster's procedure does not allow for amplitude modulations of the coupled waves, and is therefore restricted in scope. Actually, DeCoster's calculation will be invalid if the system of coupled waves in question exhibits internal resonances with the concomitant modulations in the amplitudes of the coupled waves. We give here a generalized perturbation theory that allows for both amplitude — and phase modulations of the coupled waves. This theory can therefore, successfully deal with internal resonances if they arise in the system in question. This theory also recovers results of the known special cases in the appropriate limit.

2A. Basic equations

Consider an intense electromagnetic wave propagating along the x-axis in a plasma. Under the action of this wave field, let us assume that the electrons form a cold fluid (i. e., their directed speeds are much larger than their random speeds) undergoing nonlinear oscillations, whereas the ions which cannot respond to these high-frequency oscillations make up an immobile neutralizing positive-charge background. The basic equations describing the propagation of this electromagnetic wave coupled nonlinearly to the Langmuir wave in the plasma are

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{B} = -\frac{4\pi}{c} ne\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{E} = -4\pi e (n - n_0) \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{p} = -\, e\mathbf{E} - \frac{e}{c}\, \mathbf{v} \times \mathbf{B} \tag{5}$$

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{6}$$

where **E** and **B** are the electric and magnetic field components of the wave, **v** is the electron-fluid velocity, n is the number density of the electrons, and **p** is the relativistic momentum of an electron. Note that the equation governing the charge conservation is contained in Eqs. (1)—(4).

Following Akhiezer and Polovin[1], let us consider stationary unidirectional propagation, and seek plane nonlinear wave solutions which depend on z and t in the form

$$\mathscr{V}(z, t) = \mathscr{V}(\eta), \qquad \eta = z - ut. \tag{7}$$

It may be noted that the bidirectional wave propagation leads to some resonant wave-wave interactions.

On introducing,

$$\varrho = \frac{\mathbf{p}}{mc}, \quad \beta = \frac{u}{c}, \quad \omega_p^2 = \frac{4\pi n_0 e^2}{m} \tag{8}$$

we can derive from Eqs. (1)—(6) (Akhiezer and Polovin[1])

$$\frac{\mathrm{d}^2 \varrho_x}{\mathrm{d}\eta^2} + \frac{\omega_p^2}{c^2} \frac{\beta \varrho_x}{(\beta^2 - 1)\, [\beta \sqrt{1 + \varrho^2} - \varrho_z]} = 0 \tag{9}$$

$$\frac{\mathrm{d}^2 \varrho_y}{\mathrm{d}\eta^2} + \frac{\omega_p^2}{c^2} \frac{\beta \varrho_y}{(\beta^2 - 1)\, [\beta \sqrt{1 + \varrho^2} - \varrho_z]} = 0 \tag{10}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} (\beta \varrho_z - \sqrt{1 + \varrho^2}) + \frac{\omega_p^2}{c^2} \frac{\varrho_z}{[\beta \sqrt{1 + \varrho^2} - \varrho_z]} = 0 \tag{11}$$

n being the unperturbed value of the number density. Let us consider the electromagnetic wave to be linearly polarised so that $\varrho_y \equiv 0$. On introducing,

$$\varkappa = \sqrt{\beta^2 - 1}\, \varrho_x, \qquad \mathscr{L} = \beta \varrho_z - \sqrt{1 + \varrho^2}$$

$$\xi = \frac{\omega_p}{c} \frac{\eta}{\sqrt{\beta^2 - 1}} \tag{12}$$

we obtain from Eqs. (9) and (11)

$$\frac{d^2\varkappa}{d\xi^2} + \frac{\beta\varkappa}{(\beta^2 - 1 + \varkappa^2 + \mathscr{Z}^2)^{1/2}} = 0 \tag{13}$$

$$\frac{d^2\mathscr{Z}}{d\xi^2} + \frac{\beta\mathscr{Z}}{(\beta^2 - 1 + \varkappa^2 + \mathscr{Z}^2)^{1/2}} + 1 = 0. \tag{14}$$

Eqs. (13) and (14) are the basic equations for investigation in the first part of this paper. We exclude from the following the case $\beta = 1$ since Eqs. (13) and (14) do not constitute the right system to treat this case.

