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INTERACTION BETWEEN ELEMENTARY EXCITATIONS IN MANY PARTICLES BOSON SYSTEM

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We used rearranged Feenberg perturbation formula which was adopted to describe the states with two quasiparticles of the momenta $\hbar\vec{k}_1$ and $\hbar\vec{k}_2$. It was noticed that two »diagrams of states« of order N^0 (N is the number of the particles in the system) in the fourth order perturbation term contained the interaction between particles out of the condensate. The interaction energy described by these diagrams was calculated for the potential $V_q = V(-q^2 + g^2)$, which had the main features of the real potential between two helium atoms, and for antiparallel momenta $\hbar\vec{k}_2 = -\hbar\vec{k}_1$. It was shown that the spectrum depended on the density strongly and in particular that the interaction changed its sign and became repulsive with the increase of density. Owing to the form of the interaction diagrams, Bogoliubov summing was possible. After numerical integration we found that this approximation had a significant influence on the spectrum, although qualitative picture of the interaction was unchanged.

1. Introduction

The energy spectrum of elementary excitations in boson systems has been theoretically studied for a long time. Since Landau spectrum was proposed for liquid helium four many researchers have tried to provide a microscopic theory of that spectrum. Among others Brillouin-Wigner perturbation calculations is one of the most useful procedures when dealing with this problem.

Relatively new experiments[1] showed that Landau spectrum was only one branch of the spectrum in liquid ^{4}He. The second branch of the excitation spectrum is located at higher energies which are above two-roton energy. This upper branch corresponds to exciting two or more quasiparticles from the condensate, and mainly they have two-roton nature for lower momenta.

Raman scattering experiments[2] suggested that roton-roton interaction was attractive. Last year in the experiments of Ohbayashi[3] density dependence of the interaction between two rotons with antiparallel momenta was studied. It was observed that roton-roton interaction energy changed sign and became repulsive when density increased.

In this paper we studied the interaction between elementary excitations in many boson system. As in our earlier papers[4,5] where ground state and single particle excitations were studied, we use Feenberg formula[4,6]. In Ref. 4 we showed that Feenberg formula (up to the second order) reproduced Bogoliubov spectrum: ground state and single particle excitation. Having in mind this advantage of Feenberg formula and by using it we dealt with interaction between two excitations in this paper. We are aware that to Bogoliubov approximation neither corresponding to Feenberg's formula can not be applied to real liquid helium. We want to obtain some qualitative insight into interaction of the elementary excitations. In Sec. 2 we stated regrouped Feenberg formula up to the fifth order, which was adopted to describe two excitations[7]. Furthermore was recognized in it the non-ladder interaction terms which were of order 1. Sec. 3 was devoted to the analysis of non-ladder interaction terms and to the discussion of the results. The values of the matrix elements used in the calculation and definitions of some symbols were given in Appendix .

2. *Interaction terms in improved Feenberg formula*

Let us consider N identical bosons in volume Ω with hamiltonian in occupation number representation

$$H = \sum_k e_k a_k^+ a_k + \frac{1}{2\Omega} \sum_{k_1 k_2 q}' V_q a_{k_1+q}^+ a_{k_1-q}^+ a_{k_2} a_{k_1}, \tag{1}$$

where

$$e_k = \frac{\hbar^2 k^2}{2m} \tag{2}$$

$$V_q = \int V(r)\, e^{-i\vec{q}\vec{r}}\, d^3 r, \tag{3}$$

a_k »annihilates« a particle in the state $\vec{k}$, a_k^+ »creates« a particle in the state $\vec{k}$ and they satisfy the commutation relations

$$[a_k, a_{k'}^+] = \delta_{kk'}, \qquad [a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0. \tag{4}$$

