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THE AXIAL ANOMALY FROM THE EQUATIONS OF MOTION OF STOCHASTIC QUANTIZATION

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We show how both the vector-current conservation and the axial-vector current anomaly follow directly from the Langevin equations of motion for spinor field theories in the stochastic quantization formalism. Our method employs Novikov's theorem for Grassmann variables and the regularization procedure of Bern, Chan and Halpern.

1. Introduction

It is well known that in spinor electrodynamics the axial vector current is not conserved even in the limit of vanishing mass terms. At the perturbative level, in the calculation of certain one-loop amplitudes, the momentum-routing is fixed in such a way that the vector Ward identity is maintained while the anomaly is associated with the axial-vector current. Among the non-perturbative derivations of such anomalies are those based on the path-integral quantization and the stochastic quantization[1]. In the former approach[2], the anomaly arises because of the non-invariance of the fermionic path-integral measure under chiral transformations, whereas in the latter, the anomaly can be derived from the steady-state noise average of the variation in the action under chiral transformations[3].

There is another derivation of the anomaly by Namiki and others[4], who have used Ito calculus to show that the anomaly arises in the stochastic quantization scheme due to the breaking of the naive Leibnitz formula. This equations of motion approach has also been used by Namiki et al.[5] in deriving the conformal anomaly, and by Reuter[6] in deriving parity-violating anomalies in odd dimensions.

In this paper, we have followed the equations of motion approach of Namiki et al. but have not made use of Ito calculus employed by them for deriving the axial anomaly from the Langevin equations of the stochastic quantization formalism for spinors. We have adopted the recently developed regularization procedure of Bern, Chan and Halpern[7], and derived the anomaly in a much simpler way.

2. *Derivation of the normal and the anomalous Ward identities*

In the method of stochastic quantization of Parisi and Wu[1], one starts by introducing an extra dimension into the Euclidean field theory, called the fictitious time, τ, and postulates a stochastic Langevin dynamics for the system to study its approach to the equilibrium configuration. Parisi and Wu showed that in the steady state limit, i. e., in the $\tau \to \infty$ limit, the usual quantum field theory is reproduced.

To obtain the regulated version of the Ward identities, we make use of the covariant derivative regularization procedure of Bern et al.[7], in which the noise-structure of the Langevin equations are modified in a covariant way to:

$$\frac{\partial \Psi(x, \partial)}{\partial \tau} = -\frac{\delta S[\Psi, \overline{\Psi}]}{\delta \overline{\Psi}} + \int d^4y\, R(x, y)\, \eta(y, \tau) \tag{1}$$

$$\frac{\partial \overline{\Psi}(x, \tau)}{\partial \tau} = \frac{\delta S[\Psi, \overline{\Psi}]}{\delta \overline{\Psi}} + \int d^4y\, \overline{\eta}(y, \tau)\, R(x, y) \tag{2}$$

where $\Psi(x, \tau)$ and $\overline{\Psi}(x, \tau)$ are independent stochastic Grassmann fields and $\overline{\eta}$ and η are independent Grassmann random noise sources with the statistical properties:

$$\langle \eta(x, \tau) \rangle = \langle \overline{\eta}(x, \tau) \rangle = 0$$

$$\langle \eta(x, \tau)\, \overline{\eta}(x', \tau') \rangle = -\langle \overline{\eta}(x', \tau')\, \eta(x, \tau) \rangle = 2\delta^4(x - x')\, \delta(\tau - \tau')$$

$$\langle \eta(x, \tau)\, \eta(x', \tau') \rangle = \langle \overline{\eta}(x, \tau)\, \overline{\eta}(x', \tau') \rangle = 0$$

where the angular brackets denote average over both η and $\overline{\eta}$ with respect to a Gaussian distribution:

$$\exp\left(-\frac{1}{2} \int dx\, d\tau\, \overline{\eta}(x, \tau)\, \eta(x, \tau)\right).$$

$R(x, y)$ is the fermionic regulator. It is a function of the covariant fermionic Laplacian:

$$R(x, y) = (\exp[-(\hat{D}/\Lambda)^2])(x, y)$$

where

$$\hat{D}(x, y) = \hat{D}_x\, \delta^4(x - y) = (D_x)_\mu\, \gamma^\mu\, \delta^4(x - y).$$