2B. Generalized perturbation theory

Let us now assume that the coupling between the electromagnetic wave and the Langmuir wave is weak, and for bookkeeping purposes, let us introduce a small parameter ε which may be considered to be a typical wave amplitude. Let us put

$$\varkappa = \varepsilon X, \qquad \mathscr{Z} = -1 + \varepsilon Z. \tag{15}$$

Eqs. (13) and (14) then give on expansion of the radicals

$$\frac{d^2X}{d\xi^2} + X = -\varepsilon\left(\frac{1}{\beta^2}XZ\right) + \frac{\varepsilon^2}{2\beta^2}\left[X^3 + \left(1 - \frac{3}{\beta^2}\right)XZ^2\right] + 0(\varepsilon^3) \tag{16}$$

$$\frac{d^2Z}{d\xi^2} + \left(1 - \frac{1}{\beta^2}\right)Z = -\varepsilon\left[\frac{3}{2\beta^2}\left(1 - \frac{1}{\beta^2}\right)Z^2 + \frac{X^2}{2\beta^2}\right] + \frac{\varepsilon^2}{2\beta^2}\left[\left(1 - \frac{6}{\beta^2} + \right.\right.$$

$$\left.\left. + \frac{5}{\beta^4}\right)Z^3 + \left(1 - \frac{3}{\beta^2}\right)X^2Z\right] + 0(\varepsilon^3). \tag{17}$$

Note that X corresponds to the electromagnetic wave, and Z corresponds to the Langmuir wave. ε turns out to be a kind of coupling parameter.

Let us seek solutions to Eqs. (16) and (17) of the form

$$X = A_1(\xi_1, \xi_2)\cos\varphi_1(\xi_1, \xi_2) + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots$$

$$Z = A_2(\xi_1, \xi_2)\cos\varphi_2(\xi_1, \xi_2) + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \tag{18}$$

where,

$$\varphi_1(\xi_1, \xi_2) = \xi - \Theta_1(\xi_1, \xi_2)$$

$$\varphi_2(\xi_1, \xi_2) = \sqrt{1 - \frac{1}{\beta^2}}\,\xi - \Theta_2(\xi_1, \xi_2) \tag{19}$$

$$\xi_1 = \varepsilon\xi, \qquad \xi_2 = \varepsilon^2\xi.$$

The motivation behind the prescription (18) is the expectation that solutions of Eqs. (16) and (17) for $\varepsilon \neq 0$ (but for small ε) are very nearly simple harmonics of the form $A_1 \cos\varphi_1$, $A_2 \cos\varphi_2$ which they would identically be if $\varepsilon = 0$. The perturbations induced by the terms with $\varepsilon \neq 0$ on the right hand side of Eqs. (16) and (17) are then expected to be reflected in slow changes (characterized by the slow scales ξ_1, ξ_2) in the amplitudes A_1, A_2 and phases Θ_1, Θ_2 of the near harmonics and in higher harmonics through the u's and v's. This is a slight generalization of the method given by Struble[8].

Using (18) and (19), we obtain from Eqs. (16) and (17)

