YU ISSN 0015—3206
FZKAAA 22 (1) 89 (1990)

REALIZATION OF Sp (4, *R*) MODEL USING THE GENERATING INVARIANT TECHNIQUE

GENNADY F. FILIPPOV

Institute for Theoretical Physics, Metrologicheskaya, 14-b, Kiev-130, USSR

Received 25 September 1989

UDC 539.142

Original scientific paper

The complete basis of the Sp (4, *R*) model for closed shells is constructed in the space of generating parameters by means of the generating invariant technique. The matrix elements of many-body microscopic Hamiltonian are calculated between the basis functions of this model.

Having accepted the kind proposal by Prof. V. Paar to participate in the Memorial volume, I got the possibility to pay the memory debt to Gaja Alaga, the excellent physicist whose important contribution into the nuclear physics is undoubted.

I had the only occasion to meet G. Alaga. That was in the spring of 1972 in Crimea at the School on nuclear spectrocsopy. Nevertheless his works had become known to me much earlier.

In those romantic days of nuclear physics when the ideas of Bohr and Mottelson had been perceived as a foundation of the consistent theory of collective excitations, Alaga introduced rather simple rules (then called the Alaga rules) which formed an essential addition to the quasimolecular model. The time passed. The primary treatment of the quasimolecular model changed but the Alaga rules conserved their significance as a considerable source of information on the collective excitation wave function for the atomic nuclei.

In recent years the symplectic models were proposed to analyse the collective motion within the microscopic approach. The dynamical equations due to these models take a simple form in the generator coordinate space. A convenient way of passing to the latter is to make use of generating functions (generalized coherent states).

Henceforth we proceed in the discussion of the problems appearing when realizing the Sp (4, R) model.

1. The Sp (4, R) model has proved to be an important intermediate stage in the developing of theoretical conceptions on the nature of quadrupole collective vibrations in atomic nuclei. As compared to the Sp (2, R) model, being well developed now, it includes into consideration not only the longitudinal but also the transversal quadrupole vibrations which can form together the structure of giant quadrupole resonance states. At the same time it does not exhaust all possible quadrupole modes which are the subject of investigation for more complete and more complex Sp (6, R) model.

The basic theoretical problem that arises when realizing the Sp (4, R) model consists in constructing the multiparticle Hamiltonian matrix elements between the basis functions of an irreducible representation (irrep) of the Sp (4, R) group which is induced by a definite intrinsic function[1]. If we know the Hamiltonian matrix elements between the basis functions of the model, we can proceed with solving its dynamical equations and discussing, in what follows, the adequacy of its postulates. In Ref. 2 it was shown that a convenient tool to realize this programme proved to be the generating invariants (the generating functions) that yield the multiparticle oscillator basis of the corresponding irreducible representation. The problem of constructing the Hamiltonian matrix elements (see Ref. 3) on the basis functions of the Sp (4, R) model is then reduced to constructing the Hamiltonian matrix elements on basis functions of the Sp (4, R) model. The purpose of our paper is to find the form of these matrix elements for the magic nuclei.

We remind first the basic ideas of the symplectic approach. Among all collective vibrations of nucleon systems the Sp (6, R) model selects those which correspond to variations only in the components of the mass quadrupole momentum. The description of collective quadrupole modes is usually achieved by the Slater determinants composed of orbitals of the anisotropic harmonic oscillator with different frequencies ω_1, ω_2, ω_3 along three principal directions. The simplest example of anisotropic field orbitals is the states of the Nilsson scheme[4].

The restrictions on the nucleon system wave function due to the space homogeneity and isotropy lead to the necessity of performing additional transformations of Slater determinants. First, one should separate the centre-of-mass motion and then perform a rotation of the principal axes of the harmonic oscillator deformation. As a result we get the multiparticle wave function

$$\Phi(\{x_i\}; \omega_1, \omega_2, \omega_3; \varphi, \Theta, \Psi)$$

containing six parameters — three frequencies and three Euler angles. This function is capable to reproduce the quadrupole vibrations of nucleon systems with isotropic internal state (such as ^{4}He, ^{16}O, ^{40}Ca). Arbitrary quadrupole excitation $\Psi(\{x_i\})$ of these systems can be represented in the form of the Hill-Wheeler integral[5]

$$\Psi(\{x_i\}) = \int\int\int d\omega_1 d\omega_2 d\omega_3 \int d\Omega C(\omega_1, \omega_2, \omega_3; \varphi, \Theta, \Psi) \cdot \\ \cdot \Phi(\{x_i\}; \omega_1, \omega_2, \omega_3; \varphi, \Theta, \Psi) \qquad (1)$$

of the function Φ and coefficients C. The differential wave equation for the functions Ψ is then reduced to the integral equation of the generator coordinates method for the coefficients C. The dynamic symmetry group of the wave equation for quadrupole excitations is the symplectic Sp (6, R) group. That explains the appearance of the Sp (6, R) model of collective quadrupole vibrations[6,7].

