

EXACT SOLUTION OF THE SCHRÖDINGER EQUATION FOR THE
CENTRAL NONPOLYNOMIAL POTENTIAL $V(r) = r^2 + \lambda r^2/(1 + gr^2)$ IN
TWO AND THREE DIMENSIONS*

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We derive here a class of exact solution of the Schrödinger equation for the central nonpolynomial potential $V(r) = r^2 + \lambda r^2/(1 + gr^2)$ in two and three dimensions by using two suitable ansätze. This potential is relevant in the context of laser physics and quantum field theory. The eigenfunctions and eigenvalues obtained here may, therefore, be useful for application in lasers and field theory.

1. Introduction

During the recent years, there have been several investigations on the solution of Schrödinger equation with the potential

$$V(x) = x^2 + \lambda x^2/(1 + gx^2) \quad (1)$$

in one dimension¹⁻¹⁶⁾. This type of potential is of interest in several areas of physics. In particular, this type of potential occurs when considering models in laser physics¹⁷⁾ and quantum field theory with nonlinear lagrangian¹⁸⁾.

The exact solutions of the Schrödinger equation with potential (1) in one dimension have been obtained for particular values of the parameters λ and g . The

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supersymmetric quantum mechanics has also been used to construct the exact solution of the Schrödinger equation for this case when the parameters λ and g are related through a relation¹⁵⁾. This problem has, however, not been studied much in higher (two and three) dimensions, which are more interesting for actual application in physical problems. Certain aspects of this problem has been studied in three dimension, with central symmetry, with N expansion in conjunction with supersymmetric quantum mechanics¹⁹⁾. We construct here a class of exact solutions of the Schrödinger equation with central nonpolynomial potential

$$V(r) = r^2 + \lambda r^2 / (1 + gr^2) \quad (2)$$

in two and three dimensions by using two suitable ansätze.

2. Two dimension case

The Schrödinger equation in polar coordinates with the central potential (2) can be written as

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \Theta^2} + E - r^2 - \lambda r^2 / (1 + gr^2) \right] \Psi(r, \Theta) = 0 \quad (3)$$

Using the separation of variables method

$$\Psi(r, \vartheta) = R(r) \cdot \Theta(\vartheta), \quad (4)$$

the $\Theta(\vartheta)$ admits the solution

$$\Theta(\vartheta) = \frac{1}{\sqrt{2\pi}} \cos m \vartheta, \quad (5)$$

where m is an integer, whereas $R(r)$ satisfies the differential equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{m^2 R}{r^2} + \{E - r^2 - \lambda r^2 / (1 + gr^2)\} R = 0. \quad (6)$$

Let us now seek a solution to the second order differential equation (6) by using the following ansätze, which generate two infinite sets of solutions.

(i) First set of solutions:

We use the ansatz

$$R(r) = e^{-r^2/2} \sum_{n=0}^{\infty} a_n r^{2n+\nu} \quad (7)$$

to solve the differential equation (6). The coefficients should then satisfy the three term recurrence relation

$$A_{n-1}^r a_{n-1}^e + B_n^r a_n + C_{n+1}^r a_{n+1} = 0, \quad (8)$$

where

$$A_n^r = g \{E - (4n + 2\nu + 2)\} - \lambda, \quad (9a)$$

$$B_n^r = g \{(2n + \nu)^2 - m^2\} + E - (4n + 2\nu + 2), \quad (9b)$$

$$C_n^r = (2n + \nu)^2 - m^2. \quad (9c)$$

For a_p to be the last nonvanishing coefficient in the series (7), it is necessary that the next two successive coefficients a_{p+1} and a_{p+2} vanish. Since $a_p \neq 0$ and $a_{p+1}, a_{p+2} = 0$, it is necessary that

$$A_p^r = 0 \quad \text{or} \quad E_p^r = \frac{\lambda}{g} + 4p + 2\nu + 2. \quad (10)$$

For the first coefficient to be nonzero, it follows from equation (8) that

$$C_0^r = 0 \quad \text{i. e.} \quad \nu = \pm m, \quad (11)$$

of which only $\nu = m$ is physically allowed.

