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SPHERICALLY SYMMETRIC CONFIGURATION OF ELECTRIC AND MAGNETIC CHARGE IN GENERAL RELATIVITY

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By assuming the dual symmetry of Maxwell's equations in the presence of magnetic charge we calculate the electromagnetic and gravitational fields for a spherically symmetric configuration of electric and magnetic charge. If we assume the matter carrying electric and magnetic charge admits the condition of vanishing normal pressure, we may match the interior and exterior solutions to obtain a mass charge relation. Astrophysical searches for such bound configurations would entail looking for correlations between the integrated energy released when such a configuration collapses and the total magnetic and electric charge of the configuration.

1. Introduction

The introduction of magnetic charge and its subsequent influence on both particle theory and cosmology has had a long and interesting history in physics. Dirac introduced the idea that due to the quantization of angular momentum in quantum theory the interaction of an electric charge with a magnetic charge leads to the relation

$$\frac{eq}{\hbar c} = \frac{n}{2} \qquad (n = 1, 2 - -),$$

(e = electronic charge, q = magnetic charge of interacting particle) and thus the existence of one unit of magnetic charge implies quantization of electric charge[1]. This is certainly an alternative to the G. U. T. explanation of charge quantization originating from the fact that the trace of the electric charge operator must vanish for a G. U. T. group such as SU (5)[2]. Following Dirac, Schwinger derived the fact that for two dyons whose field momentum is quantized their electric and magnetic charges must admit the relation

$$\frac{e_1 q_2 - e_2 q_1}{\hbar c} = \frac{n}{2} \qquad (n = 1, - - -)^{3)}.$$

(A dyon being a particle carrying both electric and magnetic charge.) Magnetic charge quantization follows from non-abelian gauge theory for dyon solutions from the overall consistency of the solution with the magnetic charge having topological significance[4]. When CP violating interactions are present in a gauge theory, Witten has demonstrated that the electric charge of a dyon need not be integral which suggests an intimate relation between the integral nature of electric charge and time reversal symmetry in the universe[5]. On the scale of elementary particle interactions, there have been numerous developments recently discussing the dynamics of interacting dyons or monopoles with fermions and vector particles. Olsen et al. have discussed the bound states of vector particles in the field of a monopole[6], and Tolkachev et al.[7] have discussed the Zeeman splitting of a dyon fermion system which has characteristics stemming from the parity violating nature of the dyon solution. Gal'tsov and Ershov[8] have discussed the dynamics of a scalar particle in the field of a SU (2) dyon and Frampton et al.[9] and J. zu Zhang[10] have discussed the energy levels and CP violating dipole moments of the dyon fermion system which could shed light on the general problem of CP violation in particle theory. Quite long ago Pati[11] stressed that due to the fact that the magnetic fine structure constant obeys the relation

$$\frac{q^2}{\hbar c} \approx \frac{1}{2} (137),$$

magnetic binding might provide an excellent binding mechanism for the internal constituents of elementary particles. In such models the total magnetic charge of the composite particle would have to vanish but the individual preons could be magnetically charged. In an astrophysical setting, we have discussed the far-field solution of the abelian dyon in the bi-metric theory of gravitation[12], and Prasad and Sommerfield have discussed the general problem of non-abelian monopoles and dyons irrespective of their influences in particle theory (Ref. 4). Long ago Vinciarelli had stressed that the dual structure of Maxwell's equations in the presence of magnetic charge is a unique consequence of the four dimensional spacetime we live in[13], and in this note we assume the dual structure of Maxwell's equations and calculate the internal electric, magnetic and gravitational fields of a configuration of electric and magnetic charge. Assuming the normal pressure vanishes we match the internal solution to the external solution. This matching condition serves to determine a mass charge relation and allows for the determination of integration constants.

It is well known that G. U. T. dyons appear as abelian in the far field region after the colour and electroweak fields of the dyon die off (Ref. 4). Our model could thus serve to describe a uniform configuration of G. U. T. dyons wherein the individual dyons of the charge configuration are far enough apart so as to only experience electromagnetic forces. Though it seems unlikely that dyons or monopoles could have survived inflation, they could have been produced by energetic collisions around the time of helium synthesis[14]. If such monopoles combined with anti-monopoles, they could from a bound state of monopoloniun that might survive until the present epoch. The decay of monopoloniun could be observed through its decay into a whole spectrum of G. U. T. particles which would finally be manifested in the quark lepton signatures produced. If an ensemble of such monopoles were localized near an accumulation of electric charge, we would have the necessary ingredients for the model discussed in this paper. Recently Dimopoulous et al.[15] and De Rujula et al.[16] have suggested the presence of CHAMPS, (charged massive particles between 20 and 1000 TeV predicted by particle theory) as a dark matter component, if such electrically charged particles accumulate near the residual monopoles produced around the time of helium synthesis we would have the necessary ingredients for a macroscopic distribution of electric and magnetic charge. Finally we suggest certain signatures to look for in an astrophysical setting that might be characteristic of gravitationally bound configurations of electric and magnetic charge.