Within the framework of the symplectic approach the function Φ has the sense not only as the Hill-Wheeler integral transformation kernel but also as the generating invariant yielding the basis of the $\left[\left(\frac{1}{2}f + \frac{1}{4}(A-1)\right)^3\right]$ irrep of the Sp (6, R) group (f is the number of oscillator quanta along each of three Cartesian axes of the shell-model space of magic nucleus with A nucleons). If instead of frequencies $\omega_1, \omega_2, \omega_3$ one introduces the new generator parameters

$$\beta_1 = \frac{\omega_1 - 1}{\omega_1 + 1}, \quad \beta_2 = \frac{\omega_2 - 1}{\omega_2 + 1}, \quad \beta_3 = \frac{\omega_3 - 1}{\omega_3 + 1}, \tag{2}$$

then the coefficients of expansion of the function Φ in powers of $\beta_1, \beta_2, \beta_3$ will be the basis states of multidimensional isotropic harmonic oscillator with the definite quantum numbers of the Sp (6, R) group reduced into its subgroups.

2. The function Φ generating the basis of the $\left[\left(\frac{1}{2}f + \frac{A-1}{4}\right)^3\right]$ irrep of the Sp (6, R) group maps the basis functions defined in the coordinate, spin and isospin space onto the basis functions in the space of generator parameters β_1, β_2, β_3, φ, Θ, Ψ. The latter represent the expressions of the form

$$U^{N(\lambda\mu)}_{\tau LM} = \sum_K u^{N(\lambda\mu)}_{\tau LK}(\beta_1, \beta_2, \beta_3)\, D^L_{MK}(\varphi, \Theta, \Psi), \tag{3}$$

where $U^{N(\lambda\mu)}_{\tau LM}$ are the homogeneous polynomials composed of integer powers of β_1, β_2, β_3, whose homogeneity degree equals to the total number of oscillator quanta of the basis state excitation; $(\lambda\mu)$ are the quantum numbers of $SU(3)$ symmetry of the basis state; L is the total orbital momentum, M is its projection onto the external axis and K is the projection of L onto the internal axis in the space of generator parameters, τ is the additional quantum number.

The basis functions $U^{N(\lambda\mu)}_{\tau LM}$ depend on such number of variables which is exactly equal to the number of quadrupole degrees of freedom* and therefore they are much more simple than their originals $\Phi^{N(\lambda\mu)}_{\tau LM}(\{x_i\})$ in a configuration space of nucleon coordinates $\{x_i\}$. We remind that the originals $\Phi^{N(\lambda\mu)}_{\tau LM}$ are represented by the integrals of complex superpositions of Slater determinants. They depend on the great number of variables and their description in the explicit form turns out to be an uneasy task. Meanwhile one manages to reduce the polynomials $u^{N(\lambda\mu)}_{\tau LK}(\beta_1$, β_2, $\beta_3)$ to standard expressions, with the guarantee of the fulfilment of all conservation laws and a priori requirements of permutational symmetry (nucleon

* There are six degree of freedom altogether — five properly quadrupole and one monopole.

permutations and permutations of axis of the internal reference frame) and then to investigate them with simpler means. The most important practical consequence of a transition to the space of generator parameters is the possibility to find a simple form for the dynamic equations of collective quadrupole excitations. To realize this possibility it is necessary to calculate firstly the overlap integrals of unit and Hamiltonian operator, $\langle \Phi | \tilde{\Phi} \rangle$ and $\langle \Phi | H | \tilde{\Phi} \rangle$, between the generating invariants

$$\Phi = \Phi (\{x_i\}; \beta_1, \beta_2, \beta_3; \varphi, \Theta, \Psi), \quad \tilde{\Phi} = \Phi (\{x_i\}; \beta_1, \beta_2, \beta_3; \tilde{\varphi}, \tilde{\Theta}, \tilde{\Psi})$$

which differ in the generator parameters values. The overlap integral $\langle \Phi | \tilde{\Phi} \rangle$ is by itself the generating invariant and generates the basis $U_{\tau LM}^{N(\lambda\mu)} = U_{\tau LM}^{(\lambda N\mu)} (\beta_1, \beta_2, \beta_3; \varphi, \Theta, \Psi)$, if we use β_1, β_2, β_3, φ, Θ, Ψ, as the generating parameters, and the basis $\tilde{U}_{\tau LM}^{N(\lambda\mu)} = U_{\tau LM}^{N(\lambda\mu)} (\beta_1, \beta_2, \beta_3; \varphi, \Theta, \tilde{\Psi})$ if the generator parameters $\tilde{\beta}_1$, $\tilde{\beta}_2$, $\tilde{\beta}_3$, $\tilde{\varphi}$, $\tilde{\Theta}$, $\tilde{\Psi}$ are employed. Therefore there holds the equality