$$\begin{aligned}
&(2\varepsilon A_1 \Theta_{1\xi_1} + 2\varepsilon^2 A_1 \Theta_{1\xi_2} - \varepsilon^2 A_1 \Theta_{1\xi_1}^2 + \varepsilon^2 A_{1\xi_1\xi_1}) \cos\varphi_1 + \\
&+ (-2\varepsilon A_{1\xi_1} - 2\varepsilon^2 A_{1\xi_2} + 2\varepsilon^2 A_{1\xi_1} \Theta_{1\xi_1} + \varepsilon^2 A_1 \Theta_{1\xi_1\xi_1}) \sin\varphi_1 + \\
&+ \varepsilon (u_{1\xi\xi} + u_1) + \varepsilon^2 (2u_{1\xi\xi_1} + u_{2\xi\xi} + u_2) + \dots = \\
&= -\frac{\varepsilon}{\beta^2} (A_1 A_2 \cos\varphi_1 \cdot \cos\varphi_2) + \\
&+ \varepsilon^2 [-\frac{1}{\beta^2} (A_1 \cos\varphi_1 \cdot v_1 + A_2 \cos\varphi_2 \cdot u_1) + \\
&+ \frac{1}{2\beta^2} \{A_1^3 \cos^3\varphi_1 + \left(1 - \frac{3}{\beta^2}\right) A_1 A_2^2 \cos\varphi_1 \cdot \cos^2\varphi_2\}] + \dots . \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&(2\varepsilon \sqrt{1 - 1/\beta^2}\, A_2 \Theta_{2\xi_1} + 2\varepsilon^2 \sqrt{1 - 1/\beta^2}\, A_2 \Theta_{2\xi_2} + \varepsilon^2 A_2 \Theta_{2\xi_1}^2 + \\
&+ \varepsilon^2 A_{2\xi_1\xi_1}) \cos\varphi_2 + (-2\varepsilon \sqrt{1 - 1/\beta^2}\, A_{2\xi_1} - 2\varepsilon^2 \sqrt{1 - 1/\beta^2}\, A_{2\xi 2} + \\
&+ 2\varepsilon^2 A_{2\xi_1} \Theta_{2\xi_1} + \varepsilon^2 A_2 \Theta_{2\xi_1\xi_1}) \sin\varphi_2 + \\
&+ \varepsilon \left[v_{1\xi\xi} + \left(1 - \frac{1}{\beta^2}\right) v_1\right] + \varepsilon \left[2v_{1\xi\xi_1} + v_{2\xi\xi} + \left(1 - \frac{1}{\beta^2}\right) v_2 + \dots = \right. \\
&= -\varepsilon \left[\frac{3}{2\beta^2}\left(1 - \frac{1}{\beta^2}\right) A_2^2 \cos^2\varphi_2 + \frac{1}{2\beta^2} A_1^2 \cos^2\varphi_1\right] + \\
&+ \varepsilon^2 \left[-\left\{\frac{3}{2\beta^2}\left(1 - \frac{1}{\beta^2}\right) A_2 \cos\varphi_2 \cdot v_1 + \frac{1}{\beta^2} A_1 \cos\varphi_1 \cdot u_1\right\} + \right. \\
&+ \frac{1}{2\beta^2} \left\{\left(1 - \frac{6}{\beta^2} + \frac{5}{\beta^4}\right) A_2^3 \cos^3\varphi_2 + \right. \\
&\left.\left. + \left(1 - \frac{3}{\beta^2}\right) A_1^2 A_2 \cos^2\varphi_1 \cdot \cos\varphi_2\right\}\right] + \dots \qquad (21)
\end{aligned}$$

We obtain from Eqs. (20) and (21), to $0(\varepsilon)$

$$\Theta_{1\xi_1} = 0 \tag{22}$$

$$A_{1\xi_1} = 0 \tag{23}$$

$$u_{1\xi\xi} + u_1 = -\frac{1}{\beta^2} A_1 A_2 \cos\varphi_1 \cdot \cos\varphi_2 \tag{24}$$

$$\Theta_{2\xi_1} = 0 \tag{25}$$

$$A_{2\xi_1} = 0 \tag{26}$$

$$v_{1\xi\xi} + \left(1 - \frac{1}{\beta^2}\right) v_1 = -\frac{3}{2\beta^2}\left(1 - \frac{1}{\beta^2}\right) A_2^2 \cos^2\varphi_2 - \frac{1}{2\beta^2} A_1^2 \cos^2\varphi_1. \tag{27}$$

