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A MOMENTUM SPACE FOKKER-PLANCK EQUATION FOR THE DEEP INELASTIC SCATTERING OF HEAVY IONS*

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A momentum space Fokker-Planck equation for deep inelastic scattering is developed starting from the statistical multi-step direct theory of nuclear reactions. Approximate solutions of this equation are obtained and discussed. One finds in agreement with experiment that the »temperature« decreases with increasing angle.

There are a host of formalisms[1] which have been proposed for the analysis of heavy ion deep inelastic scattering. All of these require extensive numerical calculation, a requirement which presents a formidable barrier for their use in the quantitative analysis of experimental data. It is the objective of this paper to provide analytical expressions for the cross-section of experimental interest permitting direct comparison with experiment as well as with the underlying dynamics. These are approximate but corrections can be calculated if needed.

We envisage the collision between the heavy ions as proceeding through a series of steps in which energy and particle are exchanged. It can be anticipated that a diffusion description can be developed if each step involve small changes in the system. We shall make the assumption derived from experiment that the final state is a two-body system and shall extend that assumption to all intermediate

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steps. Under these circumstances the statistical multi-step formalism of FKK[2] applies. The differential cross-section for the multi-step components is given by

$$\left[\frac{\mathrm{d}^2\sigma\,(\vec{k}_f\eta_f,\vec{k}_i\eta_i)}{\mathrm{d}\Omega_f\,\mathrm{d}U_f}\right]_{\mathrm{msd}} = \sum_{\substack{\alpha,m\\ m=\alpha\pm1}} \int\frac{\mathrm{d}\vec{k}_1}{(2\pi)^3}\,\mathrm{d}\eta_1\ldots\int\frac{\mathrm{d}\vec{k}_\alpha\,\mathrm{d}\eta_\alpha}{(2\pi)^3}\left[\frac{\mathrm{d}^2\omega_{m,\alpha}\,(\vec{k}_f,\eta_f;\vec{k}_\alpha,\eta_\alpha)}{\mathrm{d}\Omega_f\,\mathrm{d}U_f\,\mathrm{d}\eta_f}\right]\times$$

$$\times\left[\frac{\mathrm{d}^2\omega_{\alpha,\alpha-1}(\vec{k}_\alpha,\eta_\alpha;\vec{k}_{\alpha-1},\eta_{\alpha-1})}{\mathrm{d}\Omega_\alpha\,\mathrm{d}U_\alpha\,\mathrm{d}\eta_\alpha}\right]\ldots\left[\frac{\mathrm{d}^2\omega_{2,1}\,(\vec{k}_2,\eta_2;\vec{k}_1\eta_1)}{\mathrm{d}\Omega_2\,\mathrm{d}U_2\,\mathrm{d}\eta_2}\right]\times$$

$$\times\left[\frac{\mathrm{d}^2\sigma_{1i}\,(\vec{k}_1,\eta_1;\vec{k}_i\eta_i)}{\mathrm{d}\Omega_1\,\mathrm{d}U_1\,\mathrm{d}\eta_1}\right]. \tag{1}$$

In this equation, α numbers the step, $\hbar\vec{k}$ is the relative momentum of the two interacting nuclei, while α also includes their internal quantum numbers and energies. Energy is conserved at each step so that for example U_f, the total excitation energy, is given by

$$U_f = E - \frac{\hbar^2}{2\mu}k_f^2. \tag{2}$$

The quantity η designates two parameters:

$$\eta^{(A)} = \frac{A_t - A_p}{A_t + A_p} \quad\text{and}\quad \eta^{(Z)} = \frac{Z_t - Z_p}{Z_t + Z_p}. \tag{3}$$

where the subscripts t and p refer to the target and projectile, respectively. Equation (1) assumes that η varies continuously. The differential transition probabilities are defined as follows:

$$\frac{\mathrm{d}^2\omega_{\alpha,\alpha-1}\,(\vec{k}_\alpha,\eta_\alpha;\vec{k}_{\alpha-1},\eta_{\alpha-1})}{\mathrm{d}\Omega_\alpha\,\mathrm{d}U_\alpha} = 2\pi^2\varrho\,(k_\alpha)\,\varrho\,(U_\alpha)\,|\overline{v}\,(\vec{k}_\alpha,\eta_\alpha;\vec{k}_{\alpha-1},\eta_{\alpha-1})|^2 \tag{4}$$

