ISSN 0015—3206
FZKAAA 23 (4) 385 (1991)

YUKAWA MODEL AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL: A G. E. P. APPROACH

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Received 2 October 1991

UDC 536.758

Original scientific paper

We have applied Gaussian effective potential approach at finite temperature and chemical potential in $3 + 1$ dimension. The effect of fermions on the effective potential is studied in detail. It is found that the presence of fermions destabilizes the theory beyond some critical values of the chemical potential μ. Behaviour of the effective potential for different values of T and μ have been studied numerically.

1. Introduction

Of late Gaussian effective potential (GEP) method has been applied to various problems in field theory both for $T = 0$ and finite temperature cases[1-6]. Finite temperature quantum field theory has been studied in Minkowski space by Niemi and Semenoff in a series of papers using loop expansions[7,8]. However it is argued that GEP has several advantages over loop expansion methods. Also, it has been shown that it reproduces the $\frac{1}{N}$ expansion results in the context of $O(N)$ theory. Though quite a number of works on GEP approach in field theory have appeared recently, relatively few works have been done[9,10] on GEP approach to theories containing both fermions and scalars at finite temperature.

In the present paper we apply this method to Yukawa model at finite temperature and chemical potential μ in $3 + 1$ dimensions. Our main aim is to study the effect of fermions on the effective potential. We shall also study spontaneous symmetry breaking at different T and μ.

2. *Yukawa model and GEP*

We take the simplest model when the fermion field is coupled through Yukawa term in 3 + 1 dimensions. To be precise, we study the Hamiltonian given by

$$\mathscr{H} = \bar{\psi}(-\mathrm{i}\gamma_\nu \partial_\nu + \mu_B)\psi + g_B \varphi \bar{\psi} \psi + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2} m_B \varphi^2 + \lambda_B \varphi^4 \tag{1}$$

where μ_B and m_B are fermionic and bosonic masses, respectively, and g_B is the Yukawa coupling constant.

Following Stevenson et al.[4] we write the fermion field as a free field with a variable mass M i. e.,

$$\psi = \int (\mathrm{d}k)_M \sum_\lambda [u_M^\lambda(\mathbf{k})\, b_M(\mathbf{k}, \lambda)\, \mathrm{e}^{-\mathrm{i}kx} + v_\lambda^M(\mathbf{k})\, d_M^+(\mathbf{k}, \lambda)\, \mathrm{e}^{\mathrm{i}kx}] \tag{2}$$

where, in $\nu + 1$ dimension,

$$(\mathrm{d}k)_M \equiv \frac{\mathrm{d}^\nu k}{(2\pi)^\nu\, 2w_k(M)}, \quad w_k(M) \equiv (k^2 + M^2)^{1/2}. \tag{3}$$

The bosonic field φ is written as $\varphi = \varphi_0 + \hat{\varphi}$, where φ_0 is a constant classical field and $\hat{\varphi}$ is a quantum free field of mass Ω

$$\hat{\varphi} = \int (\mathrm{d}k)_\Omega\, [a_\Omega(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}kx} + a_\Omega^+(\mathbf{k})\, \mathrm{e}^{\mathrm{i}kx}]. \tag{4}$$

The state $|0\rangle_{\Omega,\varphi_0}$ is the vacuum state of this free field satisfying the conditions

$$\begin{aligned} {}_{\varphi_0,\Omega}\langle 0|0\rangle_{\Omega,\varphi_0} &= 1 \\ {}_{\varphi_0,\Omega}\langle 0|\,\varphi\,|0\rangle_{\Omega,\varphi_0} &= \varphi_0. \end{aligned} \tag{5}$$

We introduce the chemical potential μ, utilising the conservation of fermion numbers in the following way[19]

$$\bar{\psi}(-\mathrm{i}\gamma_\nu\partial_\nu)\psi \text{ is replaced by}$$

$$\bar{\psi}\{-\mathrm{i}\gamma_k\partial_k - \mathrm{i}\gamma_0(\partial_0 - \mathrm{i}\mu)\}\psi \equiv \bar{\psi}(-\mathrm{i}\gamma_\nu\partial_\nu)\psi - \mu\psi^+\psi. \tag{6}$$