Consider the Euclidean action:

$$S = - \int \mathrm{d}^4x \, \mathrm{d}\tau \, \overline{\Psi}(x, \tau) \, [\mathrm{i}\gamma_\mu (\partial_\mu - \mathrm{i}g \, A_\mu) - m] \, \Psi(x, \tau) \tag{3}$$

from which follow the equations:

$$\frac{\mathrm{d}\Psi}{\partial\tau} = [\mathrm{i}\gamma_\mu (\vec{\partial}_\mu - \mathrm{i}g \, A_\mu) - m] \, \Psi + \int \mathrm{d}^4y \, R(x, y) \, \eta(y, \tau) \tag{4}$$

$$\frac{\partial\overline{\Psi}}{\partial\tau} = \overline{\Psi} \, [\mathrm{i}\gamma_\mu (-\overleftarrow{\partial}_\mu - \mathrm{i}g \, A_\mu) - m] + \int \mathrm{d}^4y \, \overline{\eta}(y, \tau) \, R(x, y). \tag{5}$$

Multiplying (4) from the left by $\overline{\Psi}\gamma_5$ and (5) from the right by $\gamma_5\Psi$, adding the two, and taking the noise average, we get:

$$\frac{\partial}{\partial\tau} \langle \overline{\Psi}\gamma_5\Psi \rangle = \mathrm{i}\partial_\mu \langle \overline{\Psi}\gamma_5\gamma_\mu\Psi \rangle - 2m \langle \overline{\Psi}\gamma_5\Psi \rangle +$$

$$+ \langle \int \mathrm{d}^4y \, \overline{\Psi}(x, \tau) \, \gamma_5 \, R(x, y) \, \eta(y, \tau) \rangle + \langle \int \mathrm{d}^4y \, \overline{\eta}(y, \tau) \, R(x, y) \, \gamma_5 \, \Psi(x, \tau) \rangle. \tag{6}$$

Similarly, multiplying (4) from the left by $\overline{\Psi}$ and (5) from the right by Ψ, subtracting the two and taking the noise-average we get:

$$\langle \Psi\partial_\tau\overline{\Psi} \rangle - \langle (\partial_\tau\overline{\Psi}) \Psi \rangle = \mathrm{i}\partial_\mu \langle \overline{\Psi}\gamma_\mu\Psi \rangle + \langle \int \mathrm{d}^4y \, \overline{\Psi}(x, \tau) \, R(x, y) \, \eta(y, \tau) \rangle -$$

$$- \langle \int \mathrm{d}^4y \, \overline{\eta}(y, \tau) \, R(x, y) \, \gamma_5 \, \Psi(x, \tau) \rangle. \tag{7}$$

Taking the limit $\tau \to \infty$ will give us the corresponding quantum averages; in particular, the first term on the right-hand sides of (6) and (7) will become the divergence of the axial-vector current and the vector current, respectively.

To evaluate the left-hand side of (6), we consider any arbitrary function F of Ψ and $\overline{\Psi}$. Then,

$$\lim_{\tau\to\infty} \frac{\partial}{\partial\tau} \langle F[\Psi, \overline{\Psi}, \tau] \rangle = \lim_{\tau\to\infty} \int \mathrm{d}\overline{\Psi} \, \mathrm{d}\Psi \, F[\Psi, \overline{\Psi}] \, \partial P =$$

$$= - \lim_{\tau\to\infty} \int \mathrm{d}\Psi \, \mathrm{d}\overline{\Psi} \, H_{FP} \, P[\Psi, \overline{\Psi}, \tau] = 0$$

where H_{FP} is the Fokker-Planck *hamiltonian* for the system and the last step follows because in the steady state limit, the probability distribution $P[\Psi, \overline{\Psi}, \tau]$

relaxes to the steady-state distribution $e^{-S[\Psi,\overline{\Psi},\tau]}$ with zero eigenvalue. So, using this stationarity property of the steady-state noise-averages, we have:

$$\lim_{\tau\to\infty} \frac{\partial}{\partial\tau} \langle \overline{\Psi}\gamma_5 \Psi\rangle = 0.$$

In this limit, the left hand side of (7) also vanishes because of the symmetry property of the correlation function under the interchange of the arguments τ and τ'.