On solving Eqs. (22)—(27), we obtain

$$u_1 = \frac{A_1 A_2/2\beta^2}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} \cos(\varphi_1 + \varphi_2) + \frac{A_1 A_2/2\beta^2}{(1 - \sqrt{1 - 1/\beta^2})^2 - 1} \cos(\varphi_1 - \varphi_2)$$

$$v_1 = \frac{-1/4\beta^2}{(1 - 1/\beta^2)} [3(1 - 1/\beta^2) A_2^2 + A_1^2] + \frac{A_2^2}{4\beta^2} \cos 2\varphi_2 + - \frac{A_1^2/4\beta^2}{(1 - 1/\beta^2) - 4} \cos 2\varphi_1 \tag{28}$$

where the A's and Θ's are constants to $0(\varepsilon)$. Thus, in general, the two waves propagate without any appreciable energy exchange between them. Now, the presence of an internal resonance in the system is identified when there is a small divisor in the expressions for u's and v's. Observe in (28) that there are no small divisors in the expressions for u_1 and v_1 so that there are no internal resonances to $0(\varepsilon)$ in the present system. This agrees with DeCoster's result that at the $(2n + m - 1)$ th order (m and n are two integers) there are resonances for $(1 - 1/\beta^2)^{1/2} = 2m/n$, according to which there are no resonances for the first order.

Next, using (22)—(28), we obtain from Eqs. (20) and (21) to $0(\varepsilon^2)$:

$$2A_1\Theta_{1\xi_2} \cos\varphi_1 - 2A_{1\xi_2} \sin\varphi_1 + u_{2\xi\xi} + u_2 =$$

$$= -\frac{1}{\beta^2}\left[\frac{-1/4\beta^2}{(1 - 1/\beta^2)}\left\{3\left(1 - \frac{1}{\beta^2}\right) A_2^2 + A_1^2\right\} + \frac{A_1^2/8\beta^2}{(1 - 1/\beta^2) - 4} + \right.$$

$$+ \frac{A_2^{2\,2}/4\beta^2}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} + \frac{A_2^2/4\beta^2}{(1 - \sqrt{1 - 1/\beta^2})^2 - 1} - \frac{3A_1^2}{8} -$$

$$- \frac{1}{4}\left(1 - \frac{3}{\beta^2}\right) A_2^2\Big] A_1 \cos\varphi_1 + \left[- \frac{A_1 A_2^2}{8\beta^4} - \frac{A_1 A_2^2/4\beta^4}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} + \right.$$

$$+ \frac{1}{8\beta^2}\left(1 - \frac{3}{\beta^2}\right) A_1 A_2^2\Big] \cos(\varphi_1 + 2\varphi_2) + \left[- \frac{A_1 A_2^2}{8\beta^4} - \frac{A_1 A_2^2/4\beta^4}{(1 - \sqrt{1 - 1/\beta^2})^2 - 1} + \right.$$

$$+ \frac{1}{8\beta^2}\left(1 - \frac{3}{\beta^2}\right) A_1 A_2^2\Big] \cos(\varphi_1 - 2\varphi_2) + \left[\frac{A_1^3}{8\beta^2} + \frac{A_1^3/8\beta^4}{(1 - 1/\beta^2) - 4]}\right] \cos 3\varphi_1 \tag{29}$$

$$2A_2 \sqrt{1 - 1/\beta^2}\, \Theta_{2\xi_2} \cos\varphi_2 - 2A_{2\xi_2} \sqrt{1 - 1/\beta^2} \sin\varphi_2 + v_{2\xi\xi} +$$

$$+ \left(1 - \frac{1}{\beta^2}\right) v_2 = \left[\frac{3}{4\beta^4}\left\{3\left(1 - \frac{1}{\beta^2}\right) A_2^2 + A_1^2\right\} - \frac{3}{8\beta^4}\left(1 - \frac{1}{\beta^2}\right) A_2^2 - \right.$$

$$- \frac{A_1^2/4\beta^4}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} - \frac{A_1^{2}/4\beta^4}{(1 - \sqrt{1 - 1/\beta^2})^2 - 1} +$$