The expectation values for quantities like $\langle 0|\,\varphi^2\,|0\rangle$, $\langle 0|\,\bar{\psi}\psi\,|0\rangle$ etc. can now be calculated in a straightforward manner, and we just give the results.

$$\langle 0|\,\bar{\psi}(-\mathrm{i}\gamma_\nu\partial_\nu)\psi\,|0\rangle = -2\,[\sum_\lambda]\,(I_1' - M^2 I_0') \tag{7}$$

$$\langle 0| \bar{\psi}\psi |0\rangle = -2 \, [\sum_{\lambda}] \, I'_0 M \tag{8}$$

$$\langle 0| \psi^{+}\psi |0\rangle = 2 \, [\sum_{\lambda}] \, I'_{1/2} \tag{9}$$

where $[\Sigma]$ is the number of helicity states, and is equal to 2 in $3 + 1$ dimensions. For the scalar fields we have,

$$\langle 0| \left(\frac{1}{2} [\dot{\varphi}^2 + (\vec{\nabla}\varphi)^2]\right) |0\rangle = I_1 - \frac{1}{2} \Omega^2 I_0 \tag{10}$$

$$\langle 0| \varphi^2 |0\rangle = \varphi_0 + I_0 \tag{11}$$

$$\langle 0| \varphi^4 |0\rangle = \varphi_0^4 + 6\varphi_0^2 I_0 + 3I_0^2. \tag{12}$$

We have used the notations of Stevenson et al. and our I_1, I_0, I'_1, I'_0 are the same as theirs. $I'_{1/2}$ is the new integral which appears due to the chemical potential μ. $I'_{1/2}$ is given by

$$I'_{1/2} = \int \frac{d^3k}{(2\pi)^3 \, 2w_k} \, [w_k^2]^{1/2}. \tag{13}$$

In covariant form, which will be used to calculate its value at finite temperature, it is the integral

$$I'_{1/2} = i \int \frac{d^4k}{(2\pi)^4} \frac{k_0}{k_0^2 - \vec{k}^2 - M^2}. \tag{14}$$

The GEP is defined as

$$\overline{V}_G (\varphi_0, \Omega, M) \equiv \langle 0| \mathscr{H} |0\rangle_{min} \tag{15}$$

where *min* stands for minimised with respect to Ω and M. $\overline{V}_G$ can be calculated easily using the relations (1) to (12). After some algebra we get

$$\overline{V}_G^{FT} (\varphi_0, \Omega, M) = I_1^{FT} + \frac{1}{2} (m_B^2 - \Omega^2) \, I_0^{FT} + \frac{1}{2} m_B^2 \varphi_0^2 +$$
$$+ \lambda_B [\varphi_0^2 + 6\varphi_0^2 I_0^{FT} + 3 (I_0^{FT})^2] - 4 [I_1'^{FT} - M (M - \mu_B) I_0'^{FT} +$$
$$+ g_B M \varphi_0 I_0'^{FT}] - 4\mu \, I_{1/2}'^{FT} \tag{16}$$

where I_N^{FT} indicates that the integrals I_N are calculated at finite temperature.

To minimise $\overline{V}_G$ with respect to Ω and M, we put $\frac{\partial V_G^{FT}}{\partial M} = 0$ and $\frac{\partial V_G^{FT}}{\partial \Omega} = 0$ and write the solutions for M and Ω as $\overline{M}$ and $\overline{\Omega}$, respectively. $\overline{M}$ and $\overline{\Omega}$ are given by

$$\overline{M} = \mu_B + g_B \varphi_0 \tag{17}$$

$$\overline{\Omega}^2 = m_B^2 + 12\lambda_B [\varphi_0 + I_0^{FT}(\overline{\Omega})]. \tag{18}$$

The integrals I_N^{FT} can be written as

$$I_N^{FT} \equiv I_N + I_N^\beta \tag{19}$$

where I_N is the temperature independent part and is a divergent integral and I_N^β is the temperature dependent part which is finite.

