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WILSON LOOP WITH LIGHT-LIKE SIDES TO ORDER g^4 IN THE FEYNMAN GAUGE

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A Wilson loop with light-like sides to order g^4 contains quartic divergences, ε^{-4}, in dimensional regularization with $d = 4 - \varepsilon$. We evaluate the leading ε^{-4} and first non-leading ε^{-3} divergences. The part proportional to $C^2(R)$ obeys the exponentiation theorem. The leading non-Abelian $C(R)\,C(G)$ divergences cancel out.

1. Introduction

We study a Wilson loop with sides in the directions of two light-like vectors

$$\begin{aligned} n &= (1,\ 0,\ 0,\ -1), \\ n^* &= (1,\ 0,\ 0,\ 1), \end{aligned} \tag{1}$$

where the length of the side in the direction n is T, while the length of the side in the direction n^* is L. To order g^2, there are four diagrams with one-gluon exchange at each cusp. Their sum for the leading ε^{-2} and ε^{-1} divergences[1] is

$$W = 32 g^2 C(R)\,(TL)^{\varepsilon/2} \frac{1}{\varepsilon^2}\left(1 + \frac{i\pi\varepsilon}{4}\right)(2\pi)^{-4}. \tag{2}$$

The difference between this loop and the loop studied in the literature[2] is obvious. The latter had sides in the directions of space and time, contained only single divergences and was real. The former, which we study here, is doubly logarithmically divergent already at the one-gluon exchange level and contains an imaginary part.

In this paper we want to investigate what happens at order g^4 in perturbation theory. We form eight sets of diagrams with the same group and topological structure, and study each set separately. The derivations are given in the Appendix.

2. Results

We list our results for the sets of diagrams shown in Figs. 1—8.

(a) There are two diagrams of the type shown in Fig. 1. Their contribution is

$$W_a = 2^7 g^4 C^2(R)\, \pi^{4-\varepsilon} (TL)^{\varepsilon}\, \varepsilon^{-4} \left(1 + \frac{i\pi\varepsilon}{2}\right) (2\pi)^{-8}. \tag{3}$$

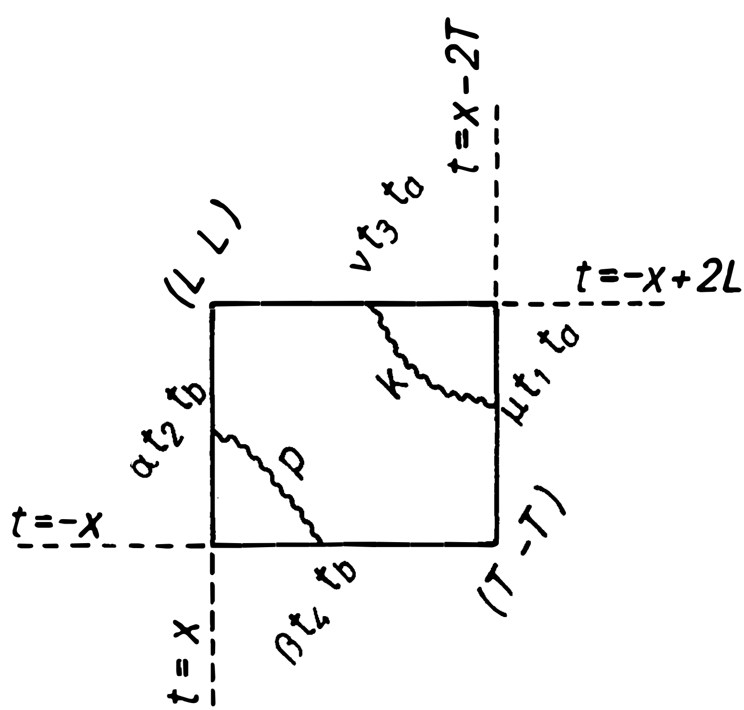
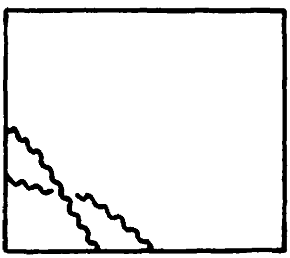
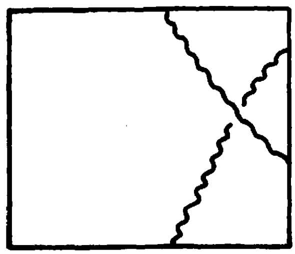


Fig. 1.