For a nontrivial solution, the coefficients m_p^r , B_p^r and C_p^r in the recurrence relation (8) should satisfy the relation

$$\Delta_{p+1}^r = \begin{vmatrix} B_0^r & C_1^r & & & & \\ A_0^r & B_1^r & C_2^r & & & 0 \\ & A_1^r & B_2^r & C_3^r & & \\ & & & & & \\ 0 & & & & & A_{p-1} B_p \end{vmatrix} = 0. \quad (12)$$

We now construct exact solutions for a general value of $\nu = m = q$ (say). The relation (10) yields the energy eigenvalue

$$E_{p,q} = \frac{\lambda}{g} + 4p + 2q + 2. \quad (13)$$

The precise relations satisfied by E and λ for different values of p can be obtained as follows:

(a) For $p = 0$,

$$E_{0,q} = \frac{\lambda}{g} + 2q + 2. \quad (14)$$

In addition, we must have

$$\Delta_1^q = 0 \quad \text{or} \quad E_{0,q} = 2(q + 1). \quad (15)$$

Combining the relations (14) and (15), it implies a condition for

$$\lambda = 0. \quad (16)$$

The corresponding eigenfunction is

$$\Psi_{0,q} = \frac{1}{\sqrt{2\pi}} a_0 r^q e^{-r^2/2} \cos q\vartheta. \quad (17)$$

(b) For $p = 1$,

$$E_{1,q} = \frac{\lambda}{g} + 2q + 6 \quad (18)$$

and

$$\Delta_2^q = 0. \quad (19)$$

The last relations, in conjunction with the relation (18), gives a condition for

$$\lambda = 0, \quad -g [(q + 1)g + 1]. \quad (20)$$

The eigenfunction in this case is given by

$$\Psi_{1,q} = \frac{1}{\sqrt{2\pi}} (a_0 + a_1 r^2) r^q e^{-r^2/2} \cos q\vartheta,$$

where

$$a_1 = \frac{-4g^2 a_0}{4g^2 (q + 1) + \lambda}. \quad (21)$$

This procedure can be continued to obtain an infinite set of eigenfunctions and their corresponding eigenvalues provided λ and g satisfy appropriate algebraic conditions.

(ii) Second set of solutions:

Here we use the ansatz

$$R(r) = e^{-r^2/2} \sum_{n=0}^{\infty} a'_n (1 + gr^2) r^{2n+r} \quad (22)$$

to solve the second order differential equation (6). The coefficients a'_n should then satisfy the three term recurrence relation

$$A'_{n+1} a'_{n+1} + B'_n a'_n + C'_{n-1} a'_{n-1} = 0, \quad (23)$$

where

$$A'_n = (2n + \nu)^2 - m^2 \quad (24a)$$

$$B'_n = \{(2n + \nu + 2)^2 - m^2\}g + E - 2(2n + \nu + 1) \quad (24b)$$

$$C'_n = Eg - \{2g(2n + \nu + 3) + \lambda\}. \quad (24c)$$

For a'_p to be the last nonvanishing coefficient in the series (22), it is necessary that the next two successive coefficients a'_{p+1} and a'_{p+2} vanish. Since $a'_p \neq 0$ and $a'_{p+1}, a'_{p+2} = 0$, it is necessary that

$$C'_p = 0 \quad \text{or} \quad E'_p = \frac{\lambda}{g} + 2(2p + \nu + 3). \quad (25)$$

For the first coefficient a'_0 to be nonzero, it follows from the relation (24a) that

$$A'_n = 0 \quad \text{or} \quad \nu = \pm m \quad (26)$$

of which only $\nu = m$ is physically allowed.

For a nontrivial solution, the coefficients A'_n, B'_n and C'_n in the recurrence relation (23) should satisfy the condition

$$\Delta'_{p+1} = \begin{vmatrix} B'_0 & A'_1 & & & 0 \\ C'_0 & B'_1 & A'_2 & & \\ & C'_1 & B'_2 & A'_3 & \\ 0 & & & & C'_{p-1}B'_p \end{vmatrix} = 0. \quad (27)$$

The exact solutions of the Schrödinger equation (3) for a general value of ν , $\nu = m = q$ (say) can be constructed as follows. Firstly, the relation (25) yields the energy eigenvalue

$$E_{p,q} = \frac{\lambda}{g} + 2(2p + q + 3). \quad (28)$$

The precise relations that E and λ must satisfy for different values of p in order that exact solution exist can be derived as follows.