The presence of the divergent integrals I_1, I_1', I_0, I_0' and $I_{1/2}'$ indicates that some renormalisation is necessary. The renormalisation is done following Stevenson et al.[4]. The renormalised masses $\overline{M}|_{\varphi_0=0}$ and $\overline{\Omega}^2|_{\varphi_0=0}$ are, respectively, given by

$$\overline{M}|_{\varphi=0} \equiv \mu_R = \mu_B \tag{20}$$

$$\overline{\Omega}^2|_{\varphi_0=0} \equiv m_R^2 = m_B^2 + 12\lambda_B I_0(m_R) \tag{21}$$

and the equation for $\overline{\Omega}^2$ is

$$(x-1)[1 + 6\lambda_B I_{-1}(m_R)] = \frac{3\lambda_B}{4\pi^2}\left[L_2(x) + \frac{16\pi^2}{m_R^2}(\varphi_0^2 + I_0^\beta)\right] \tag{22}$$

with

$$x \equiv \overline{\Omega}^2/m_R^2. \tag{23}$$

Here we have used the results[1]

$$I_1(\overline{\Omega}) - I_1(m) = \frac{1}{2}(\overline{\Omega}^2 - m^2)I_0(m) - \frac{1}{8}(\overline{\Omega}^2 - m^2)I_{-1}(m) + \frac{m^4}{32\pi^2}L_3(x) \tag{24}$$

$$I_0(\overline{\Omega}) - I_0(m) = -\frac{1}{2}(\overline{\Omega}^2 - m^2)I_{-1}(m) + \frac{m^2}{16\pi^2}L_0(x) \tag{25}$$

$$I_{-1}(\overline{\Omega}) - I_{-1}(m) = -\frac{1}{8\pi^2}L_1(x) \tag{26}$$

$$L_1(x) = \ln x \tag{27}$$

$$L_2(x) = x \ln x - x + 1 \tag{28}$$

$$L_3(x) = \frac{1}{4}\,[2x^2 \ln x - 2(x-1) - 3(x-1)^2]. \tag{29}$$

Using the above results the self-consistent equation for x can be written as

$$x = 1 + \frac{\lambda_R}{4\pi^2}\,[x \ln x - x + 1 + \frac{16}{m_R^2}\,\pi^2 (\varphi_0^2 + I_0^\theta)] \tag{30}$$

where λ_R is defined by

$$\lambda_R \equiv \frac{1}{4!}\,\frac{\mathrm{d}^4 \overline{V}_G^{FT}}{\mathrm{d}\varphi_0^4}\bigg|_{\varphi_0=0} = \lambda_B \frac{(1 - 12\lambda_B I_{-1}(m_R))}{1 + 6\lambda_B I_{-1}(m_R)}. \tag{31}$$

Now the fermionic contribution to GEP is

$$\overline{V}^{FT}(\text{fermionic}) = -4I_{1/2}^{\prime FT} - 4\mu I_{1/2}^{\prime FT}. \tag{32}$$

Using formulas for $I_1'(\overline{M})$ and $I_0'(\overline{M})$ similar to those given in (24) to (29), we obtain

$$\overline{V}_G^{FT}(\text{fermionic}) = D' - 4I_1'^{\beta} - 4\mu I_{1/2}'^{\theta} - 2g_B^2\varphi_0^2\,[I_0'(\mu_B) - \mu_B^2 I_{-1}'(\mu_B)] +$$

$$+ 2g_B^3\varphi_0^3\,\mu_B I_{-1}'(\mu_B) + \frac{1}{2}\,g_B^4\varphi_0^4\,I_{-1}'(\mu_B) - \frac{\mu_B^4}{8\pi^2}\,L_3(x') \tag{33}$$

where

$$x' \equiv \frac{\overline{M}^2}{\mu_B^2} = 1 + \frac{g_B^2\varphi_0^2}{\mu_B}. \tag{34}$$