Fig. 2. Fig. 3.





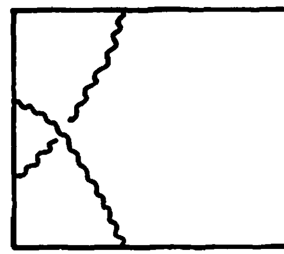
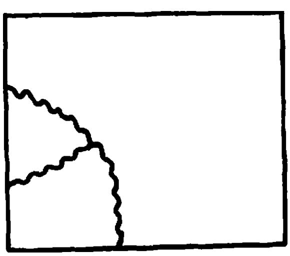


Fig. 4. Fig. 5.



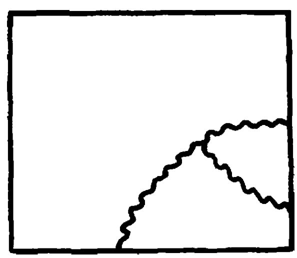
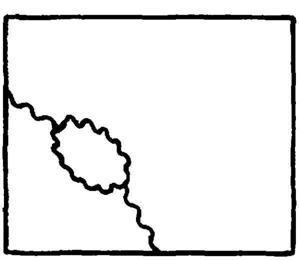


Fig. 6.



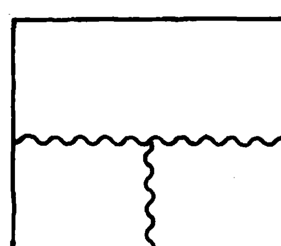


Fig. 7. Fig. 8.

Figs. 1—8. Graphs contributing to the vacuum expectation value of the Wilson-loop operator at order g^4. The sides of the loop are in the directions of the two null vectors. Wavy lines denote gluons and curved solid lines denote quarks. Dotted lines in Fig. 7 denote ghosts.

(b) Four diagrams of the type shown in Fig. 2 form a set which gives the contribution

$$W_b = 2^8 g^4 C^2 (R)\, \pi^{4-\varepsilon} (TL)^\varepsilon \varepsilon^{-4} \left(1 + \frac{i\pi\varepsilon}{2}\right) (2\pi)^{-8}. \tag{4}$$

(c) A set of four diagrams (an exmaple shown in Fig. 3) gives

$$W_c = 2^6 g^4 C^2 (R)\, \pi^{4-\varepsilon} (TL)^\varepsilon \varepsilon^{-4} \left(1 + \frac{i\pi\varepsilon}{2}\right) (2\pi)^{-8}. \tag{5}$$

The following diagrams contain the non-Abelian group factor $C(R)\, C(G)$.

(d) A set of four diagrams (the representative shown in Fig. 4) gives the contribution

$$W_d = 2^6 g^4 \left[C^2 (R) - \frac{1}{2} C(R)\, C(G)\right] \pi^{4-\varepsilon} (TL)^\varepsilon \varepsilon^{-4} \left(1 + \frac{i\pi\varepsilon}{2}\right) (2\pi)^{-8}. \tag{6}$$

(e) Diagrams of the type shown in Fig. 5 give no rise to the leading divergences:

$$W_e = 0 \cdot \varepsilon^{-4} + 0 \cdot \varepsilon^{-3} + \mathcal{O}(\varepsilon^{-2}) \tag{7}$$

(f) There are eight diagrams with the three-gluon vertex shown in Fig. 6. Their contribution is

$$W_f = 2^5 g^4\, C(R)\, C(G)\, \pi^{4-\varepsilon} (TL)^\varepsilon \varepsilon^{-4} \left(1 - \varepsilon + \frac{i\pi\varepsilon}{2}\right) (2\pi)^{-8}. \tag{8}$$

(g) The leading divergence in the set of four diagrams with the gluon self-energy insertion (Fig. 7) is cubic:

$$W_g = -\frac{160}{3\varepsilon^3} g^4 C(R)\, C(G)\, \pi^{4-\varepsilon} (TL)^\varepsilon (2\pi)^{-8}. \tag{9}$$

(h) A set of four diagrams with the three-gluon vertex (the representative shown in Fig. 8) contains neither quartic nor cubic divergences:

$$W_h = 0 \cdot \varepsilon^{-4} + 0 \cdot \varepsilon^{-3} + \mathcal{O}(\varepsilon^{-2}). \tag{10}$$