$$\langle \Phi | \tilde{\Phi} \rangle = \sum_{N} \sum_{(\lambda\mu)} \sum_{\tau} \sum_{LM} U_{\tau LM}^{N(\lambda\mu)} \tilde{U}_{\tau LM}^{N(\lambda\mu)*}. \tag{4}$$

After the basis functions $U_{\tau LM}^{N(\lambda\mu)}$ and $\tilde{U}_{\tau LM}^{N(\lambda\mu)}$ are constructed from the overlap integral $\langle \Phi | \tilde{\Phi} \rangle$, we can, by using the other overlap integral $\langle \Phi | \hat{H} | \tilde{\Phi} \rangle$, calculate the Hamiltonian matrix elements between these basis functions from the expansion

$$\langle \Phi | \hat{H} | \tilde{\Phi} \rangle = \sum_{N, \tilde{N}} \sum_{\substack{(\lambda\mu) \\ (\tilde{\lambda}, \tilde{\mu})}} \sum_{LM} \sum_{\tau, \tilde{\tau}} U_{\tau LM}^{N(\lambda\mu)} \langle N(\lambda\mu), \tau LM | \hat{H} | \tilde{N} (\tilde{\lambda} \tilde{\mu}), \tilde{\tau} LM \rangle \tilde{U}_{\tilde{\tau} LM}^{\tilde{N}(\tilde{\lambda} \tilde{\mu})*}. \tag{5}$$

The calculation of the Hamiltonian matrix elements between basis functions of the Sp (6, R) model completes the construction of dynamic equations of this model. Afterwards one can proceed with their numerical solution and the analysis of theoretical spectrum of the collective quadrupole excitations. Such is, generally speaking, the programme of a symplectic approach and it should be carried out in order to make the Sp (6, R) model a practical tool of studying collective quadrupole excitations of atomic nuclei on the basis of microscopic theory.

3. The Sp (4, R) model can be considered as a particular case of the Sp (6, R) model transition to which is performed by assuming

$$\beta_3 = \tilde{\beta}_3 = 0, \qquad \varphi = \Theta = \tilde{\Theta} = \tilde{\varphi} = 0$$

in the generating functions Φ and $\tilde{\Phi}$ of the Sp (6, R) model.

For the light magic nuclei (^{4}He, ^{16}O, ^{40}Ca) the overlap integral of the Sp (4, R) model generating invariants Φ and $\tilde{\Phi}$ has the following form

$$\langle \Phi | \tilde{\Phi} \rangle = \Delta^{-I}, \quad I = f + \frac{A-1}{2},$$

$$\Delta = 1 - \sum_{\nu \tilde{\nu}} \beta_\nu \beta_{\tilde{\nu}}^2 d_{\nu\tilde{\nu}}^2 + a\tilde{a}, \quad \nu, \tilde{\nu} = 1, 2, \tag{6}$$

where $d_{11}^2 = d_{22}^2 = \cos^2 \Psi$, $d_{12}^2 = d_{21}^2 = \sin^2 \Psi$, $a = \beta_1 \beta_2$, $\tilde{a} = \tilde{\beta}_1 \tilde{\beta}_2$. The formulae (6) follow immediately from the expression obtained in Ref. 2 for the overlap integrals of the Sp (6, R) model, as a particular case. It is known[8] that the quantum numbers of the Sp (4, R) model basis states coincide with those of the three-dimensional harmonic oscillator states (n, l, m, where n is the total number of oscillator quanta of the state, l is the orbital angular momentum, m is its projection onto the selected direction). Therefore the following expansion should take place

$$\Delta^{-I} = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{m=-(n-2k)}^{n-2k} B(I, nk)\, r^n\, Y_{n-2k,m}(\Theta, \varphi)\, \tilde{r}^n\, Y_{n-2k,m}(\tilde{\Theta}, \tilde{\varphi}). \tag{7}$$