Thus we are forced to renormalise the coupling constant g_B as

$$g_B^2 = \frac{G^2}{I_0'(\mu_B)} \tag{35}$$

so that g_B is infinitesimal and G is finite while $I_0'(\mu_B)$ is a quadratically divergent integral. This gives,

$$\overline{V}_G^{FT}(\text{fermionic}) = D' - 4I_1'^{\theta} - 4\mu\, I_{1/2}'^{\theta} - 2G^2\varphi_0^2 \tag{36}$$

where D' is the divergent vacuum energy constant due to fermions. Finally,

$$\overline{V}_G^{FT} - D = I_1^\beta + \frac{1}{2}\overline{\Omega}^2 \varphi_0^2 - \frac{1}{16\lambda_R}(\overline{\Omega}^2 - m_R^2)^2 + \\ + \frac{m_R^4}{128\pi^2}\left[\frac{2\overline{\Omega}^4}{m_R^4}\ln\frac{\overline{\Omega}^2}{m_R^2} - 2\left(\frac{\overline{\Omega}^2}{m_R^2} - 1\right) - \\ - 3\left(\frac{\overline{\Omega}^2}{m_R^2} - 1\right)^2\right] - 4I_1'^\beta - 2G^2\varphi_0^2 - 4\mu I_{1/2}'^\beta. \tag{37}$$

D is the overall divergent vacuum energy constant.

Taking $m_R^2 = 1$

$$\frac{d\overline{V}_G^{FT}}{d\varphi_0} = \{\varphi_0^2 + I_0^\beta + \frac{1}{16\pi^2}[\overline{\Omega}^2 \ln\overline{\Omega}^2 - \overline{\Omega}^2 + 1] - \\ - \frac{1}{4\lambda_R}(\overline{\Omega}^2 - 1)\}\overline{\Omega}\frac{d\overline{\Omega}}{d\varphi_0} + (\overline{\Omega}^2 - 4G^2)\varphi_0. \tag{38}$$

Symmetry breaking takes place if

$$\frac{d\overline{V}_G^{FT}}{d\varphi_0} = 0 \text{ at } \varphi_0 \neq 0.$$

From (38) we get

$$\overline{\Omega}^2 = 4G^2. \tag{39}$$

From (39) it follows that for $\overline{\Omega}^2 \geq 1$,

$$G^2 \geq 0.25.$$

3. *Large temperature behaviour of $I_1'^\beta$ and $I_{1/2}'^\beta$*

(i) $I_{1/2}'^\beta$

We write $I_{1/2}'^\beta$ in the covariant form

$$I_{1/2}'^\beta = i\int\frac{d^4k}{(2\pi)^4}\frac{k_0}{k_0^2 - \vec{k}^2 - M^2}. \tag{40}$$

To perform finite temperature calculations in the presence of a chemical potential μ, we make the following substitutions

$$-ik_0 \to (2n+1)\,\pi T - i\mu$$

and

$$\int \frac{dk_0}{2\pi} \to iT \sum_{n=-\infty}^{\infty} . \tag{41}$$

After some simple algebra, we get,

$$I_{1/2}^{\prime\beta} = \frac{i}{2} \sum_n \int_0^\infty \frac{d^3k}{(2\pi)^3} \left[\frac{1}{A - iX} + \frac{1}{A + iX} \right] \tag{42}$$

where

$$A = (2n+1)\,\pi - i\bar{\mu} \tag{43}$$

$$X = \frac{(k^2 + M^2)^{1/2}}{T} = \frac{E}{T} \tag{44}$$

$$\bar{\mu} = \frac{\mu}{T}. \tag{45}$$

Next we follow the properties of generalised ζ functions. (For details please see Roy and Roychoudhury[12]).