3. *Discussion*

The total contribution to order g^4 in perturbation theory is the sum of equations from Eq. (3) to Eq. (10):

$$W = 2^9 g^4 C^2 (R)\, \pi^{4-\varepsilon} (TL)^\varepsilon \varepsilon^{-4} \left(1 + \frac{i\pi\varepsilon}{2}\right) (2\pi)^{-8} - $$
$$- \frac{256}{3} g^4 C(R)\, C(G)\, \pi^{4-\varepsilon} (TL)^\varepsilon \varepsilon^{-3} (2\pi)^{-8}. \tag{11}$$

The exponentiation theorem[3)] is obeyed, as the Abelian part with $C^2(R)$ is really a half of a square of Eq. (2).

The leading ε^{-4} divergence with the group factor $C(R)\,C(G)$ cancels out.

Appendix (a)

The graph shown in Fig. 1 reads

$$G = g^4 i^6 (2\pi)^{-8}\,\mathrm{Tr}\,(t_a t_a t_b t_b) \int d^n p d^n k \frac{n^*_\mu n_\nu \delta_{\mu\nu}}{k^2 + i\eta} \cdot \frac{n^*_\alpha n_\beta \delta_{\alpha\beta}}{p^2 + i\eta} \times$$
$$\times \int_0^L dt_2 \int_{-T}^0 dt_4\, e^{i[p_0(t_2 - t_4) - p_3(x_2 - x_4)]} \times \tag{a.1}$$
$$\times \int_L^{L-T} dt_3 \int_{L-T}^{-T} dt_1\, e^{i[k_0(t_1 - t_3) - k_3(x_1 - x_3)]}.$$

We introduce the Mandelstam variables

$$\begin{aligned} n \cdot p &= p_+ = p_0 + p_3, \\ n^* \cdot p &= p_- = p_0 - p_3. \end{aligned} \tag{a.2}$$

Adding its counterpart and symmetrizing under $p_-, k_- \to -p_-, -k_-$, we obtain

$$W_a = -8 g^4 C^2(R) (2\pi)^{-8} \int d^n p d^n k \frac{1}{(k^2 + i\eta)(p^2 + i\eta)\, p_- p_+ k_- k_+} \times$$
$$\times \{(\cos p_- L - 1)(\cos k_- L - 1) + \sin p_- L \sin k_- L\} \times$$
$$\times \{(\cos p_+ T - 1)(\cos k_+ T - 1) + \sin p_+ T \sin k_+ T\}. \tag{a.3}$$

We first perform the integration in $(n-2)$ dimensions, leaving the integration over Mandelstam's vairables last:

$$\int d^n k \ldots = \frac{1}{2} \int dk_+ dk_- d^{2-\varepsilon} K \ldots. \tag{a.4}$$

Therefore,

$$W_a = -2 g^4 C^2(R)\, \pi^{2-\varepsilon}\, \Gamma^2\left(\frac{\varepsilon}{2}\right) (2\pi)^{-8} \times$$
$$\times \int dp_+ dp_- (-p_+ p_- - i\eta)^{-\varepsilon/2} \frac{1}{p_- p_+} (1 - \cos p_- L)(1 - \cos p_+ T) \times$$
$$\times \int dk_+ dk_- (-k_+ k_- - i\eta)^{-\varepsilon/2} \frac{1}{k_- k_+} \times (1 - \cos k_- L)(1 - \cos k_+ T). \tag{a.5}$$

The terms with $\sin p_- L \sin k_- L$ do not contribute to the leading divergences, as can be checked by direct calculation. The infinitesimal parameter $i\eta$ determines the branch of integration. For example,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}p_+ \mathrm{d}p_- \, (-p_+p_- - i\eta)^{-\varepsilon/2} \frac{1}{p_-p_+} (1 - \cos p_- L)(1 - \cos p_+ T) =$$

$$= 2\int_0^{\infty}\int_0^{\infty} \mathrm{d}p_+ \mathrm{d}p_- \, (1 - \cos p_- L)(1 - \cos p_+ T) \times$$

$$\times \frac{1}{p_-p_+} \{(-p_+p_- - i\eta)^{-\varepsilon/2} - (p_+p_- - i\eta)^{-\varepsilon/2}\} =$$

$$= 2(e^{i\pi\varepsilon/2} - 1)\int_0^{\infty}\int_0^{\infty} \mathrm{d}p_+ \mathrm{d}p_- \, (p_+p_-)^{-1-\varepsilon/2} (1 - \cos p_- L)(1 - \cos p_+ T)$$

$$= 2(e^{i\pi\varepsilon/2} - 1)\, \Gamma^2\left(-\frac{\varepsilon}{2}\right) \cos^2 \frac{\varepsilon\pi}{4} (TL)^{\varepsilon/2}. \qquad \text{(a.6)}$$