$$+ \frac{3}{8\beta^2}\left(1 - \frac{6}{\beta^2} + \frac{5}{\beta^4}\right) A_2^2 + \frac{1}{4\beta^2}\left(1 - \frac{3}{\beta^2}\right) A_1^2\Big] A_2 \cos\varphi_2 +$$

$$+ \left[- \frac{3}{8\beta^4}\left(1 - \frac{1}{\beta^2}\right) A_2^3 + \frac{1}{8\beta^2}\left(1 - \frac{6}{\beta^2} + \frac{5}{\beta^4}\right) A_2^3\right] \cos 3\varphi_2 +$$

$$+ \left[\frac{\frac{3}{8\beta^2}(1 - 1/\beta^2) A_1^2 A_2}{(1 - 1/\beta^2) - 4} - \frac{A_1^2 A_2 4\beta^2}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} + \right.$$

$$+ \frac{1}{8\beta^2}\left(1 - \frac{3}{\beta^2}\right) A_1^2 A_2\Big] \cos(2\varphi_1 + \varphi_2) +$$

$$+ \left[\frac{\frac{3}{8\beta^2}(1 - 1/\beta^2) A_1^2 A_2}{(1 - 1/\beta^2) - 4} - \frac{A_1^2 A_2/4\beta^2}{(1 - \sqrt{1 - 1/\beta^2})^2 - 1} + \right.$$

$$+ \frac{1}{8\beta^2}\left(1 - \frac{3}{\beta^2}\right) A_1^2 A_2\Big] \cos(2\varphi_1 - \varphi_2) \tag{30}$$

from which it follows

$$\Theta_{1\xi_2} = - \frac{1}{2\beta^2}\left[\frac{-1/4\beta^2}{(1 - 1/\beta^2)}\left\{3\left(1 - \frac{1}{\beta^2}\right) A_2^2 + A_1^2\right\} + \right.$$

$$-\frac{A_1^2/8\beta^2}{(1-1/\beta^2)-4}+\frac{A_2^2/4\beta^2}{(1+\sqrt{1-1/\beta^2})^2-1}+\frac{A_2^2/4\beta^2}{(1-\sqrt{1-1/\beta^2})^2-1}-$$

$$-\frac{3A_1^2}{8}-\frac{1}{4}\left(1-\frac{3}{\beta^2}\right)A_2^2\Bigg] \tag{31}$$

$$A_{1\xi_2}=0 \tag{32}$$

$$u_{2\xi\xi_2}+u_2=\left[-\frac{A_1A_2^2}{8\beta^4}-\frac{A_1A_2^2/4\beta^4}{(1+\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{8\beta^2}\left(1-\frac{3}{\beta^2}\right)A_1A_2^2\right]\cos(\varphi_1+2\varphi_2)+\left[-\frac{A_1A_2^2}{8\beta^4}-\frac{A_1A_2^2/4\beta^4}{(1-\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{8\beta^2}\left(1-\frac{3}{\beta^2}\right)A_1A_2^2\right]\cos(\varphi_1-2\varphi_2)+\left[\frac{A_1^3/8\beta^4}{(1-1/\beta^2)-4}+\frac{A_1^3}{8\beta^2}\right]\cos 3\varphi_1 \tag{33}$$

$$\Theta_{2\xi_2}=\frac{1}{2\sqrt{1-1/\beta^2}}\left[\frac{3}{4\beta^4}\left\{3\left(1-\frac{1}{\beta^2}\right)A_2^2+A_1^2\right\}-\frac{3}{8\beta^4}\left(1-\frac{1}{\beta^2}\right)A_2^2-\right.$$

$$-\frac{A_1^2/4\beta^4}{(1+\sqrt{1-1/\beta^2})^2-1}-\frac{A_1^2/4\beta^4}{(1-\sqrt{1-1/\beta^2})^2-1}+$$