For the sake of brevity we just quote here the final result

$$I_{1/2}^{\prime\beta} \sim -\frac{\mu T^2}{12} + \frac{1}{12\pi^2} (\mu^2 - M^2)^{3/2} \tag{46}$$

(where we have assumed $\mu^2 > M^2$ to determine the non leading term).

(ii) $I_{1/2}^{\prime\beta}$:

In covariant form

$$I_1^{\prime\beta} = -\frac{i}{2(2\pi)^4} \int d^4k \ln (k^2 - M^2). \tag{47}$$

For finite temperature calculations in the presence of μ, we perform identical calculations as in (41) above.

Then, after some simple algebraic steps, we obtain

$$\frac{\mathrm{d}I_1'^{\beta}}{\mathrm{d}M^2} = \frac{T^2}{8\pi^2}\int\limits_0^{\infty} \frac{x\,\mathrm{d}x}{(x^2+\bar{m}^2)^{1/2}}\left[\frac{1}{\exp(\sqrt{x^2+\bar{m}^2}-r\bar{m})+1}+\right.$$

$$\left.+\frac{1}{\exp(\sqrt{x^2+\bar{m}^2}+r\bar{m})+1}\right] \tag{48}$$

where

$$x^2 = \frac{k^2-M^2}{T^2}$$

$$\bar{m} = \frac{M}{T}$$

$$\bar{\mu} = \frac{\mu}{T} \tag{49}$$

$$r = \frac{\mu}{M}.$$

These integrals can be tackled following the methods of Haber and Weldon[13]. Their calculations were for bosonic integrals where they utilised the fomula

$$\frac{1}{e^y-1} = \frac{1}{y} - \frac{1}{2} + 2\sum_{p=1}^{\infty}\frac{y}{y^2+(2\pi p)^2}. \tag{50}$$

For fermionic integrals, we use the formula given below, instead

$$\frac{1}{e^y+1} = \frac{1}{2} - 2\sum_{p=0}^{\infty}\frac{y}{y^2+(2p+1)^2\pi^2}. \tag{51}$$

Then after a long, tedious, but straightforward calculation, we get

$$I_1'^{\beta} \approx -\frac{7}{720}\pi^2T^4 + \frac{T^2}{48}(M^2-2\mu^2) + \frac{M^2\mu^2}{16\pi^2} + \frac{M^4}{64\pi^2}\left(\ln\frac{M^2}{4\mu^2}-\frac{1}{2}\right) \tag{52}$$

in the large T limit, and for $\mu > M$. The results conforms with the result of Yildiz[14].

4. *Results and discussions*

As mentioned earlier, our aim was to study the effect of fermions on the effective potential at finite temperature. To study this, we calculate $\overline{V}_G^{FT}$ for various values of μ and T.

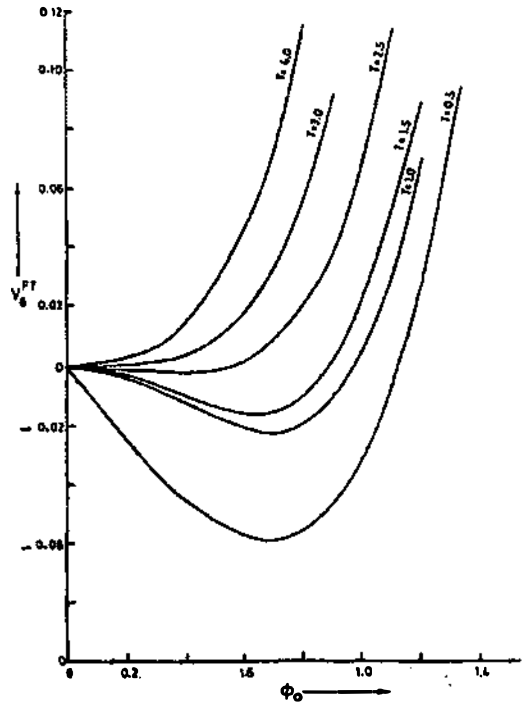

Fig. 1.