Appendix (b)

The pair of graphs shown in Fig. 2 gives

$$G = -8g^4 C^2(R)(2\pi)^{-8} \int \mathrm{d}^n p \, \mathrm{d}^n k \, \frac{1}{k^2 p^2} \frac{1}{k_+ p_+ p_-} \times$$

$$\times \{1 + \cos(p + k)_+ T - \cos p_+ T - \cos k_+ T\} \times$$

$$\times \left\{\frac{1}{k_-}(1 - \cos k_- L) - \frac{1}{(p + k)_-}(\cos p_- L - \cos k_- L)\right\}. \qquad \text{(b.1)}$$

The expression in the first bracket can be factorized:

$$1 + \cos(p + k)_+ T - \cos p_+ T - \cos k_+ T =$$

$$= (\cos p_+ T - 1)(\cos k_+ T - 1) - \sin p_+ T \sin k_+ T. \qquad \text{(b.2)}$$

As in the preceding case, we first integrate over $\mathrm{d}^{2-\varepsilon} K$, $\mathrm{d}^{2-\varepsilon} P$ and then transform all the regions of integration $(-\infty, +\infty)$ into $(0, +\infty)$, where the infinitesimal parameter $i\eta$ determines the branch of integration.

Then, the integration over $\mathrm{d}k_+$, $\mathrm{d}p_+$ is straightforward to perform using the tables of integrals, while the integration over $\mathrm{d}k_-$, $\mathrm{d}p_-$, which have not been factorized, gives

$$\int_0^{\infty} \mathrm{d}p_- p_-^{-1-\varepsilon/2} (1 - \cos p_- L) \int_0^{\infty} \mathrm{d}k_- \frac{k_-^{1-\varepsilon/2}}{(k + p)_- (k - p)_-}, \qquad \text{(b.3)}$$

leading to integral (6) in the tables. Let us note that there is actually no pole at $k_- = p_-$ (see (b.1)). The other pair of diagrams is calculated in an analogous way.

Appendix (c)

The sum of all four diagrams in the set (the representative shown in Fig. 3) amounts to

$$W_c = -16g^4C^2(R)(2\pi)^{-8}\int d^npd^nk \frac{1}{(k^2+i\eta)(p^2+i\eta)k_-k_+} \times$$
$$\times \left\{\frac{1}{p_-}(1-\cos p_-L) - \frac{1}{(p+k)_-}(1-\cos(p+k)_-L)\right\} \times$$
$$\times \left\{\frac{1}{p_+}(1-\cos p_+T) - \frac{1}{(p+k)_+}(1-\cos(p+k)_+T)\right\}. \qquad \text{(c.1)}$$

As before, we first perform integrations over $d^{2-\varepsilon}P$, $d^{2-\varepsilon}K$ and then transform the range of integration into $(0, +\infty)$. The new type of integral which we get is

$$N = \int_0^\infty dx \int_0^\infty dy\, x^{-1-\varepsilon/2} y^{-\varepsilon/2} \left\{\frac{1}{y}(1-\cos yL) - \frac{1}{x+y}(1-\cos(x+y)L)\right\}, \qquad \text{(c.2)}$$

where $x = k_\pm$, $y = p_\pm$.

We solve it by symmetrizing $x \leftrightarrow y$:

$$N = \frac{1}{2}\int_0^\infty dx \int_0^\infty dy\,(xy)^{-1-\varepsilon/2}\{(\cos yL - 1)(\cos xL - 1) - \sin yL \sin xL\}. \qquad \text{(c.3)}$$

The integral

$$N' = \int_0^\infty dx \int_0^\infty dy\, x^{-1-\varepsilon/2} y^{-\varepsilon/2} \left\{\frac{1}{y}(1-\cos yL) - \frac{1}{(y-x)} \times (1-\cos(y-x)L)\right\} \qquad \text{(c.4)}$$

has been obtained in an analogous way as that for N.