We put $l = n - 2k$ in correspondence with the fact that the orbital angular momentum l can not exceed n, and its parity coincides with that of n. The functions $r^n Y_{lm}(\Theta\varphi)$ and $\tilde{r}^n Y_{lm}(\tilde{\Theta}\tilde{\varphi})$ introduced in (7) are the images of basis functions of the three-dimensional oscillator in the space of generator parameters r, Θ, φ and $\tilde{r}$, $\tilde{\Theta}$, $\tilde{\varphi}$. The latter are connected with the generator parameters β_1, β_2, $\tilde{\beta}_1$, $\tilde{\beta}_2$, $\Psi = \Psi - \tilde{\Psi}$ by the relations

$$\begin{aligned} r \sin\Theta \cos\varphi &= \frac{1}{2}(\beta_1 - \beta_2)\cos 2\Psi, \\ r \sin\Theta \sin\varphi &= \frac{1}{2}(\beta_1 - \beta_2)\sin 2\Psi, \\ r \cos\Theta &= \frac{i}{2}(\beta_1 + \beta_2), \end{aligned} \tag{8a}$$

$$\begin{aligned} \tilde{r} \sin\tilde{\Theta} \cos\tilde{\varphi} &= \frac{1}{2}(\tilde{\beta}_1 - \tilde{\beta}_2)\cos 2\tilde{\Psi}, \\ \tilde{r} \sin\tilde{\Theta} \sin\tilde{\varphi} &= \frac{1}{2}(\tilde{\beta}_1 - \tilde{\beta}_2)\sin 2\tilde{\Psi}, \\ \tilde{r} \cos\tilde{\Theta} &= \frac{i}{2}(\tilde{\beta}_1 + \tilde{\beta}_2). \end{aligned} \tag{8b}$$

In the formulae (7), (8a), (8b) the notations φ and $\tilde{\varphi}$ having the quality, different from that in paragraph 1, are introduced.

To construct explicitly the expansion (7), we express the overlap integral (6) in terms of the new generator parameters

$$\Delta^{-I} = (1 - 2r\tilde{r}t + r^2\tilde{r}^2)^{-I}, \quad t = \sin\Theta \sin\tilde{\Theta}\cos(\varphi - \tilde{\varphi}) + \cos\Theta\cos\tilde{\Theta}. \tag{9}$$

But the r. h. s. of (9) represents the generating function for the Gegenbauer polynomials $C_n^I(t)$, therefore the overlap integral expansion in powers of $(r\tilde{r})^n$ is a trivial problem:

$$\Delta^{-I} = \sum_{n=0}^{\infty} r^n \tilde{r}^n C_n^I(t). \tag{10}$$

And now we only need to represent the Gegenbauer polynomial as series in the Legendre polynomials and then to use the familiar relation between the Legendre polynomials and the spherical functions. It is easy to show that

$$C_n^I(t) = \sum_{k=0}^{\left[\frac{n}{2}\right]} C(I, n-2k, k)\, P_{n-2k}(t), \tag{11}$$

$$C\left(p+\frac{1}{2}, n-2k, k\right) = \frac{(2n-4k+1)\,\Gamma(3/2)\,\Gamma(2p+2k)\,\Gamma\left(p+n-k+\frac{1}{2}\right)}{2^{2k}\,k!\,\Gamma\left(n-k+\frac{3}{2}\right)\Gamma(2p)\,\Gamma\left(p+k+\frac{1}{2}\right)},$$

$$p > 0$$

$$C\left(\frac{1}{2}, n-2k, k\right) = \delta_{k0}.$$

Besides, the Legendre polynomials are expressed in terms of spherical functions as follows:

$$P_l(t) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\Theta\varphi)\, Y_{lm}^{*}(\tilde{\Theta}\tilde{\varphi}). \tag{12}$$

Therefore, using (10), (11) and (12), we obtain

$$B(I, n, k) = \frac{4\pi}{2n-2k+1}\, C(I; n-2k, k). \tag{13}$$

Thus we have determined the square of the basis function norm in the generator parameter space. The normalized basis functions are written as

$$\Phi_{NkM} = \sqrt{B(I; N, k)}\,(\mathrm{i}r)^N\, Y_{N-2k,M}(\Theta\varphi),$$

$$\tilde{\Phi}_{\tilde{N}\tilde{k}\tilde{M}} = \sqrt{B(I; \tilde{N}, \tilde{k})}\,(\mathrm{i}\tilde{r})^N\, Y_{\tilde{N}-2\tilde{k},\tilde{M}}(\Theta\varphi).$$

The phase factors i^N and $\mathrm{i}^{\tilde{N}}$ are introduced in order to make all Hamiltonian matrix elements real. It is obvious that

$$\langle\Phi|\tilde{\Phi}\rangle = \sum_{N=0}^{\infty} \sum_{k=0}^{\left[\frac{N}{2}\right]} \sum_{M=-(N-2k)}^{N-2k} \Phi_{NkM}\, \tilde{\Phi}^*{}_{NkM}.$$