$$\left.+\frac{3}{8\beta^2}\left(1-\frac{6}{\beta^2}+\frac{5}{\beta^4}\right)A_2^2+\frac{1}{4\beta^2}\left(1-\frac{3}{\beta^2}\right)A_1^2\right] \tag{34}$$

$$A_{2\xi_2}=0 \tag{35}$$

$$v_{2\xi\xi}+\left(1-\frac{1}{\beta^2}\right)v_2=\left[-\frac{3}{8\beta^4}\left(1-\frac{1}{\beta^2}\right)A_2^3+\right.$$

$$\left.+\frac{1}{8\beta^2}\left(1-\frac{6}{\beta^2}+\frac{5}{\beta^4}\right)A_2^3\right]\cos 3\varphi_2+$$

$$+\left[\frac{\frac{3}{8\beta^2}\left(1-\frac{1}{\beta^2}\right)A_1^2A_2}{(1-1/\beta^2)-4}-\frac{A_1^2A^2/4\beta^2}{(1+\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{8\beta^2}\left(1-\frac{3}{\beta^2}\right)A_1^2A_2\right]\cos(2\varphi_1+\varphi_2)+$$

$$+\left[\frac{\frac{3}{8\beta^2}\left(1-\frac{1}{\beta^2}\right)A_1^2A_2}{(1-1\beta^2)-4}-\frac{A_1^2A_2/4\beta^2}{(1-\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{8\beta^2}\left(1-\frac{3}{\beta^2}\right)A_1^2A_2\right]\cos(2\varphi_1-\varphi_2). \tag{36}$$

Eqs. (32) and (35) imply that A_1 and A_2 are constants to $0(\varepsilon^2)$. Solving equations (33) and (36), we obtain

$$u_2=-\frac{A_1A_2^2}{4\beta^2}\left[-\frac{1}{2\beta^2}-\frac{1/\beta^2}{(1+\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{2}\left(1-\frac{3}{\beta^2}\right)\right]\frac{\cos(\varphi_1-2\varphi_2)}{(1+2\sqrt{1-1/\beta^2})^2-1}+-\frac{A_1A_2^2}{4\beta^2}\left[-\frac{1}{2\beta^2}-\right.$$

$$\left.-\frac{1/\beta^2}{(1-\sqrt{1-1/\beta^2})^2-1}+\frac{1}{2}\left(1-\frac{3}{\beta^2}\right)\right]\frac{\cos(\varphi_1-2\varphi_2)}{(1-2\sqrt{1-1/\beta^2})^2-1}+$$

$$+-\frac{A_1^3}{64\beta^2}\left[\frac{1/\beta^2}{(1-1/\beta^2)-4}+1\right]\cos 3\varphi_1$$

$$v_2=-\frac{A_1^2A_2}{4\beta^2}\left[\frac{3/4(1-1/\beta^2)}{(1-1/\beta^2)-4}-\frac{1}{(1+\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{2}\left(1-\frac{3}{\beta^2}\right)\right]\frac{\cos(2\varphi_1+\varphi_2)}{(2+\sqrt{1-1/\beta^2})^2-(1-1/\beta^2)}-$$

$$-\frac{A_1^2A_2}{4\beta^2}\left[\frac{3/4(1-1/\beta^2)}{(1-1/\beta^2)-4}-\frac{1}{(1-\sqrt{1-1/\beta^2})^2-1}+\right.$$

$$\left.+\frac{1}{2}\left(1-\frac{3}{\beta^2}\right)\right]\frac{\cos(2\varphi_1-\varphi_2)}{(2-\sqrt{1-1/\beta^2})^2-(1-1/\beta^2)}-$$

$$-\frac{A_2^3}{64\beta^4}\left[-\frac{3}{\beta^2}+\frac{(1-6/\beta^2+5/\beta^4)}{(1-1/\beta^2)}\cos 3\varphi_2.\right. \tag{37}$$