2. Spherically symmetric configuration of electric and magnetic charge in general relavity

We begin our analysis by assuming the spherically symmetric metric of the form

$$(\mathrm{d}S)^2 = e^{\nu} (\mathrm{d}x^4)^2 - e^{\lambda} (\mathrm{d}r)^2 - r^2 (\mathrm{d}\vartheta)^2 - r^2 \sin^2 \vartheta \, (\mathrm{d}\varphi)^2$$

$$(x^4 = ct,\ x^1 = r,\ x^2 = \vartheta,\ x^3 = \varphi). \tag{2.1}$$

For the equations of the electromagnetic field we have

$$\frac{\partial}{\partial x^{\nu}} (\sqrt{-g}\, F^{\mu\nu}) = 4\pi \sqrt{-g}\, j^{\mu}_{E} \tag{2.2}$$

$$\frac{\partial}{\partial x^{\nu}} \left(\frac{\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}}{2} \right) = 4\pi \sqrt{-g}\, j^{\mu}_{M}. \tag{2.3}$$

Eq. (2.2) and Eq. (2.3) follow from the dual symmetry of Maxwell's equations in the presence of magnetic charge with j^{μ}_{E}, j^{μ}_{M} being the electric and magnetic four currents, respectively. For the fourth component of the four currents we have

$$j^{\mu}_{E} = \varrho_{0E} \frac{\mathrm{d}x^4}{\mathrm{d}S} = \varrho_{0E} e^{-\nu/2} \tag{2.4}$$

$$j^{\mu}_{M} = \varrho_{OM}\frac{dx^4}{dS} = \varrho_{OM}\, e^{-\nu/2}. \tag{2.5}$$

Here ϱ_{OE}, ϱ_{OM} are the proper electric and magnetic charge densities.

As a particular charge distribution to simplify the analysis, we take

$$\varrho_{OE} = \overline{\varrho}_{OE}\, e^{-\lambda/2} \tag{2.6}$$

$$\varrho_{OM} = \overline{\varrho}_{OM}\, e^{-\lambda/2} \tag{2.7}$$

where $\overline{\varrho}_{OE}$, $\overline{\varrho}_{OM}$ are constant.

Eq. (2.2) gives for ($F_{14} = E(r)$),

$$E(r) = \frac{1}{r^2}\, e^{(\lambda+\nu)/2} \int_0^r 4\pi\overline{\varrho}_{OE} r^2 dr = \frac{4\pi\overline{\varrho}_{OE} r e^{(\lambda+\nu)/2}}{3} \tag{2.8}$$

and Eq. (2.3) gives

$$B(r) = \frac{4\pi\overline{\varrho}_{OM} r}{3}. \tag{2.9}$$

For the energy momentum tensor of the electromagnetic field we have the same form as is used when magnetic charge is not present as discussed by Semiz[17]

$$T_{\mu\nu} = \frac{g_{\mu\nu}}{16\pi}(F_{\alpha\beta}F^{\alpha\beta}) - \frac{1}{4}F_{\mu\alpha}F^{\alpha}_{\nu} \tag{2.10}$$

giving

$$T^4_1 = T^1_1 = \frac{1}{8\pi}\left[\frac{4\pi\overline{\varrho}_{OE}\, r}{3}\right]^2 + \frac{1}{8\pi}\left[\frac{4\pi\overline{\varrho}_{OM}\, r}{3}\right]^2 \tag{2.11}$$

for $R > r$ (R = radius of configuration). For the matter component of the energy momentum tensor, we have

$$T^4_4 = \varepsilon_0$$

$$T^1_1 = 0 \tag{2.12}$$

$$T^2_2 = T^3_3 = -P.$$

Here we have considered the case of vanishing normal pressure and constant energy density ε_0. To calculate the metric we take the $\binom{4}{4}$ component of the Einstein equation to read

$$\frac{d}{dr}(re^{-\lambda}) = 1 - \frac{8\pi G}{c^4} r^2 T^4_4. \tag{2.13}$$

Here T_4^4 represents the energy momentum tensor of matter plus electromagnetic field

$$T_4^4 = \varepsilon_0 + \frac{2\pi (\overline{\varrho}_{0E})^2 r^2}{9} + \frac{2\pi (\overline{\varrho}_{0M})^2 r^2}{9}. \tag{2.14}$$