Knowing the normalization coefficient, one can proceed with extracting the Hamiltonian matrix elements between the Sp (4, R) model basis functions from the overlap integral of generating invariants with the Hamiltonian

$$\hat{H} = \hat{T} + \hat{V}.$$

4. The principal difficulty to be considered is the calculation of matrix elements of the potential energy operator. The overlap integral of generating invariants Φ and $\tilde{\Phi}$ with the potential energy operator

$$V = \sum_{i<j}^{A} V_0 \exp\left[-\frac{1}{b_0^2}(\vec{r}_i - \vec{r}_j)^2\right]$$

is reduced to a superposition of the derivatives of different order over the dynamic parameter $\gamma = -\frac{2r_0^2}{b_0^2}$, where r_0 is the oscillator radius, of the expression

$$\langle\Phi|V_\gamma(I)|\Phi\rangle = \Delta_\gamma^{-1/2}\,\Delta^{-I+\frac{1}{2}}, \tag{14}$$

obtained in Ref. 2. The simplest definition is given by the relation

$$\Delta_\gamma = (1-\gamma^2)^{-2}\bar{\Delta}, \quad \bar{\Delta} = 1 - \sum_{\nu\tilde{\nu}} \bar{\beta}_\nu \tilde{\bar{\beta}}_{\tilde{\nu}}\, d^2_{\nu\tilde{\nu}} + \bar{\alpha}\tilde{\bar{\alpha}}$$

$$\bar{\beta}_\nu = (1+\gamma)\left(\beta_\nu - \frac{\gamma}{1+\gamma}\right), \quad \tilde{\bar{\beta}}_\nu = (1+\gamma)\left(\tilde{\beta}_\nu - \frac{\gamma}{1+\gamma}\right) \tag{15}$$

$$\bar{\alpha} = \bar{\beta}_1\bar{\beta}_2, \quad \tilde{\bar{\alpha}} = \tilde{\bar{\beta}}_1\tilde{\bar{\beta}}_2.$$

The overlap integral (14) should be written as the expansion in basis functions of the Sp (4, R) model

$$\Delta_{\gamma}^{-1/2}\,\Delta^{-I+1/2} = \sum_{N=0}^{\infty}\sum_{\widetilde{N}=0}^{\infty}\sum_{p=0}^{\left[\frac{N}{2}\right]}\sum_{\widetilde{p}=0}^{\left[\frac{\widetilde{N}}{2}\right]}\sum_{M=-\mu}^{\mu}\left[B(I;N,p)\,B(I;\widetilde{N},\widetilde{p})\right]^{1/2}\times$$

$$\times\,(\mathrm{i}r)^{N}\,Y_{N-2p,M}(\Theta\varphi)\,(-\mathrm{i}\widetilde{r})^{\widetilde{N}}\,Y_{\widetilde{N}-2\widetilde{p},M}{}^{*}(\widetilde{\Theta}\widetilde{\varphi})\times \tag{16}$$

$$\times\,\langle N, N-2p, M\,|\,\hat{V}_{\gamma}(I)\,|\,\widetilde{N}, \widetilde{N}-2\widetilde{p}, M\rangle;$$

$$\mu = \min(\widetilde{N}-2\widetilde{p}, N-2p),$$

and as a result the matrix elements $\langle N, N-2p, M|\,V_{\gamma}(I)\,|\widetilde{N}, \widetilde{N}-2\widetilde{p}, M\rangle$ should be obtained.

First we take up the multiplier $\Delta_{\gamma}^{-1/2}$. We note that

$$\Delta_{\gamma}^{-1/2} = (1-\gamma^{2})\sum_{n=0}^{\infty}\frac{4\pi}{2n+1}\sum_{m=-n}^{n}(\overline{r}\,\overline{\widetilde{r}})^{n}\,Y_{nm}(\overline{\Theta}\overline{\varphi})\,Y_{nm}^{*}(\overline{\widetilde{\Theta}}\,\overline{\widetilde{\varphi}}). \tag{17}$$