(c) For $p = 0$,

$$E_{0,q} = \frac{\lambda}{g} + 2(q + 3). \quad (29)$$

In addition, we must have

$$\Delta'_1 = 0 \quad (30)$$

Combining the relations (29) and (30), it implies a condition for

$$\lambda = -4g [g(q+1) + 1]. \quad (31)$$

The eigenfunction, in this case, can be written as

$$\Psi_{0,q} = \frac{a'_0}{\sqrt{2\pi}} e^{-r^2/2} (1 + gr^2) r^q \cos q\vartheta. \quad (32)$$

(d) For $p = 1$,

$$E_{1,q} = \frac{\lambda}{g} + 2(q+5) \quad (33)$$

and

$$A_2^q = 0. \quad (34)$$

The last relation, in conjunction with the relation (33), gives a condition for

$$\lambda = 2g [-\{g(3q+5) + 3\} \pm \{g^2(q+3)^2 + 2g(q-1) + 1\}^{1/2}]. \quad (35)$$

The eigenfunction, in this case, can be written as

$$\Psi_{1,q} = \frac{1}{\sqrt{2\pi}} (a'_0 + a'_1 r^2) (1 + gr^2) r^q e^{-r^2/2} \cos q\vartheta,$$

where

$$a'_1 = - \left[\frac{\lambda}{4g(q+1)} + g + \frac{2}{q+1} \right] a'_0. \quad (36)$$

As before, the procedure can be continued to obtain infinite set of eigenfunctions and their corresponding eigenvalues provided E and λ satisfy appropriate algebraic conditions.

3. Three dimensional case

The Schrödinger equation with the central nonpolynomial potential (2) can be written as

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} + \right. \\ \left. + E - r^2 - \lambda r^2 / (1 + gr^2) \right] \Psi(r, \vartheta, \varphi) = 0. \quad (37)$$

Using the separation of variables method

$$\Psi(r, \vartheta, \varphi) = R(r) \Theta(\vartheta, \varphi) \tag{38}$$

the $\Theta(\vartheta, \varphi)$ admits the solution

$$\Theta(\vartheta, \varphi) = Y_l^m(\vartheta, \varphi) = \left[\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!} \right]^{1/2} P_l^m(\cos \vartheta) e^{im\varphi},$$

$$\left(\begin{matrix} l = 0, 1, 2, \dots \\ m = l, l-1, \dots, -l \end{matrix} \right) \tag{39}$$

whercas $R(r)$ satisfies the second order differential equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \frac{l(l+1)}{r^2} R + \{E - r^2 - \lambda r^2 / (1 + gr^2)\} R = 0. \tag{40}$$

Let us now seek a solution to the second order differential equation (40) by using the following ansätze which generate two infinite sets of solutions

(i) First set of solutions

We use the ansatz

$$R(r) = e^{-r^2/2} \sum_{n=0}^{\infty} b_n r^{2n+\nu} \tag{41}$$

to solve the second order differential equation (40). The coefficients b_n should then satisfy the three term recurrence relation as follows

$$D_{n-1}^r b_{n-1} + E_n^r b_n + F_{n+1}^r b_{n+1} = 0, \tag{42}$$

where

$$D_n^r = \{E - (4n + 2\nu + 3)\} g - \lambda, \tag{43a}$$

$$E_n^r = g \{(2n + \nu)(2n + \nu + 1) - l(l + 1)\} + \{E - (4n + 2\nu + 3)\}, \tag{43b}$$

$$F_n^r = (2n + \nu)(2n + \nu + 1) - l(l + 1). \tag{43c}$$

Now for the first coefficient b_0 to be nonzero i. e. $b_0 \neq 0$, we should have

$$F_0^r = 0 \quad \text{or} \quad (\nu - l)(\nu - l - 1) = 0. \tag{44}$$

Thus $\nu = l, -(l + 1)$ in which only $\nu = l$ is physically acceptable. Again, for b_p to be the last nonvanishing coefficient in the series (41), it is necessary that the