In Fig. 1, $\overline{V}_G^{FT}$ is plotted against φ_0 for various temperatures, for $G^2 = 0.3$, $\lambda_R = 0.1$ and $\mu = 0.02$. For these values of the parameter, spontaneous symmetry breaking occurs below $T = 2.87$.

In Fig. 2, the mass gap equation is shown by plotting $\overline{\Omega}^2$ against T. There is a discontinuity at $T = 2.87$ as expected.

To study the effect of μ, in Fig. 3 we have plotted $\overline{V}_G^{FT}$ against φ_0 at $T = 0.5$ and $T = 0.2$ for different values of μ, ranging from 0.02 to 1.0. (For $\mu = 1.0$, only the $T = 0.5$ curve has been drawn.) The solid lines depict the $T = 0.5$ curves, whereas the dotted lines are for $T = 0.2$. For $T = 0.5$, the critical value of μ is about 0.2 above which the symmetry breaking disapears.

Fig. 4 shows the plot $\overline{V}_G^{FT}$ against μ for $T = 0.5$, $\varphi_0 = 0.2$ and $G^2 = 0.3$.

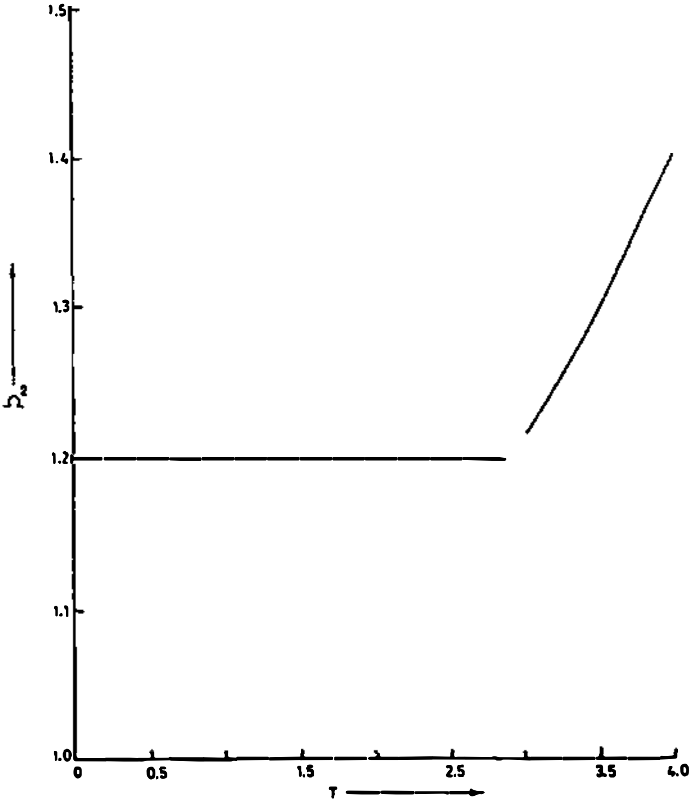

Fig. 2.

To summarize, it is found that for $G^2 > 0.25$ symmetry breaking occurs at low values of T and μ but symmetry is restored at higher values of these parameters. For $G^2 < 0.25$, no symmetry breaking is found from numerical analysis supporting our theoretical calculations. Of course, in the absence of fermions, no symmetry breaking would occur, as

$$V(\varphi_0) = \frac{1}{2} m_B^2 \varphi_0^2 + \lambda_B \varphi_0^3$$

has no minimum except at $\varphi_0 = 0$. Thus spontaneous symmetry breaking occurs due to the presence of fermion terms through Yukawa coupling. Thus in $3 + 1$ dimensions, the presence of fermions destabilizes the φ^4 Yukawa theory. This has been noticed by authors in Ref. 4 also. GEP gives an overall qualitative view of how the symmetry breaking occurs.