Appendix (d)

The sum of all four diagrams (the representative shown in Fig. 4) gives

$$W_d = -16g^4\,\mathrm{Tr}(t_at_bt_at_b)(2\pi)^{-8}\int d^npd^nk \frac{1}{k^2p^2k_-p_+} \times$$
$$\times \left\{\frac{1}{p_-}(1-\cos p_-L) - \frac{1}{(p+k)_-}(1-\cos(p+k)_-L)\right\} \times$$
$$\times \left\{\frac{1}{k_+}(1-\cos k_+T) - \frac{1}{(p+k)_+}(1-\cos(p+k)_+T)\right\}. \qquad \text{(d.1)}$$

This is evaluated in the same way as W_c.

Appendix (e)

The sum of two diagrams in Fig. 5 is

$$W_e = -8g^4 \operatorname{Tr}(t_a t_b t_a t_b)(2\pi)^{-8} \int d^n p d^n k \frac{1}{k^2 p^2 k_- k_+ p_+} \times$$
$$\times \{1 + \cos(p+k)_+ T - \cos p_+ T - \cos k_+ T\} \times$$
$$\times \left\{\frac{1}{(k+p)_-}(\cos k_- L - \cos p_- L) - \frac{1}{p_-}(\cos k_- L - \cos(p-k)_- L)\right\}. \quad \text{(e.1)}$$

Combined knowledge of Appendix (b) and Appendix (c) helps in solving (e.1).

Appendix (f)

The pair of graphs in Fig. 6 gives the contribution

$$G = -4ig^4 f_{abc} \operatorname{Tr}(t_a t_b t_c) \int d^n p d^n k \frac{(p-k)_-}{k^2 p^2 (p+k)^2} \cdot \frac{\cos(p+k)_+ T - 1}{(p+k)_+} \times$$
$$\times \frac{1}{k_-}\left\{\frac{1}{(p+k)_-}(\cos(p+k)_- L - 1) - \frac{1}{p_-}(\cos p_- L - 1)\right\}. \quad \text{(f.1)}$$

We shift the integration variables:

$$p + k = p',$$
$$k = k', \quad \text{(f.2)}$$

and integrate over $dk_+ d^{2-\varepsilon} K$ using integral (1) from the tables. This leads to

$$G = 4g^4 \pi^{2-\varepsilon/2} \Gamma\left(\frac{\varepsilon}{2}\right) f_{abc} \operatorname{Tr}(t_a t_b t_c)(2\pi)^{-8} \times$$
$$\times \int_0^1 dx \int d^n p\,(2x-1)\, x^{-1-\varepsilon/2}(1-x)^{-\varepsilon/2} \frac{1}{p_- p_+}(\cos p_+ T - 1)(-p^2 - i\eta)^{-1-\varepsilon/2} \times$$
$$\times \{(\cos p_- L - 1) - \frac{1}{1-x}(\cos p_- L(1-x) - 1)\}. \quad \text{(f.3)}$$

Now the integration over $d^{2-\varepsilon} P$ is straightforward following integral (2). The integration over $dp_+ dp_-$ is performed as in (a 6). The final integral over the parameter x reads

$$I = \int_0^1 dx\, x^{-1-\varepsilon/2} \{(1-x)^{\varepsilon/2} - (1-x)^{1-\varepsilon/2}\} + \int_0^1 dx\, x^{-\varepsilon/2}(1-x)^{-\varepsilon/2} -$$
$$- \int_0^1 dx\, x^{\varepsilon/2-1}(1-x)^{-\varepsilon/2}. \quad \text{(f.4)}$$

For the leading and first non-leading divergences, this is

$$I = -\frac{2}{\varepsilon} + 2. \qquad \text{(f.5)}$$

Appendix (g)

The diagram with the gluon self-energy insertion is trivial to work out using the knowledge of (a.6).

Appendix (h)

The diagram in Fig. 8 is

$$G = -2ig^4 f_{abc}\,\mathrm{Tr}\,(t_a t_b t_c)\,(2\pi)^{-8} \int d^n p d^n k \frac{(2p-k)_-}{k^2 (p-k)^2 p^2} \cdot \frac{e^{ip_+ T}}{k_+ p_- (p-k)_-} \times$$
$$\times\,(e^{i(p-k)_- L} - 1)(1 - e^{-ik_+ T})(1 - e^{-ip_- L}). \qquad \text{(h.1)}$$

There is actually no pole at $k_+ = 0$ since it is protected by the factor $(1 - e^{-ik_+ T})$.