Now, to evaluate the last two terms on the right hand sides of (6) and (7), we make use of Novikov's theorem for Grassmann variables[8]. Then,

$$\langle \int d^4y \overline{\Psi}(x,\tau)\gamma_5 R(x,y)\,\eta(y,\tau)\rangle = -\langle \int d^4y\, \eta(y,\tau)\overline{\Psi}(x,\tau)\gamma_5 R(x,y)\rangle =$$

$$= -\langle x| \int d^4y \,\mathrm{Tr}\,\gamma_5 \exp\left[-2\frac{\hat{D}(x)}{\Lambda}\frac{\hat{D}(y)}{\Lambda}\delta^4(x-y)\right]|x\rangle =$$

$$= \langle \int d^4y\, \overline{\eta}(y,\tau) R(x,y)\gamma_5 \Psi(x,\tau)\rangle = -\frac{1}{16\pi^2}\mathrm{Tr}\,({}^*F^{\mu\nu}F_{\mu\nu}) \qquad (8)$$

where the last step follows after going through the same steps as done by Fujikawa[2] and taking the limit $\Lambda\to\infty$. Similarly,

$$\langle \int d^4y \overline{\Psi}(x,\tau) R(x,y)\,\eta(y,\tau)\rangle = -\langle \int d^4y\, \eta(y,\tau)\overline{\Psi}(x,\tau) R(x,y)\rangle =$$

$$= -\langle x| \int d^4y \,\mathrm{Tr} \exp\left[-2\frac{\hat{D}(x)}{\Lambda}\frac{\hat{D}(y)}{\Lambda}\delta^4(x-y)\right]|x\rangle =$$

$$= \langle \int d^4y\, \overline{\eta}(y,\tau) R(x,y)\Psi(x,\tau)\rangle \qquad (9)$$

so that we are left with:

$$i\partial_\mu(\overline{\Psi}\gamma_5\gamma_\mu\Psi) = 2m\overline{\Psi}\gamma_5\Psi + \frac{1}{8\pi^2}\mathrm{Tr}\,({}^*F^{\mu\nu}F_{\mu\nu})$$

$$i\partial_\mu(\Psi\gamma_\mu\overline{\Psi}) = 0 \qquad (10)$$

which are the anomalous and the normal Ward identities in field theory. Thus, we are able to obtain both these identities in a very direct and natural manner in the stochastic quantization formalism, without even having to explicitly solve the Langevin equations for Ψ and $\overline{\Psi}$.

However, the explicit time dependence of $\langle\overline{\Psi}\gamma_5\Psi\rangle$ can be seen by solving the Langevin equations. We then get:

$$\langle \Psi\gamma_5\Psi\rangle = \langle \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \int d^4y\, \bar{\eta}(y,\tau_1) R(x,y) \exp[-(\hat{D}+m)(\tau-\tau_1)] \times$$

$$\times\, \gamma_5 \int d^4\beta \exp[(\hat{D}-m)(\tau-\tau_2)] R(x,\beta)\, \eta(\beta,\tau_2)\rangle =$$

$$= -2 \int_0^\tau d\tau_1 \int d^4y\, R(x,y) \exp[-(\hat{D}+m)(\tau-\tau_1)] \times$$

$$\times\, \gamma_5 \exp[(\hat{D}-m)(\tau-\tau_1)]\, R(x,y)$$

$$\frac{\partial}{\partial\tau}\langle\bar{\Psi}\gamma_5\Psi\rangle = \frac{\partial}{\partial\tau}\int d^4y\, R(x,y)\, \gamma_5 \frac{(1-\exp[2(\hat{D}-m)\tau])}{(\hat{D}-m)} R(x,y) =$$

$$= -2\int d^4y\, \gamma_5 \exp\left[-2\left(\frac{\hat{D}_x}{\Lambda}\right)^2 \delta^4(x-y)\right] \exp[2(\hat{D}-m)\tau] =$$

$$= \lim_{y\to x} \mathrm{Tr}\left(\gamma_5 \exp\left[-2\left(\frac{\hat{D}}{\Lambda}\right)^2\right] \exp[2(\hat{D}-m)\tau]\, \delta^4(x-y)\right)$$

which vanishes as $\exp(-2m\tau)$ because of the oscillating nature of $\exp(2\hat{D}\tau)$.

Our approach is similar in spirit to that adopted by Mamiki et al.[4], but bypasses the need to use Ito calculus.

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AKSIJALNA ANOMALIJA IZ JEDNADŽBI GIBANJA STOHASTIČKE KVANTIZACIJE

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Pokazano je da sačuvanje vektorske struje i aksijalno-vektorske anomalije direktno slijede iz Langevinove jednadžbe gibanja za spinorne teorije polja u formalizmu stohastičke kvantizacije. Prikazana metoda koristi Novikov teorem za Grassmannove varijable i regulacionu proceduru Berna, Chana i Halperna.