The definition of $\overline{\gamma}, \overline{\Theta}, \overline{\varphi}$ $(\overline{\widetilde{r}}, \overline{\widetilde{\Theta}}, \overline{\widetilde{\varphi}})$ via the parameters $\overline{\beta}_1, \overline{\beta}_2, \overline{\Psi}$ $(\overline{\widetilde{\beta}}_1, \overline{\widetilde{\beta}}_2, \overline{\widetilde{\Psi}})$ is analogous to the definition of γ, Θ, φ $(\widetilde{r}, \widetilde{\Theta}, \widetilde{\varphi})$ through β_1, β_2, Ψ $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\Psi})$ (see (8a), (8b)). Therefore

$$\overline{r}\sin\overline{\Theta}\cos\overline{\varphi} = (1+\gamma)\,r\sin\Theta\cos\varphi,$$

$$\overline{r}\sin\overline{\Theta}\sin\overline{\varphi} = (1+\gamma)\,r\sin\Theta\sin\varphi, \tag{18a}$$

$$\overline{r}\cos\overline{\Theta} = (1+\gamma)\left(r\cos\Theta - \frac{\mathrm{i}\gamma}{1+\gamma}\right),$$

$$\overline{r}\sin\overline{\widetilde{\Theta}}\cos\overline{\widetilde{\varphi}} = (1+\gamma)\,\widetilde{r}\sin\widetilde{\Theta}\cos\widetilde{\varphi},$$

$$\overline{r}\sin\overline{\widetilde{\Theta}}\sin\overline{\widetilde{\varphi}} = (1+\gamma)\,\widetilde{r}\sin\widetilde{\Theta}\sin\widetilde{\varphi},$$

$$\overline{\widetilde{r}}\cos\overline{\widetilde{\Theta}} = (1+\gamma)\left(\widetilde{r}\cos\widetilde{\Theta} - \frac{\mathrm{i}\gamma}{1+\gamma}\right). \tag{18b}$$

The return from parameters $\overline{r}, \overline{\Theta}, \overline{\varphi}$ to r, Θ, φ is equivalent to the change of the radial scale by a factor of $1+\gamma$ and the subsequent translation along the Z axis by the distance of $i\gamma/(1+\gamma)$. This transition to the parameters r, Θ, φ corresponds to the familiar transformation of the solid spherical harmonics

$$\bar{r}^n\, Y_{nm}(\bar{\Theta}\bar{\varphi}) =$$

$$= \sum_{l=0}^{n} (1+\gamma)^n \left[\frac{(2n+1)(n+m)!\,(n-m)!}{(2l+1)(l+m)!\,(l-m)!}\right]^{1/2} \frac{1}{(n-l)!}\, a^{n-l} r^l\, Y_{lm}(\Theta\varphi), \quad (19a)$$

$$a = -\frac{i\gamma}{1+\gamma}.$$

The transformation of solid spherical harmonics $\tilde{\bar{r}}^n\, Y_{nm}(\tilde{\bar{\Theta}}\tilde{\bar{\varphi}})$ obeys the same law

$$\tilde{\bar{r}}^n\, Y_{nm}(\tilde{\bar{\Theta}}\tilde{\bar{\varphi}}) =$$

$$= \sum_{l=0}^{n} (1+\gamma)^n \left[\frac{(2n+1)(n+m)!\,(n-m)!}{(2l+1)(l+m)!\,(l-m)!}\right]^{1/2} \frac{1}{(n-l)!}\, a^{n-l} \tilde{r}^l\, Y_{lm}(\tilde{\Theta}\tilde{\varphi}). \quad (19b)$$

After the simple calculations one can write the expansion of the overlap integral $\Delta_\gamma^{-1/2}$ through the basis states of the Sp (4, R) model

$$\Delta_\gamma^{-1/2} = \sum_{l=0}^{\infty} \sum_{\tilde{l}=0}^{\infty} \sum_{m=-\bar{l}}^{\bar{l}} D(l, \tilde{l}, m, \gamma)\,(ir)^l\, Y_{lm}(\Theta\varphi)\, Y^*_{\tilde{l}m}(\tilde{\Theta}\tilde{\varphi})\,(-i\tilde{r})^{\tilde{l}},$$

$$D(l, \tilde{l}, m, \gamma) = 4\pi \left(\frac{\gamma}{1-\gamma}\right)^{l+\tilde{l}} \gamma^{2m}\,(l+\tilde{l})! \times$$

$$\times \left[(2l+1)(2\tilde{l}+1)(l+m)!\,(l-m)!\,(\tilde{l}+m)!\,(\tilde{l}-m)!\right]^{-1/2} \times$$

$$\times F\left(-l-m, -\tilde{l}-m, -l-\tilde{l}; 1-\frac{1}{\gamma^2}\right), \quad (20)$$

$$\bar{l} = \min(l, \tilde{l}).$$

It immediately follows from (7) that

$$\Delta^{-I+1/2} = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \sum_{m'=-(n-2k)}^{n-2k} B\left(I-\frac{1}{2};\, n, k\right) \times$$

$$\times\, r^n\, Y_{n-2k,m'}(\Theta\varphi)\, \tilde{r}^n\, Y_{n-2k,m'}(\tilde{\Theta}\tilde{\varphi}). \quad (21)$$