After performing the usual integrations, we get

$$W_h = 16ig^4 C(R)\, C(G)\, \pi^{3-\varepsilon} (e^{i\pi\varepsilon} - 1)(TL)^\varepsilon \cos^2\frac{\varepsilon\pi}{2}\, \Gamma^2(-\varepsilon)(2\pi)^{-8} \times$$
$$\times \frac{1}{\varepsilon}\int_0^1 dx\,(1+x)\,[1 + x^\varepsilon - (1-x)^\varepsilon]\, x^{-1-\varepsilon/2} \times$$
$$\times \left\{ \frac{1}{1+\frac{\varepsilon}{2}} (1-x)^{-\varepsilon/2}\, {}_2F_1\left(1,1;2+\frac{\varepsilon}{2};1-x\right) - B\left(1+\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\right) \times \right.$$
$$\left. \times\, {}_2F_1\left(1,1+\frac{\varepsilon}{2};2+\varepsilon, 1-x\right) \right\}. \qquad \text{(h.2)}$$

The main divergences arise from $x \sim 0$. Transforming the hypergeometric functions of the argument $1 - x$ into functions of the argument x, we can convince ourselves that this diagram behaves as ε^{-2} and does not contribute to the leading divergences.

Tables of integrals

(1) $\int d^n k \frac{1}{(k^2 + i\eta)\,[(p-k)^2 + i\eta]} = i\pi^{2-\varepsilon/2}\,\Gamma\left(\frac{\varepsilon}{2}\right)(-p^2 - i\eta)^{-\varepsilon/2} \times \int_0^1 dx\,[x(1-x)]^{-\varepsilon/2},$

where $x = k_-/p_-$.

(2) $\int d^{2-\varepsilon} K \frac{1}{K^2 - k_+ k_- - i\eta} = \pi^{1-\varepsilon/2}\,\Gamma\left(\frac{\varepsilon}{2}\right)(-k_+ k_- - i\eta)^{-\varepsilon/2},$

(3) $$\int d^{2-\varepsilon}K \frac{1}{(-k^2 - i\eta)^{1+\varepsilon/2}} = \int d^{2-\varepsilon} K \frac{1}{(K^2 - k_+k_- - i\eta)^{1+\varepsilon/2}}$$

$$= \pi^{1-\varepsilon/2} \frac{\Gamma(\varepsilon)}{\Gamma\left(1 + \frac{\varepsilon}{2}\right)} (-k_+k_- - i\eta)^{-\varepsilon}$$

4) $$\int_0^\infty x^{\mu-1} \sin^2 ax \, dx = - \frac{\Gamma(\mu)\cos \mu\pi/2}{2^{\mu+1} a^\mu}, \qquad [a > 0, \ -2 < \mathrm{Re}\,\mu < 0],$$

(5) $$\int_0^\infty x^{\mu-1} \sin ax \, dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2}, \qquad [a > 0, \ 0 < |\mathrm{Re}\,\mu| < 1],$$

(6) $$\int_0^\infty \frac{dp \, p^{1-\varepsilon/2}}{(k-p)(k+p)} = \frac{\pi}{2} k^{-\varepsilon/2} \left[\mathrm{cosec}\left(2 - \frac{\varepsilon}{2}\right)\pi + \mathrm{ctg}\left(2 - \frac{\varepsilon}{2}\right)\pi\right],$$

$$[k > 0, \ 0 < \mathrm{Re}\,\varepsilon/2 < 2],$$

(the pole at $k = p$ is taken in the sense of principal value).

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WILSONOVA PETLJA SA SVJETLOLIKIM STRANAMA DO REDA g^4 U FEYNMANOVOM BAŽDARNOM UVJETU

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Wilsonova petlja sa svjetlolikim stranama do reda g^4 sadrži kvartičke divergencije, ε^{-4}, u dimenzionalnoj regularizaciji sa $d = 4 - \varepsilon$. Izračunate su vodeće ε^{-4} i prve nevodeće ε^{-3} divergencije. Dio proporcionalan $C^2(R)$ slijedi teorem eksponencijacije. Vodeće neabelovske $C(R)\,C(G)$ divergencije se poništavaju.