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CORRECTIONS TO LIÈNARD-WIECHERT POTENTIALS

MARIJAN RIBARIČ and LUKA ŠUŠTERŠIČ

Jožef Stefan Institute, University of Ljubljana, Jamova 39, 61111 Ljubljana, Slovenija

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We calculate two terms of a particular multipole expansion of the retarded Lorentz-gauge scalar and vector potentials with respect to a time-dependent trajectory. For a trajectory coinciding with the average position of the electric charges, the first term implies the Liènard-Wiechert potentials, and the next, higher-order term vanishes.

1. Introduction

The scalar and vector potentials, and the electromagnetic fields of localized electric charges and currents that are steady or vary sinusoidally in time can be expanded in terms of multipoles; see, e. g., Jackson[1]. For very localized, pointlike densities we may decide to approximate their scalar and vector potentials by the Liènard-Wiechert potentials in terms of the total charge q and its trajectory $\mathbf{r}_0(t)$, see, e. g., Panofsky and Phillips[2]. If so, the question arises of how adequate is such an approximation by the Liènard-Wiechert potentials, and how best to choose the trajectory $\mathbf{r}_0(t)$. To answer this question we are going to construct such a multipole expansion that its first term gives the Liènard-Wiechert potentials, and calculate the next higher-order term. We will use rationalized SI units.

2. Basic assumptions

We assume that the densities of electric charges and currents $\varrho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ satisfy the continuity equation

$$\nabla \cdot \mathbf{j} + \partial \varrho / \partial t = 0, \tag{1}$$

and are localized within a moving sphere S_t, centred at $\mathbf{r} = \mathbf{r}_0(t)$, with a constant radius R_0, i.e.,

$$\varrho(\mathbf{r}, t) = 0 \text{ and } \mathbf{j}(\mathbf{r}, t) = 0 \text{ if } |\mathbf{r} - \mathbf{r}_0(t)| \geq R_0. \tag{2}$$

Multiplying (1) by a scalar-valued function $g(\mathbf{r})$ of $r \in \mathbb{R}^3$, integrating by parts, and taking account of (2), we can conclude that for localized densities, the continuity equation (1) is equivalent to relation

$$\int_{S_t} \mathbf{j}(\mathbf{r}, t) \cdot \nabla g(\mathbf{r})\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_t} g(\mathbf{r})\, \varrho(\mathbf{r}, t)\, \mathrm{d}V \tag{3}$$

valid for any t and for any differentiable scalar function $g(\mathbf{r})$ of $\mathbf{r}$. Due to (1), (2), and (3) with $g \equiv 1$, the total charge

$$q \equiv \int_{S_t} \varrho(\mathbf{r}, t)\, \mathrm{d}V \tag{4}$$

is constant, i. e.. $\mathrm{d}q/\mathrm{d}t = 0$.

Let us assume that with reference to any time instant t' we can express the densities $\varrho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$ in terms of material coordinates as follows:

$$\varrho(\mathbf{r}, t) = \int_{S_t} \varrho(\mathbf{r}', t')\, \delta(\mathbf{r} - \mathbf{r}(\mathbf{r}', t))\, \mathrm{d}V'$$

and (5)

$$\mathbf{j}(\mathbf{r}, t) = \int_{S_t} \varrho(\mathbf{r}', t') \frac{\mathrm{d}\mathbf{r}(\mathbf{r}', t)}{\mathrm{d}t}\, \delta(\mathbf{r} - \mathbf{r}(\mathbf{r}', t))\, \mathrm{d}V' = \varrho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t)$$

with

$$\mathbf{v}(\mathbf{r}, t) \equiv \left. \frac{\mathrm{d}\mathbf{r}(r', t)}{\mathrm{d}t} \right|_{r' = a(r,t)} = \mathbf{j}(\mathbf{r}, t)/\varrho(\mathbf{r}, t) \tag{6}$$