where $\varrho\,(k_\alpha)$ and $\varrho\,(U_\alpha)$ are the density of states to be used in the transition from stage $\alpha - 1$ to stage α. Finally changes in the deformation of the nuclei in the transitions may occur so that deformation parameters should also be included in Eq. (1) and Eq. (4). We note that $\mathrm{d}^2\omega_{\alpha,\alpha-1}/\mathrm{d}\Omega_\alpha\mathrm{d}U_\alpha$ is proportional to the single step cross-section which dominate the quasielastic process in heavy ion collisions.

For reasons of clarity we shall suppress for the time being the dependence on η. We now introduce the transfer function $Y_\alpha(\vec{k}_\alpha)$:

$$Y_\alpha(\vec{k}_\alpha) \equiv \int\frac{\mathrm{d}\vec{k}_1}{(2\pi)^3}\cdots\int\frac{\mathrm{d}\vec{k}_{\alpha-1}}{(2\pi)^3}\frac{\mathrm{d}^2\omega_{\alpha,\alpha-1}}{\mathrm{d}\Omega_\alpha\,\mathrm{d}U_\alpha}\cdots\frac{\mathrm{d}^2\omega_{2,1}}{\mathrm{d}\Omega_2\mathrm{d}U_2}\frac{\mathrm{d}^2\omega_{1i}}{\mathrm{d}\Omega_1\,\mathrm{d}U_1}. \tag{5}$$

The cross-section is then given by

$$\left[\frac{\mathrm{d}^2\sigma(\vec{k}_f,\vec{k}_i)}{\mathrm{d}\Omega_f\,\mathrm{d}U_f}\right]_{\mathrm{msd}} = \frac{2\pi\mu}{\hbar^2 k_i}\sum_{m,\nu}\int\frac{\mathrm{d}\vec{k}_\nu}{(2\pi)^3}\,\frac{\mathrm{d}^2\omega_{m,\nu}}{\mathrm{d}\Omega_f\,\mathrm{d}U_f}\,Y_\nu(\vec{k}_\nu)=$$

$$=\frac{2\pi\mu}{\hbar^2 k_i}\sum_\nu\left[Y_{\nu+1}(\vec{k}_f)+Y_{\nu-1}(\vec{k}_f)\right]. \tag{6}$$

The function Y_ν satisfied the equation

$$Y_\nu(\vec{k}_\nu)=\int\frac{\mathrm{d}\vec{k}_{\nu-1}}{(2\pi)^3}\,\frac{\mathrm{d}^2\omega_{\nu,\nu-1}(\vec{k}_\nu,\vec{k}_{\nu-1})}{\mathrm{d}\Omega_\nu\,\mathrm{d}U_\nu}\,Y_{\nu-1}(\vec{k}_{\nu-1}). \tag{7}$$

The assumption is now made that the change in momentum in the transition from stage $\nu-1$ to stage ν is small. One may therefore expand $Y_{\nu-1}(\vec{k}_{\nu-1})$ in terms of $Y_{\nu-1}(\vec{k}_\nu)$:

$$Y_{\nu-1}(\vec{k}_{\nu-1})=Y_{\nu-1}(\vec{k}_\nu)+(\vec{k}_{\nu-1}-\vec{k}_\nu)\cdot\nabla Y_{\nu-1}+\frac{1}{2}\left[(\vec{k}_{\nu-1}-\vec{k}_\nu)\cdot\nabla\right]^2 Y_{\nu-1}+\dots$$