Using (22), (23), (25), (26), (31), (32), (34) and (35), we obtain from (19)

$$\varphi_1(\xi_1,\xi_2)=\xi+\frac{1}{2\beta^2}\left[\frac{-1/4\beta^2}{(1-1/\beta^2)}\{3(1-1/\beta^2)A_2^2+A_1^2\}-\right.$$

$$- \frac{A_1^2 A_2/4\beta^2}{(1 - 1/\beta^2) - 4} + \frac{A_2^2/4\beta^2}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} - \frac{3}{8} A_1^2 + - \frac{1}{4} \left(1 - \frac{3}{\beta^2}\right) A_2^2 \Big] \xi_2$$

$$\varphi_2 (\xi_1, \xi_2) = \sqrt{1 - 1/\beta^2}\, \xi + - \frac{1}{2\sqrt{1 - 1/\beta^2}} \left[\frac{3}{4\beta^4} \left\{ 3 \left(1 - \frac{1}{\beta^2}\right) A_2^2 + A_1^2 \right\} - \right.$$

$$- \frac{3}{8\beta^4} \left(1 - \frac{1}{\beta^2}\right) A_2^2 - \frac{A_1^2/4\beta^2}{(1 + \sqrt{1 - 1/\beta^2})^2 - 1} - \frac{A_1^2/4\beta^4}{(1 - \sqrt{1 - 1/\beta^2})^2 - 1} +$$

$$\left. + \frac{3}{8\beta^2} \left(1 - \frac{6}{\beta^2} + \frac{5}{\beta^4}\right) A_2^2 + \frac{1}{4\beta^2} \left(1 - \frac{3}{\beta^2}\right) A_1^2 \right] \xi_2.$$

Observe in (37) that there are again no small divisors in the expressions for u_2 and v_2 — so there are no internal resonances to $0(\varepsilon^2)$ in the present system. This also agrees with DeCoster's result. It is quite possible that internal resonances may arise at higher orders.

2C. *Special cases*

Let us now examine the results in Sect. (2B) for some known special cases. First, let us consider a quasi-transverse wave, i. e., $A_2 = 0$, because then, from (12), to $0(\varepsilon)$, $\varrho_z = 0$. Wave-coupling still exists because now $v_1 \neq 0$ (see (28)). Also, observe that now $u_2 \sim A_1^3$ (the second-harmonic terms are not present and are not needed anyway for the second-order frequency shift). Then, (31) gives

$$\Theta_{1\varepsilon_2} = - \frac{(9\beta^2 - 1)}{16(\beta^2 - 1)(3\beta^2 + 1)} A_1^2 \tag{38}$$

The wave frequency ω ($= uk$, k being the wavenumber) is then given, on using (19) and (38), by

$$\omega = \varphi_{1\xi} \cdot \xi_t = \frac{\omega_p \cdot \omega}{\sqrt{\omega^2 - k^2c^2}} \left[1 - \varepsilon^2 \frac{(9\beta^2 - 1)}{16(\beta^2 - 1)(3\beta^2 + 1)} A_1^2 + \dots \right]. \tag{39}$$

Using $V_x \approx eE_x/m\,\omega$, we obtain from Eq. (39) an expression showing the amplitude -dependent frequency shift

$$\omega^2 = (\omega_p^2 + k^2c^2) - \varepsilon^2 \left[\frac{e^2\omega_p^2 E_x^2}{m^2\omega^2c^2} \frac{8\omega^2 + \omega_p^2}{8(4\omega^2 - \omega_p^2)} \right] + 0(\varepsilon^3) \tag{40}$$

which (when ε is set equal to unity) is identical to the result of Sluijter and Montgomery[7]. The above treatment gives a systematic deduction of the latter result through a consistent treatment of the coupling of the electromagnetic transverse wave to a Langmuir wave of weaker strength. This derivation also shows that the frequency shift is actually characterized by the weak coupling parameter which when ε is set equal to unity may be taken to be the quantity $\delta \equiv eE_x/m\omega c$. The result of Sluijter and Montgomery[7] is thus valid to $0(\delta^2)$. The above treatment also gives a systematic procedure to derive the higher-order corrections.