in which $\mathbf{r}(\mathbf{r}', t)$ is the trajectory of the »charge« $\varrho(\mathbf{r}', t')$ whose position was $\mathbf{r}'$ at the reference time instant t', and $a(\mathbf{r}, t)$ is the so-called material coordinate that gives the position at the reference time instant t' of the »charge« $\varrho(\mathbf{r}, t)$ located at $\mathbf{r}$ at time instant t, so that $\mathbf{r}(\mathbf{r}', t') = \mathbf{r}'$ and $a(\mathbf{r}(\mathbf{r}', t), t) = \mathbf{r}'$, see, e. g., Soper[3] and Ribarič and Šušteršič[4]. Relations (5) are continuous generalizations of the case with several non-colliding point charges, a fact directly evident from (6), with $\mathbf{v}(\mathbf{r}_1, t_1) = \mathbf{j}(\mathbf{r}_1, t_1)/\varrho(\mathbf{r}_1, t_1)$ being the velocity at the point $\mathbf{r}_1$ and time t_1 of the trajectory $\mathbf{r}(\mathbf{a}(\mathbf{r}_1, t_1), t)$ of »charge« $\varrho(\mathbf{a}(\mathbf{r}_1, t_1), t')$. Note that

$$\frac{\mathrm{d}\mathbf{r}(\mathbf{r}', t)}{\mathrm{d}t} = \mathbf{v}(\mathbf{r}(\mathbf{r}', t), t). \tag{7}$$

Using (5) and (7), we can express the retarded Lorentz-gauge scalar and vector potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ as integrals of the Liènard-Wiechert scalar and vector potentials of »charges« $\varrho(\mathbf{r}', t')$ moving along trajectories $\mathbf{r}(\mathbf{r}', t)$ with velocities $\mathbf{v}(\mathbf{r}(\mathbf{r}', t), t)$:

$$\Phi(\mathbf{r}, t) \equiv \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \frac{\varrho(\mathbf{r}'', t - |\mathbf{r} - \mathbf{r}''|/c)}{|\mathbf{r} - \mathbf{r}''|} \mathrm{d}V'' =$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \frac{\varrho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}_{ret}| - (\mathbf{r} - \mathbf{r}_{ret}) \cdot \mathbf{v}_{ret}/c} \mathrm{d}V' \tag{8}$$

and

$$\mathbf{A}(\mathbf{r}, t) \equiv \frac{1}{4\pi\varepsilon_0 c^2} \int_{\mathbb{R}^3} \frac{\mathbf{j}(\mathbf{r}'', t - |\mathbf{r} - \mathbf{r}''|/c)}{|\mathbf{r} - \mathbf{r}''|} \mathrm{d}V'' =$$

$$= \frac{1}{4\pi\varepsilon_0 c^2} \int_{\mathbb{R}^3} \frac{\varrho(\mathbf{r}', t')\, \mathbf{v}_{ret}}{|\mathbf{r} - \mathbf{r}_{ret}| - (\mathbf{r} - \mathbf{r}_{ret}) \cdot \mathbf{v}_{ret}/c} \mathrm{d}V' \tag{9}$$

in which

$$\mathbf{r}_{ret} \equiv \mathbf{r}(\mathbf{r}', t_{ret}) \text{ and } \mathbf{v}_{ret} \equiv \mathbf{v}(\mathbf{r}_{ret}, t_{ret}) \tag{10}$$

and the retarded time $t_{ret} = t_{ret}(\mathbf{r}', \mathbf{r}, t)$ is defined implicitly by

$$t_{ret} = t - |\mathbf{r} - \mathbf{r}(\mathbf{r}', t_{ret})|/c. \tag{11}$$

3. *Multipole expansion*

Assume that we have so chosen the centre $\mathbf{r}_0(t)$ of the sphere S_t containing $\varrho(\mathbf{r}, t)$ that