Inserting this into Eq. (7) yields:

$$Y_\nu(\vec{k}_\nu)=W_\nu^{(0)}\,Y_{\nu-1}(\vec{k}_\nu)+\vec{W}_\nu^{(1)}(\vec{k}_\nu)\cdot\nabla Y_{\nu-1}(\vec{k}_\nu)+\frac{1}{2}\sum W_{ab}\nabla_a\nabla_b Y_{\nu-1}(\vec{k}_\nu) \tag{8}$$

where

$$W_\nu^{(0)}=\int\frac{\mathrm{d}\vec{k}_{\nu-1}}{(2\pi)^3}\,\frac{\mathrm{d}^2\omega_{\nu,\nu-1}(\vec{k}_\nu,\vec{k}_{\nu-1})}{\mathrm{d}\Omega_\nu\,\mathrm{d}U_\nu} \tag{9a}$$

$$\vec{W}_\nu^{(1)}=\int\frac{\mathrm{d}\vec{k}_{\nu-1}}{(2\pi)^3}(\vec{k}_{\nu-1}-\vec{k}_\nu)\frac{\mathrm{d}^2\omega_{\nu,\nu-1}(\vec{k}_\nu,\vec{k}_{\nu-1})}{\mathrm{d}\Omega_\nu\,\mathrm{d}U_\nu} \tag{9b}$$

and

$$W_{ab}=\int\frac{\mathrm{d}\vec{k}_{\nu-1}}{(2\pi)^3}(\vec{k}_{\nu-1}-\vec{k}_\nu)_a(\vec{k}_{\nu-1}-\vec{k}_\nu)_b\,\frac{\mathrm{d}^2\omega_{\nu,\nu-1}(\vec{k}_\nu,\vec{k}_{\nu-1})}{\mathrm{d}\Omega_\nu\,\mathrm{d}U_\nu}. \tag{9c}$$

The subscripts a and b refer to Cartesian components.

We simplify Eq. (8) by introducing the quantity

$$f_\nu=\prod_1^\nu W_\alpha^{(0)}(\vec{k}_\alpha),\qquad \nu\leq 1,\qquad f_0=1. \tag{10}$$

Note that

$$W_\nu^0 = \frac{f_\nu}{f_{\nu-1}}. \tag{11}$$

We also introduce the new dependent variable

$$Z_\nu \equiv \frac{Y_\nu}{f_\nu}. \tag{12}$$

Equation (8) then becomes

$$W_\nu^{(0)}(Z_\nu - Z_{\nu-1}) = \frac{1}{f_{\nu-1}} [\vec{W}_\nu^{(1)} \cdot \nabla (f_{\nu-1} Z_{\nu+1})] + \frac{1}{2f_{\nu-1}} \sum_{ab} W_{ab} \nabla_a \nabla_b (f_{\nu-1} Z_{\nu-1}). \tag{13}$$

We now take the continuum limit in the variable ν by introducing the continuous variable τ such that $\Delta\tau = 1/W_\nu^0$. Then in Eq. (13) the left-hand side becomes $\Delta Z/\Delta\tau$ which in the limit is $\partial Z/\partial\tau$. Note that

$$\tau = \sum_{\alpha=1}^{\nu} \frac{1}{W_\nu^0}. \tag{14}$$

Qualitatively τ is proportional to the number of steps and can be thought of as the interaction time. It is a function of ν and $\vec{k}_\nu$. In this limit the equation for Z is

$$\frac{\partial Z}{\partial\tau} = \frac{1}{f(\tau, \vec{k})} \left\{ \vec{\omega}_1 \cdot \nabla (fZ) + \frac{1}{2} \sum \omega_{ab} \nabla_a \nabla_b (fZ) \right\} \tag{15}$$

where

$$\vec{\omega}_1 (\tau, \vec{k}) = W_\nu^{(1)} (\vec{k}_\nu), \text{ etc.} \tag{16}$$

Equation (15) is the Fokker-Planck diffusion equation. We approximate it further to obtain an analytically solvable equation. Corrections to these approximations can be readily developed. First we make the reasonable assumption that f varies slowly with $\vec{k}$:

$$f(\tau, \vec{k}) \simeq f(\tau).$$

Second, the simplifying assumption that ω_{ab} is diagonal and independent of $\vec{k}$:

$$\omega_{ab} = \delta_{ab}\omega.$$

Then Eq. (15) becomes

$$\frac{\partial Z}{\partial\tau} = \vec{\omega}_1 \cdot \nabla Z + \frac{1}{2} \omega \nabla^2 Z. \tag{17}$$