Acknowledgement

The authors are thankful to Professor R. Tarrach of University of Barcelona, Spain for some very useful comments.

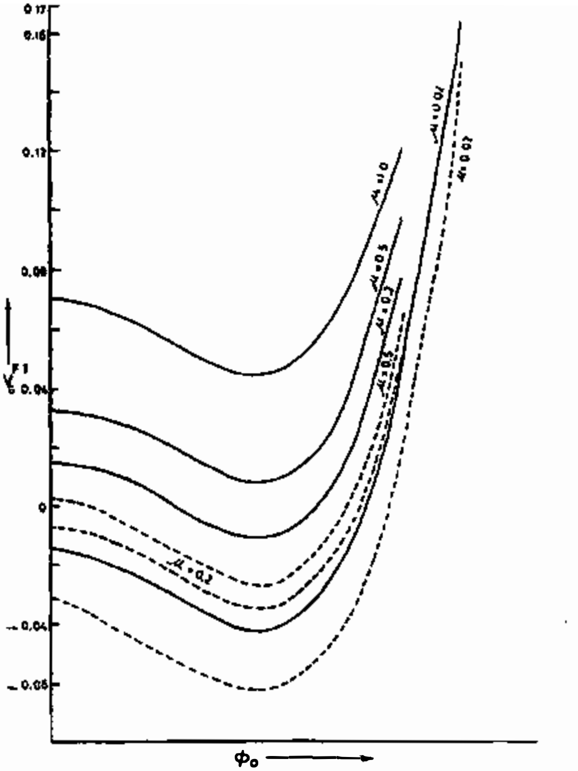

Fig. 3.

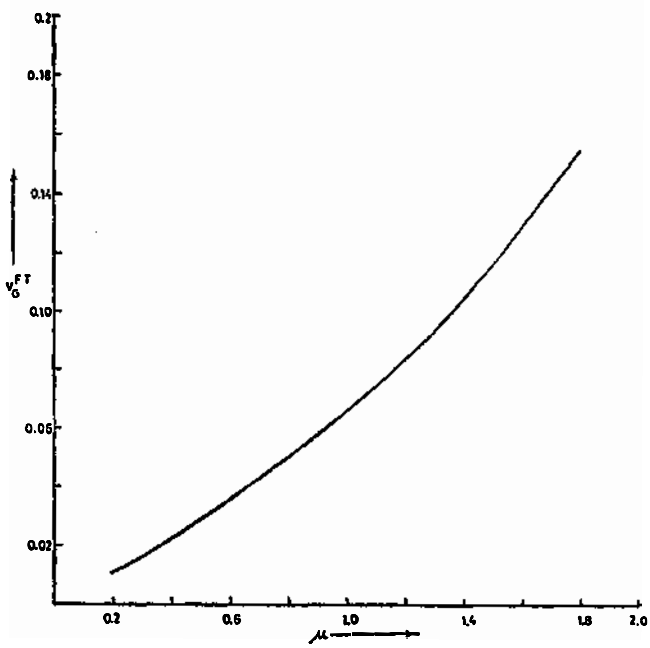

Fig. 4.

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YUKAWA-MODEL KOD KONAČNIH TEMPERATURA I KEMIJSKOG POTENCIJALA

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Received 2 October 1991

UDK 536.758

Originalni znanstveni rad

Primijenili smo metodu Gaussova efektivnog potencijala kod konačne temperature i kemijskog potencijala u 3 + 1 dimenziji. Studiran je detaljno utjecaj fermiona na efektivni potencijal. Nađeno je da prisustvo fermiona destabilizira teoriju nakon neke određene kritične vrijednosti kemijskog potencijala μ. Studirano je numerički ponašanje efektivnog potencijala pri različitim vrijednostima T i