$$\varrho(\mathbf{r}_0(t), t) \neq 0 \text{ for any } t. \tag{12}$$

If so, $\mathbf{r}_0(t')$ is the material coordinate of the trajectory $\mathbf{r}(\mathbf{a}(\mathbf{r}_0(t'), t'), t)$ passing through the point $\mathbf{r}_0(t')$ at $t = t'$, i. e., $\mathbf{a}(\mathbf{r}_0(t'), t') = \mathbf{r}_0(t')$ and $\mathbf{r}(\mathbf{r}_0(t'), t') = \mathbf{r}_0(t')$. Let us calculate the first two terms of the multipole expansion of the scalar potential $\Phi(\mathbf{r}, t)$ for the given $\mathbf{r}$ and t, by calculating the first two terms of the Taylor series, with respect to the point $\mathbf{r}_0(t')$, of the denominator in the second integral in (8) as a function of $\mathbf{r}'$. Defining the retarded time t_- implicitly by

$$t_- = t - |\mathbf{r} - \mathbf{r}_0(t_-)|/c, \tag{13}$$

choosing $t' = t_-$, so that $t_- = t_{ret}(\mathbf{r}_0(t_-), \mathbf{r}, t)$, and using (7), we get

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{q + \mathbf{p}(t_-) \cdot \mathbf{f}(\mathbf{r}_0(t_-), t_-)}{|\mathbf{r} - \mathbf{r}_0(t_-)| - (\mathbf{r} - \mathbf{r}_0(t_-)) \cdot \mathbf{v}(\mathbf{r}_0(t_-), t_-)/c} + \dots \tag{14}$$

with

$$\mathbf{p}(t)\cdot f(\mathbf{r}',t) \equiv \frac{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{pF}_1) + [c((\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|) - \mathbf{v}(\mathbf{r}',t)]\cdot(\mathbf{pF}_2)}{c\,|\mathbf{r}-\mathbf{r}'| - (\mathbf{r}-\mathbf{r}')\cdot\mathbf{v}(\mathbf{r}',t)}$$

$$\mathbf{p}(t)\,\mathbf{F}_1(\mathbf{r}',t) \equiv [(\mathbf{pF}_2)\cdot\nabla']\,\mathbf{v}(\mathbf{r}',t) + \frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}{c\,|\mathbf{r}-\mathbf{r}'| - (\mathbf{r}-\mathbf{r}')\cdot\mathbf{v}(\mathbf{r}',t)}\,\frac{\partial\mathbf{v}(\mathbf{r}',t)}{\partial t} \tag{15}$$

$$\mathbf{p}(t)\,\mathbf{F}_2(\mathbf{r}',t) \equiv \mathbf{p} + \frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}{c\,|\mathbf{r}-\mathbf{r}'| - (\mathbf{r}-\mathbf{r}')\cdot\mathbf{v}(\mathbf{r}',t)}\,\mathbf{v}(\mathbf{r}',t)$$

and

$$\mathbf{p}(t) \equiv \int_{\mathbb{R}^3} (\mathbf{r}-\mathbf{r}_0(t))\,\varrho(\mathbf{r},t)\,\mathrm{d}V \tag{16}$$

in which $+\dots$ denotes terms of the order of R_0^2, depending on higher order, Cartesian moments of $\varrho(\mathbf{r},t)$, and on $\mathbf{v}(\mathbf{r},t)$ and its derivatives. Proceeding analogously, we can calculate the first two terms of the multipole expansion of the vector potential:

$$\mathbf{A}(\mathbf{r},t) = \frac{[q+\mathbf{p}(t_-)\cdot f(\mathbf{r}_0(t_-),t_-)]\,\mathbf{v}(\mathbf{r}_0(t_-),t_-) + \mathbf{p}(t_-)\,\mathbf{F}_1(\mathbf{r}_0(t_-),t_-)}{4\pi\varepsilon_0 c^2\,[|\mathbf{r}-\mathbf{r}_0(t_-)| - (\mathbf{r}-\mathbf{r}_0(t_-))\cdot\mathbf{v}(\mathbf{r}_0(t_-),t_-)/c]} + \dots . \tag{17}$$