In order to identify the relativistic effects in (40), let us write it in the form

$$\omega^2 = (\omega_p^2 + k^2c^2) - \varepsilon^2 \frac{e^2\omega_p^2 E_x^2}{2m^2\omega^2c^2}\left(\frac{3}{4} - \frac{\omega^2 - \omega_p^2}{4\omega^2 - \omega_p^c}\right).$$

The first term in the amplitude-dependent correction to freqeuncy is due to relativistic effects and leads to modulational instability of the electromagnetic wave (Max et al.[10] and Shivamoggi[11]).

Next, let us consider a quasi-longitudinal wave, i. e., $A_1 = 0$. Wavecoupling does not exist now to $0(\varepsilon^2)$ because $u_1, u_2 = 0$ (see (28) and (37)). Then, (34) gives

$$\Theta_{2\xi_2} = \frac{3A_2^2/\beta^6}{16\sqrt{1 - 1/\beta^2}}(\beta^4 - \beta^2). \tag{41}$$

The wave-frequency ω is then given, on using (19) and (41), by

$$\omega = \frac{\omega_p \cdot \beta}{\sqrt{\beta^2 - 1}}\left[\sqrt{1 - 1/\beta^2} - \varepsilon^2 \frac{3}{16}\frac{\sqrt{1 - 1/\beta^2}}{\beta^2} A_2^2 + 0(\varepsilon^3)\right]$$

or

$$\omega = \omega_p\left[1 - \varepsilon^2 \frac{3}{16}\left|\frac{v_\beta}{c}\right|^2 + 0(\varepsilon^3)\right] \tag{42}$$

which (when ε is set equal to unity) is identical to the result deduced for longitudinal waves by Akhiezer and Polovin[1].

3. *Discussion*

The generalized coupled-wave solution given in the foregoing shows that the effect of coupling is to produce nonlinear frequency shifts and higher harmonics in the individual wave solutions. The higher harmonics are of importance because of their ability to propagate into the plasma more readily than the fundamental — a useful feature from the point of laser heating of plasmas. Larger the coupling parameter ε, larger will be the frequency shifts and the amplitudes of the higher harmonics. To second order in the coupling parameter ε, there are no amplitude modulations of the individual wave solutions which is consistent with the fact that there are no internal resonances in this coupled system to second order in ε. Observe that the foregoing solutions are not valid when $\beta = 1$, because as mentioned before equations (13) and (14) are not the right system to treat this case. It is of interest to note that this generalized solution also gives a unified treatment of the previously known specialized nonlinear solutions for the two individual waves.

It remains to be pointed out that we have not obtained still the most general solution of the coupled nonlinear equations but only the *stationary* solutions. Further, there remains the issue of the stability of these stationary solutions. In this context, it is of interest to note that equations (13) and (14) may be derived from a Hamiltonian (Kaw et al.[9])

$$H = \frac{1}{2}\left(\frac{\mathrm{d}X}{\mathrm{d}\xi}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\mathscr{X}}{\mathrm{d}\xi}\right)^2 + \beta\left(\beta^2 - 1 + X^2 + \mathscr{X}^2\right)^{1/2} + \mathscr{X}. \quad (43)$$

Thus, the coupled-wave problem is equivalent to that of the motion of a fictitious particle in a two-dimensional potential. If $\beta > 1$, the Hamiltonian is positive definite everywhere, and the motion is bounded (this is also apparent from equation (17)).

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RELATIVISTIČKO NELINEARNO MEĐUDJELOVANJE ELEKTROMAGNETSKOG I LANGMUIROVOG VALA

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Poopćena teorija smetnje koristi se za razmatranje nelinearnog međudjelovanja intenzivnog elektromagnetskog vala sa Langmuirovim valom u homogenoj plazmi, u relativističkom području. Razmatra se problem odgovarajućeg frekventnog pomaka i amplitudne modulacije za osnovni val i za njegove harmonike.