It is only necessary now to single out the Sp (4, R) basis states from the product of the overlap integrals $\Delta_{\gamma}^{-1/2}$ and $\Delta^{-1/2}$. It can be done by the recoupling of spherical harmonics using the Clebsch-Gordan coefficients $C^{LM}_{l_1 m_1 l_2 m_2}$. One obtains:

$$\langle N, N-2p, M|\, \hat{V}_{\gamma}(I)\,|\tilde{N}, \tilde{N}-2\tilde{p}, M\rangle = [B(I;\, n, p)\, B(I; \tilde{N}, \tilde{p})]^{-1/2} \times$$

$$\times \sum_{k=0}^{\min(p,\tilde{p})} \sum_{q=\max(p,\tilde{p})}^{\min(N-p,\tilde{N}-\tilde{p})} \sum_{m=-(\tilde{N}-k-q)}^{\tilde{N}-k-q} \sum_{m'=-(q-k)}^{q-k} \delta_{m+m',n} \times$$

$$\times \left[\frac{2N-2k-2q+1}{2N-4p+1} \cdot \frac{2\tilde{N}-2k-2q+1}{2\tilde{N}-4\tilde{p}+1}\right]^{1/2} (2q-2k+1)\times \tag{22}$$

$$\times\, C^{N-2p,0}_{N-k-q,0\; q-k,0}\; C^{\tilde{N}-2\tilde{p},0}_{\tilde{N}-k-q,\,0\; q-k,\,0}\; C^{N-2p\,M}_{N-k-q,\,m\; q-k,\,m'}\; C^{\tilde{N}-2\tilde{p}\,M}_{\tilde{N}-k-q,\,m\; q-k,m'} \times$$

$$\times\, D(N-q-k, \tilde{N}-q-k, m, \gamma)\; B\left(I-\frac{1}{2};\; q+k, k\right).$$

For the ^{4}He nucleus the matrix elements of the potential energy operator $\hat{V}(A=4)$ differ from those of the operator $\hat{V}_{\gamma}(3/2)$ only by the factor of

$$3\,(V_{31}+V_{13})\,(1-\gamma)^{-3/2},$$

$$\langle N, N-2p, M|\, \hat{V}(A=4)\,|\tilde{N}, \tilde{N}-2\tilde{p}, M\rangle = 3\,(V_{31}+V_{13})\,(1-$$

$$-\gamma)^{3/2}\; \langle N, N-2p, M|\, \hat{V}_{\gamma}\left(\frac{3}{2}\right) |\tilde{N}, \tilde{N}-2\tilde{p}, M\rangle. \tag{23}$$

As usually, $V_{2S+1,2T+1}$ is the intensity of the Gauss potential describing the interaction of the nucleon pair having total spin S and isospin T. In general for the light magic nuclei there holds the following formula[2)]

$$\langle N, N-2p, M|\, \hat{V}(A)\,|\tilde{N}, \tilde{N}-2\tilde{p}, M\rangle =$$

$$= V_+ \left[L_n^3\left(-\gamma\frac{\mathrm{d}}{\mathrm{d}\gamma}\right)\right]^2 (1-\gamma)^{3/2}\, \langle N, N-2p, M|\, \hat{V}_{\gamma}(I)\,|\tilde{N}, \tilde{N}-2\tilde{p}, M\rangle -$$

$$- V_- \left[L_n^3\left(-\frac{\eta^2}{\gamma}\frac{\partial}{\partial\eta}\right)\right]^2 (1-\eta)^{-3/2} \times$$

$$\times\, \eta^3\, \langle N, N-2p, M|\, \hat{V}_{\gamma}(I)\,|\tilde{N}, \tilde{N}-2\tilde{p}, M\rangle|_{\eta=\gamma}, \tag{24}$$

where $L_n^3(x)$ is the Laguerre polynomial and V_+ and V_- are the known superpositions of Gauss potential intensities for the integral of the direct interaction

$$V_+ = \frac{1}{2}(9V_{33} + 3V_{31} + 3V_{13} + V_{11})$$

and that of the exchange one

$$V_- = \frac{1}{2}(9V_{33} - 3V_{31} - 3V_{13} + V_{11}).$$

In the case of the ^{16}O nucleus we should put $A = 16$, $n = 1$, $I = \frac{23}{2}$ and for the nucleus ^{40}Ca — $A = 40$, $n = 2$, $I = \frac{79}{2}$.