To solve this equation we let

$$Z(\vec{k}, \tau) = \left(\frac{1}{2\pi\omega\,\tau}\right)^{3/2} \mathrm{e}^{-\frac{1}{2\tau\omega}\left(\vec{k}-\vec{k}_\tau\right)^2} \tag{18}$$

where $\vec{k}_\tau$ is a function of τ reducing to $\vec{k}_0$ for the collision of light nuclei. For heavy nuclei when the collision is Coulomb dominated, $\vec{k}_0$ is to be taken equal in magnitude to $\vec{k}_i$ but with the direction given by the grazing Coulomb orbit at the point of grazing. The vector $\vec{k}_\tau$ gives the position of the center of a Gaussian. It is a function of τ which in turn is related to the number of steps. As we shall see, it gives the trajectory of the maxima of cross-section in a Wilcyznski plot.

To determine $\vec{k}_\tau$ we calculate

$$\langle\vec{k}\rangle \equiv \int \vec{k} Z(\vec{k}, \tau)\, \mathrm{d}\vec{k}. \tag{19}$$

Equation (17) will then yield a differential equation for $\vec{k}_\tau$.

To be concrete (the choice can be generalized) we assume that $\vec{\omega}_1$ is a linear function of $\vec{k}$:

$$\vec{\omega}_1 = \vec{\omega}_C + \omega_R (\hat{\vec{k}}_0 \times \vec{k}) + \frac{1}{4}\omega_D \vec{k} \tag{20}$$

where $\vec{\omega}_C$ is a constant vector. Multiplying Eq. (17) by $\vec{k}$ and integrating yields

$$\frac{\mathrm{d}\vec{k}_\tau}{\mathrm{d}\tau} = -\left[\vec{\omega}_C + (\hat{\vec{k}} \times \vec{k}_\tau)\,\omega_R + \omega_D \vec{k}_\tau\right]. \tag{21}$$

These equations are be integrated subject to the initial condition $\vec{k}_\tau\,(\tau = 0) = \vec{k}_0$. Let the direction along $\vec{k}_0$ be designated by the 0 subscript. One obtains

$$(\vec{k}_\tau)_0 = \vec{k}_0\, \mathrm{e}^{-\omega_D\tau} - \frac{(\omega_C)_0}{\omega_D}(1 - \mathrm{e}^{-\omega_D\tau}) \tag{22}$$

$$\begin{aligned}(\vec{k}_\tau)_\perp = \frac{1}{\omega_D^2 + \omega_R^2} \{\hat{\vec{k}}_0 \times (\hat{\vec{k}}_0 \times \vec{\omega}_C)\,[\omega_D (1 - \cos\omega_R\tau\, \mathrm{e}^{-\omega_D\tau}) + \\ + \omega_R \sin\omega_R\tau\, \mathrm{e}^{-\omega_D\tau})] + (\hat{k}_0 \times \vec{\omega}_C)\,[\omega_R (1 - \cos\omega_R\tau\, \mathrm{e}^{-\omega_D\tau}) - \\ - \omega_D \sin\omega_R\tau\, \mathrm{e}^{-\omega_D\tau}]\}.\end{aligned} \tag{23}$$

The vector $(\vec{k}_\tau)_1$ is the component of $\vec{k}_\tau$ in the plane perpendicular to $\vec{k}_\nu$.

The sinusoidal terms described a damped rotation of $(k_\tau)_\perp$ with a radius given by $(\omega_C)_\perp/\sqrt{\omega_D^2 + \omega_R^2}$. The value of k_τ^2 which is proportional to the average kinetic energy is given by $(k_\tau)_\perp^2 + (k_\tau)_0^2$

$$(k_\tau)_\perp^2 = \frac{\omega_{C\perp}^2}{\omega_D^2 + \omega_R^2} [1 - 2 \cos \omega_R \tau \, e^{-\omega_D \tau} + e^{-2\omega_D \tau}] \tag{24}$$

$$(k_\tau)_0^2 = \left[\vec{k}_0 \, e^{-\omega_D \tau} - \frac{\omega_{C0}}{\omega_D} (1 - e^{-\omega_D \tau})\right]^2. \tag{25}$$

The Wilczynski plot gives the cross-section contours in the kinetic energy-ϑ plane where ϑ is observation angle. The trajectory maxima of the maxima is given by $\vec{k}_\tau$ where the angle ϑ is given by

$$\tan(\vartheta - \Theta_0) = \frac{(k_\tau)_\perp}{(k_\tau)_0}, \tag{26}$$

where Θ_0 is the direction of $\vec{k}_0$.