Inserting Eq. (2.14) into Eq. (2.13) gives

$$\frac{d}{dr}(re^{-\lambda}) = 1 - \frac{8\pi G r^2}{c^4}\left[\varepsilon_0 + \frac{2\pi}{9}(\overline{\varrho}_{0E})^2 r^2 + \frac{2\pi}{9}(\overline{\varrho}_{0M})^2 r^2\right]. \tag{2.15}$$

Integrating from 0 to R (the radius of the charged configuration) we have

$$(Re^{-\lambda})_R = R - \frac{8\pi G}{c^4}\left[\frac{\varepsilon_0 R^3}{3} + \frac{2\pi (\overline{\varrho}_{0E})^2 R^5}{45} + \frac{2\pi (\overline{\varrho}_{0M})^2 R^5}{45}\right]$$

or

$$e^{-\lambda} = 1 - \frac{8\pi G}{c^4}\left[\frac{\varepsilon_0 R^2}{3} + \frac{2\pi (\overline{\varrho}_{0E})^2 R^4}{45} + \frac{2\pi (\overline{\varrho}_{0M})^2 R^4}{45}\right]. \tag{2.16}$$

For the $\binom{1}{1}$ component of the Einstein equations we have

$$R_1^1 - \frac{1}{2} R\delta_1^1 = -\frac{8\pi G}{c^4} T_1^1$$

or from Eq. (2.1)

$$\frac{1}{r^2}(e^{-\lambda} - 1) + \frac{1}{r} e^{-\lambda} \nu' = -\frac{8\pi G}{c^4} T_1^1 \tag{2.17}$$

or

$$\nu' = -\frac{8\pi G}{c^4} \Omega\, e^{\lambda} T_1^1 + \frac{1}{r}(e^{\lambda} - 1).$$

Integrating to give for $r < R$

$$\nu(r) = -\frac{8\pi G}{c^4}\int^r r e^{\lambda} T_1^1\, dr + \int^r \frac{1}{r}(e^{\lambda} - 1)\, dr + \overline{C}. \tag{2.18}$$

Here T_1^1 is given by Eq. (2.11) for $r < R$, e^{λ} is given by Eq. (2.16) with r replaced by R for $r < R$. We will evaluate $\overline{C}$ by matching to the exterior solution at $r = R$. For $r > R$ we have the electromagnetic field equation

$$\frac{\partial}{\partial x^{\nu}}(\sqrt{-g}\, F^{\mu\nu}) = 0 \tag{2.19}$$

with the condition

$$\frac{\partial}{\partial x^{\nu}} (\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}) = 0 \tag{2.20}$$

when no magnetic charge is present. For $r > R$, $F_{14} = E(r)$, $F_{23} = r^2 \sin \vartheta B_r$, Eq. (2.19), Eq. (2.20) give

$$E(r) = \frac{e}{r^2} \tag{2.21}$$

$$B(r) = \frac{q}{r^2}, \tag{2.22}$$

matching Eq. (2.21), Eq. (2.22) to Eq. (2.8) and Eq. (2.9) at $r = R$, gives

$$e = \frac{4}{3} \pi R^3 \bar{\varrho}_{0E},$$

$$q = \frac{4}{3} \pi R^3 \bar{\varrho}_{0M}.$$

Here e, q are total electric and magnetic charge within the configuration of charge density. Also $T^1_1 = T^4_4$ for $r > R$ from Eq. (2.10) which gives $\lambda + \nu = 0$ from the $(^1_1)$, $(^4_4)$ component of the Einstein equation. Using Eq. (2.21), Eq. (2.22) and Eq. (2.10) we have

$$T^1_1 = T^4_4 = \frac{e^2}{8\pi r^4} + \frac{q^2}{8\pi r^4}. \tag{2.23}$$

For the $(^4_4)$ component of the Einstein equation we have for $r > R$

$$\frac{\mathrm{d}}{\mathrm{d}r} (r\, \mathrm{e}^{-\lambda}) = 1 - \frac{8\pi G}{c^4} r^2 \left[\frac{e^2}{8\pi r^4} + \frac{q^2}{8\pi r^4} \right] \tag{2.24}$$

upon integration we have

$$\mathrm{e}^{-\lambda} = \mathrm{e}^{\nu} = 1 - \frac{2GM}{rc^2} + \frac{Gq^2}{r^2c^4} + \frac{Ge^2}{r^2c^4}. \tag{2.25}$$

Matching Eq. (2.16) and Eq. (2.25) at $r = R$ yields an expression for M (the mass of the configuration of electric and magnetic charge with non-electromagnetic energy density ε_0). Also from Eq. (2.25) we have at $r = R$

$$\nu = \ln \left(1 - \frac{2GM}{Rc^2} + \frac{Gq^2}{R^2c^4} + \frac{Ge^2}{R^2c^4} \right). \tag{2.26}$$

Matching Eq. (2.26) and Eq. (2.18) at $r = R$ will determine the constant of integration in Eq. (2.18).