4. *Comments*

When the total charge $q \neq 0$, the preceding results simplify considerably if we so choose the centre $\mathbf{r}_0(t)$ of the sphere containing the density $\varrho(\mathbf{r},t)$ that the electric dipole moment $\mathbf{p}(t)$ is identically zero, i.e., if

$$\mathbf{r}_0(t) = \mathbf{r}_q(t) \equiv \int_{\mathbb{R}^3} \mathbf{r}\varrho(\mathbf{r},t)\,\mathrm{d}V/q \tag{18}$$

provided, of course, that the average position of electric charges, $\mathbf{r}_q(t)$, satisfies condition (12); note that (3) with $g = x, y, z$, (18), (4) and (5) imply that

$$\int_{\mathbb{R}^3} \mathbf{j}(\mathbf{r},t)\,\mathrm{d}V = q\dot{\mathbf{r}}_q(t) = q\mathbf{v}(\mathbf{r}_q(t),t) + \dots . \tag{19}$$

We get

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{|\mathbf{r}-\mathbf{r}_q(t_-)| - (\mathbf{r}-\mathbf{r}_q(t_-))\cdot\dot{\mathbf{r}}_q(t_-)/c} + \dots \tag{20}$$

and

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0 c^2}\,\frac{q\dot{\mathbf{r}}_0(t_-)}{|\mathbf{r}-\mathbf{r}_q(t_-)| - (\mathbf{r}-\mathbf{r}_q(t_-))\cdot\dot{\mathbf{r}}_q(t_-)/c} + \dots \tag{21}$$

in which the given terms equal the Liènard-Wiechert potentials. Thus, if we use the average position of electric charges $\mathbf{r}_q(t)$ as an effective trajectory of the total charge q, the Liènard-Wiechert potentials, corresponding to the effective densities of electric charges and currents $q\delta(\mathbf{r} - \mathbf{r}_q(t))$ and $q\dot{\mathbf{r}}_q(t)\,\delta(\mathbf{r} - \mathbf{r}_q(t))$, so approximate scalar and vector potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ that the first corrections in terms of multipole expansions of $\varrho(\mathbf{r}, t)$ vanish, i. e., the first non-zero corrections depend on second-order spatial moments of $\varrho(\mathbf{r}, t)$. Since the Liènard-Wiechert potentials imply Liènard's generalization of the Larmor power formula, we can use the average position of electric charges (18) to calculate the approximate loss of energy and linear momentum from an extended electric charge due to electromagnetic radiation; analogously, one can also use the formula for the loss of relativistic angular momentum by an accelerated pointlike charge, see Ribarič and Šušteršič[4].

References

1) J. D. Jackson, *Classical Electrodynamics*, 2nd edn, Wiley, New York, 1975, Sects. 4.1, 5.6, 9.1, 9.2, 9.3 and 16.5;

2) W. K. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd edn, Addison-Wesley, Reading, Mass., 1969, Sect. 19.1;

3) D. E. Soper, *Classical Field Theory*, Wiley, New York, 1976, Sects. 4.1 and 8.3;

4) M. Ribarič and L. Šušteršič, *Conservation Laws and Open Questions of Classical Electrodynamics*, World Scientific, Singapore, 1990, Eqs. (4.4.1) to (4.4.13), and Sect. 9.2.

POPRAVKI K LIÈNARD-WIECHERTOVIM POTENCIALOM

MARIJAN RIBARIČ in LUKA ŠUŠTERŠIČ

Jožef Stefan Institute, University of Ljubljana, Jamova 39, 61111 Ljubljana, Slovenija

UDK 537.8

Izvirno znanstveno delo

Izračunali smo prva dva člena multipolnega razvoja retardiranih Lorentz-gauge potencialov z ozirom na časovno odvisen tir. V primeru, ko vzamemo za tir povprečni položaj električnih nabojev, nam prvi člen da Liènard-Wiechertove potenciale, in drugi člen izgine.