which together with the relation (52) gives a condition for λ as

$$\lambda = 0, \quad -2g [(2q + 3)g + 2]. \quad (54)$$

The eigenfunction for this case can be obtained as

$$\Psi_{1,q}^m = (b_0 + b_1 r^2) r^q e^{-r^2/2} Y_q^m(\vartheta, \varphi) \quad (m = q, q - 1, \dots, -q)$$

where

$$b_1 = -4g^2 b_0 \{g^2 (4q + 6) + \lambda\}^{-1}. \quad (55)$$

The procedure can be continued to obtain an infinite set of eigenfunctions and their corresponding eigenvalues provided E and λ satisfy appropriate algebraic conditions.

(ii) Second set of solutions:

Here we use the ansatz

$$R(r) = e^{-r^2/2} \sum_{n=0}^{\infty} b'_n (1 + gr^2) r^{2n+\nu} \quad (56)$$

to solve the second order differential equation (40). The coefficients b'_n should then satisfy the three term recurrence relation

$$D'_{n-1} b'_{n-1} + E'_n b'_n + F'_{n-1} b'_{n+1} = 0 \quad (57)$$

where

$$D'_n = g \{E - (4n + 2\nu + 7)\} - \lambda, \quad (58a)$$

$$E'_n = E + g \{(2n + \nu + 2)(2n + \nu + 3) - l(l + 1)\} - (4n + 2\nu + 3), \quad (58b)$$

$$F'_n = (2n + \nu)(2n + \nu + 1) - l(l + 1). \quad (58c)$$

Now for b'_p to be the last nonvanishing coefficient in the series (56), it is necessary that the next two successive coefficients b'_{p+1} and b'_{p+2} must vanish. Since $b'_p \neq 0$ and $b'_{p+1}, b'_{p+2} = 0$, it is necessary that

$$D'_p = 0 \quad \text{or} \quad E_{p,\nu} = \frac{\lambda}{g} + (4p + 2\nu + 7). \quad (59)$$

Again for the first coefficient b'_0 to be nonzero i. e. $b'_0 \neq 0$, we should have

$$F'_0 = 0 \quad \text{or} \quad (\nu - l)(\nu - l - 1) = 0. \quad (60)$$

The eigenfunction for this case can be written as

$$\Psi_{1,q}^m = (b'_0 + b'_1 r^2) (1 + gr^2) r^q e^{-r^2/2} Y_q^m(\vartheta, \varphi),$$

where

$$b'_1 = - \left[\frac{\lambda}{2g(2q+3)} + g + \frac{4}{2q+3} \right] b'_0. \quad (70)$$

Once again the procedure can be continued to obtain an infinite set of eigenfunctions and their corresponding eigenvalues provided E and λ satisfy appropriate algebraic conditions.

4. Conclusions

We have obtained here a class of exact solutions of the Schrödinger equation with central nonpolynomial potential (2) in two and three dimensions using two suitable ansätze. The class contains two sets, corresponding to two ansätze, each consisting infinite number of solutions. We notice that the parameters λ and g of the potential are interrelated through the angular momentum number m and the orbital angular momentum quantum number l in two and three dimensions, respectively. Further, in the case of first ansatz, for $p = 0$ and one of the values corresponding to $p = 1$ gives $\lambda = 0$, which correspond to the isotropic harmonic oscillator case. The eigenfunctions obtained here are normalizable and the normalization constant can be written in a closed form.

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TOČNO RJEŠENJE SCHRÖDINGEROVE JEDNADŽBE U DVIJE I TRI
DIMENZIJE ZA CENTRALNI NEPOLINOMIJALNI POTENCIJAL

$$V(r) = r^2 + \lambda r^2 / (1 + gr^2)$$

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Koristeći dva pogodna Ansatzea izvodimo klasu točnih rješenja Schrödingerove jednadžbe u dvije i tri dimenzije za centralni nepolinomijalni potencijal $V(r) = r^2 + \lambda r^2 / (1 + gr^2)$. Ovaj je potencijal relevantan u kontekstu fizike lasera i kvantne teorije polja. Dobivene vlastite funkcije i vlastite vrijednosti mogu prema tome biti korisne u primjeni na lasere i teoriju polja.