The last point is the calculation of matrix elements of the kinetic energy operator $\hat{T}$. It is performed by the scheme presented in Ref. 3. The overlap integral of generating invariants with the kinetic energy operator is written in the form

$$\langle \Phi | \hat{T} | \hat{\Phi} \rangle = \frac{\hbar^2}{\mu r_0^2} \left\{ \frac{3}{2} I \Delta^{-I} + \frac{1}{2} I \Delta^{-I-1} \left[2 \sum_{\nu\tilde{\nu}} \beta_\nu \tilde{\beta}_{\tilde{\nu}} d^2_{\nu\tilde{\nu}} - \right.\right.$$

$$\left.\left. - 4a\tilde{a} + \beta_1 + \beta_2 + \tilde{\beta}_1 + \tilde{\beta}_2 - (\beta_1 + \beta_2)\tilde{a} - (\tilde{\beta}_1 + \tilde{\beta}_2) a \right] \right\}, \quad (25)$$

where μ is the nucleon mass. Below we quote only the results of calculations performed for matrix elements between the basis functions

$$\langle N, N - 2k, M | \hat{T} | N, N - 2k, M \rangle = \frac{\hbar^2}{\mu r_0^2} \left(N + \frac{3}{2} I \right); \quad (26a)$$

$$\langle N, N - 2k, M | \hat{T} | N + 1, N + 1 - 2k - 2, M \rangle =$$

$$= \frac{1}{2} \frac{\hbar^2}{\mu r_0^2} [B(I; N, k) B(I; N + 1, k + 1)]^{-1/2} B(I + 1; N, k) \times$$

$$\times \frac{I(2I - 1)}{I + N - k} \left[\frac{(N - 2k + M)(N - 2k - M)}{(2N - 4k + 1)(2N - 4k - 1)} \right]^{1/2}, \quad (26b)$$

$$\langle N, N, M | \hat{T} | N + 1, N + 1, M \rangle =$$

$$= \frac{1}{2} \frac{\hbar^2}{\mu r_0^2} [B(I; N, 0) B(I; N + 1, 0)]^{-1/2} \times$$

$$\times B(I + 1; N, 0) \times 2I \left[\frac{(N + 1 + M)(N + 1 - M)}{(2N + 1)(2N + 3)} \right]^{1/2}; \quad (26c)$$

and if $k \neq 0$, we have instead of (26c),

$$\langle N, N-2k, M | \hat{T} | N+1, N+1-2k, M \rangle =$$

$$= \frac{1}{2} \frac{\hbar^2}{\mu r_0^2} [B(I, N, k) B(I; N+1, k)]^{-1/2} B(I+1; N, k) \times$$

$$\times \frac{2I(2I-1)}{2I+2k-1} \left[\frac{(N-2k+M+1)(N-2k-M+1)}{(2N-4k+3)(2N-4k+1)} \right]^{1/2} . \qquad (26d)$$

We have at our disposal all expressions necessary to calculate, within the Sp (4, R) model, the light magic nuclei collective excitation spectrum.

Acknowledgement

In conclusion, the author expresses his gratitude to M. Moshinsky and V. S. Vasilevsky for fruitful discussions.

References

1) G. F. Filippov, V. I. Ovcharenko and Yu. F. Smirnov, *Microscopic Theory of Collective Excitations in Nuclei*, Naukova Dumka, Kiev, 1981 (in Russian);
2) V. S. Vasilevsky, Yu. F. Smirnov and G. F. Filippov, Sov. J. Nucl. Phys. **37** (1980) 510;
3) G. F. Filippov, V. S. Vasilevsky and L. L. Chopovsky, Sov. J. Part. Nucl. **15** (1984) 600;
4) S. G. Nilsson, Mat. Fys. Medd. Dan. Vid. Selsk. **79** (1955) N16;
5) D. L. Hill and J. A. Wheeler, Phys. Rev. **89** n. 5 (1953);
6) R. M. Asherova, V. A. Knyr, Yu. F. Smirnov and V. N. Tolstoi, Sov. J. Nucl. Phys. **71** (1975) 580;
7) G. Rosensteel and D. J. Rowe, Phys. Rev. Lett. **36** (1977);
8) E. Chacon, P. Hess and M. J. Moshinsky, Math. Phys. **75** (1984) 5.

REALIZACIJA Sp (4, R) MODELA KORISTEĆI TEHNIKU GENERIRAJUĆIH INVARIJANTI

GENNADY F. FILIPPOV

Institute for Theoretical Physics, Metrologicheskaya, 14-b, Kiev-130, USSR

UDK 539.142

Originalni znanstveni rad

Konstruirana je potpuna baza Sp (4, R) modela za zatvorene ljuske pomoću tehnike generirajućih invarijanti. Izračunati su matrični elementi višečestičnog mikroskopskog hamiltonijana u bazi tog modela.