3. *Conclusion*

In the above calculation we have obtained the exact solution for the metric and the electromagnetic fields for a configuration of electric and magnetic charge when the charge densities obey Eq. (2.6) and Eq. (2.7). We have also assumed that the radial pressure vanishes which greatly simplifies the solution. If such a configuration were dispersed it would release energy since the energy of the bound configuration is greater than that of unbound configuration as emphasized in a previous note[18]. In fact the formula for the energy release would be

$$E_R = M\,c^2 - \int_0^R 4\pi\, r^2 e^{\lambda/2}\, \varrho_0\, c^2\, \mathrm{d}r$$

$$E_R = M\,c^2 - \int_0^R 4\pi\, r^2\, e^{\lambda/2}\, (\varepsilon_0 - P)\, \mathrm{d}r. \qquad (2.27)$$

In Eq. (2.27) the proper rest energy is related to the energy density and the pressure by $\varepsilon_0 = \varrho_0 c^2 + P$ since there are only two non-relativistic degrees of freedom for the KE of the particles since the radial pressure vanishes. In Eq. (2.27) M would be found by equating Eq. (2.25) and Eq. (2.16) at $r = R$, e^{λ} is found from Eq. (2.16) and P would be found from the $\binom{2}{2}$ component of the Einstein equations. A dependence of energy released on the electric and magnetic charges e, q, would provide a signature for the existence of such configurations in an astrophysical setting. It could also be that some of the γ ray bursts presently observed might be related to the collapse of such electromagnetically charged structures that we have discussed above, if the total luminosity has a dependence on the two parameters (e, q) it would provide a positive signature for macroscopic electrically and magnetically charged configurations in the cosmos.

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References

1) P. A. M. Dirac, Proc. Roy. Society **A133** (1931) 60;
2) H. Georgi and S. Glashow, Phys. Rev. Lett. **32** (1974) 438;
3) J. Schwinger, Phys. Rev. **144** (1966) 1087;
4) M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35** (1975) 760;
5) E. Witten, Phys. Lett. **B86** (1979) 287;
6) H. A. Olsen, P. Ostand and Tai Tsun Wu, Ark. Fys. Semin. Trondheim (Norway) **10** (1989) 197;

7) E. A. Tolkachev, L. M. Tomil'Chik, Ya. Shnir, Sov. J. Nucl. Phys. (U. S. A.) **50** (1989) 275;
8) D. V. Gal'tsov and A. A. Ershov, Mosc. Univ. Phys. Bull. (U. S. A.) **43** (1988) 40;
9) P. H. Frampton, Zhang-Jian-zu and Qi Yong-chang, Phys. Rev. **D40** (1989) 3533;
10) Jian-zu Zhang, J. Phys. G. (Nucl. and Part. Phys.) **16** (1990) 195;
11) J. C. Pati, Phys. Lett. **98B** (1981) 40;
12) C. Wolf, Acta Physica Slovaca **37** (1987) 369;
13) P. Vinicarelli, Proc. of the 1976 Int. School of Subnuclear Phys. (Plenum Press. N. Y., 1978), 799;
14) C. T. Hill, Proc. of a NATO Advanced Study Inst. on Magnetic Monopoles, Oct. 14—17, 1982. Wingsgread, Wisconsin. (Plenum Press, NY, 1983), p. 159;
15) S. Dimopoulous, B. Eichler, R. Esmailzadeh, G. D. Starkman: Phys. Rev. **D41** (1990) 2388;
16) A. De Rujula, S. L. Glashow and U. Sarid, Nucl. Phys. B. (Part Phys.) **133** (1990) 173;
17) I. Semiz, Class. and Quan. Gravity **7** (1990) 353;
18) C. Wolf, submitted to Pramana (1990).

SFERNO SIMETRIČNA KONFIGURACIJA ELEKTRIČNOG I MAGNETSKOG NABOJA U OPĆOJ RELATIVNOSTI

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Pretpostavivši dualnu simetriju Maxwellovih jednadžbi u prisustvu magnetskog naboja, računato je elektromagnetsko i gravitaciono polje sferno simetrične konfiguracije električnog i magnetskog naboja. Ako pretpostavimo da tvar koja nosi električni i magnetski naboj zadovoljava uvjet dokidanja normalne komponente pritiska, moguće je spojiti nutarnja i vanjska rješenja i dobiti relaciju između mase i naboja. Astrofizikalne potrage za takvim vezanim konfiguracijama vode na traženje korelacija između integrirane energije, koja se oslobodi pri kolabiranju konfiguracije, i ukupnog magnetskog i električnog naboja te konfiguracije.