The experimental data can be fitted by choosing the parameters ω_D, ω_R and $\vec{\omega}_C$. The width of the cross-section around $\vec{k}_\tau$ will then be given by $\sqrt{2\tau\omega}$ increasing with τ. Note that these phenomenological parameters can be compared with the more microscopic values obtained from Eq. (9).

The function $Y(\vec{k}, \tau)$ is obtained from Z by using Eq. (12). Making the assumption that W_α^0 is independent of α, $f_\nu = W_0^\nu$ and converting from ν dependence to τ dependence, we parametrize f_ν by

$$f_\nu = f_0 \, e^{-\gamma\tau} \tag{27}$$

since τ is proportional to ν. The cross-section can be obtained, according to Eq. (6), by summing over ν. The sum of ν is replaced by an integration:

$$\Sigma_\nu \to \frac{1}{W^0} \int d\tau \tag{28}$$

so that

$$\left[\frac{d^2\sigma}{d\Omega_f \, dU_f}\right]_{msd} = \frac{4\pi\mu}{\hbar^2 k_i} \frac{1}{W_0} \int Y(\vec{k}, \tau) \, d\tau. \tag{29}$$

We give the results of this calculation assuming

$$\omega_R = 0, \qquad \vec{k}_\tau = \vec{k}_0 + \vec{k}_1 \tau$$
$$\vec{k}_1 = -(\vec{\omega}_C + \vec{k}_0 \omega_D). \tag{30}$$

These restrictions can be readily removed. The integral can of course always be computed numerically. We obtain

$$\left[\frac{d^2\sigma}{d\Omega_f\, dU_f}\right]_{\mathrm{mds}} = \frac{4\pi\mu}{\hbar^2 k_i}\frac{1}{W_0} I(\vec{k}_f) \tag{31}$$

where

$$I(\vec{k}) = \frac{f_0}{\sqrt{2}(2\pi\omega)}\frac{1}{|\vec{k}_0 - \vec{k}|}\exp\left\{-\frac{1}{\omega}\left[K_1|\vec{k}_0 - \vec{k}| + \vec{k}_1 \cdot (\vec{k}_0 - \vec{k})\right]\right\}. \tag{32}$$

The quantity K_1 is

$$K_1^2 \equiv k_1^2 + 2\gamma\omega. \tag{33}$$

The energy distribution at each angle can be determined from Eq. (32). Generally one finds, in agreement with experiment, that as the angle increases the rate of the exponential decrease increases. One thus finds the statement in the literature that the »temperature« decreases with increasing angle. However as we see from the above calculation one cannot conclude that the system has achieved thermal equilibrium. For that conclusion to be correct the »temperature« must be independent of angle.

We conclude our discussion by extending the above analysis to include the dependence of the mass and charge parameters $\eta^{(A)}$ and $\eta^{(Z)}$. The details are similar so that we shall not give them here. We shall just quote the result analogous to Eq. (17):

$$\frac{\partial Z}{\partial \tau} = \vec{\omega}_1 \cdot \nabla Z + \omega^{(0)}\frac{\partial Z}{\partial \eta} + \frac{1}{2}\omega\nabla^2 Z + \vec{\omega}_{11} \cdot \nabla\left(\frac{\partial Z}{\partial \eta}\right) + \frac{1}{2}\omega_2\frac{\partial^2 Z}{\partial \eta^2}. \tag{34}$$

The interesting feature here is the coupling between the dependence on $\vec{k}$ and the dependence on η as given by the fourth term. As a consequence, the mass and charge distributions are no longer symmetric with respect to the maximum of the distributions.

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FOKKER-PLANCKOVA JEDNADŽBA U IMPULSNOM PROSTORU ZA DUBOKO NEELASTIČNA RASPRŠENJA TEŠKIH IONA

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Razvijena je Fokker-Planckova jednadžba u impulsnom prostoru za duboko neelastična raspršenja. Diskutirana su dobivena aproksimativna rješenja te jednadžbe. Povećanjem kuta smanjuje se »temperatura«, što je u skladu s eksperimentom.