Starting from the excited unperturbed eigenstate

$$|2\rangle = |N - 2;\, 1_{k_1} 1_{k_2}\rangle \equiv |1_{k_1} 1_{k_2}\rangle, \tag{5}$$

we performed a rearranged Feenberg formula up to the fifth order[7]

$$E_2^{(5)} = E_2^{(1)} + \sum_n{}' \frac{V_{2n} V_{n2}}{E_2^{(3)} - \varepsilon_n^{F(3)}} + \sum_{nn'}{}' \frac{V_{2n} V_{nn'} V_{n'2}}{(E_2^{(2)} - \varepsilon_n^{F(2)})(E_2^{(2)} - \varepsilon_{n'}^{F(2)})} +$$

$$+ \sum_{nn'n''}{}'^{*} \frac{V_{2n} V_{nn'} V_{n'n''} V_{n''2}}{(E_2^{(1)} - \varepsilon_n^{F(1)}(E_2^{(1)} - \varepsilon_{n'}^{F(1)})(E_2^{(1)} - \varepsilon_{n''}^{F(1)})} +$$

$$+ \sum_{nn'n''n'''}{}'^{*} \frac{V_{2n} V_{nn'}}{(E_2^{(0)} - \varepsilon_n^{F(0)})(E_2^{(0)} - \varepsilon_{n'}^{F(0)})} \frac{V_{n'n''} V_{n''n'''} V_{n'''2}}{(E_2^{(0)} - \varepsilon_{n''}^{F(0)})(E^{(0)} - \varepsilon_{n'''}^{F(0)})}, \tag{6}$$

where $E_2^{(b)}$ is the successive approximation of the order b; the mark "$'$" means that all indices are different, and »*« denotes summing over such n's that lead to dependence on N which is of order N or less. Other symbols are explained in Appendix.

The study of the interaction between two excitations with momenta $\hbar\vec{k}_1$ and $\hbar\vec{k}_2$ is possible on the basis of the relation (6). But there is a simpler way. Instead of considering the whole expression (6) it could be done by recognizing all interaction diagrams and analyzing them. For this purpose we use »diagrams of states«[4].

Diagrams of states are similar to the Goldstone's diagrams. The horizontal lines denote interactions and the vertical lines represent particles out of the condensate (one line per one particle). The space between two successive lincs coiresponds to the state which is defined by lines that cross over. The matrix elements are, then, calculated only between the neighbouring states.

Let us now recognize the interaction terms in Eq. (6) and consider only such diagrams that contain the interaction between particles which are out of the condensate. It is evident that such terms are ladder diagrams (Fig. 1). Besides ladder diagrams, there are other diagrams which describe the interaction between two particles out of the condensate; let us call them non-ladder interaction diagrams. Non-ladder diagrams begin from the fourth order term and there are just two such diagrams in the fourth order term (Fig. 2). The matrix elements between the states which appear in Fig. 2 are given in Appendix. Non-ladder diagrams are of order 1 and they amount to

$$V_i = \varrho^2 \frac{1}{2\Omega^2} \sum_p \sum_q V_q^2 (V_p + V_{p+k_1-k_2})^2 \{[e_{k_1} + e_{k_2} - e_{k_1+p} - e_{k_2-p} +$$

$$+ \varrho (V_{k_1} + V_{k_2} - V_{k_1+p} - V_{k_2-p})] \, [e_{k_1} + e_{k_2} - e_{k_1+p} - e_{k_2-p} -$$

$$- 2e_q + \varrho (V_{k_1} + V_{k_2} - V_{k_1+p} - V_{k_2-p} - 2V_q] \cdot [-2e_q - 2\varrho V_q]\}^{-1}. \tag{7}$$

Let us mention that, on the contrary, each ladder diagram is of order N^{-1}.

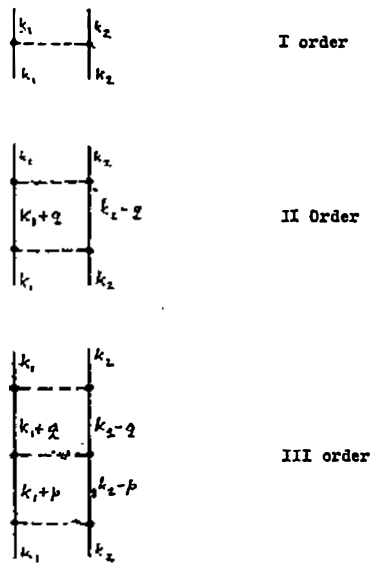

Fig. 1. First three ladder diagrams.

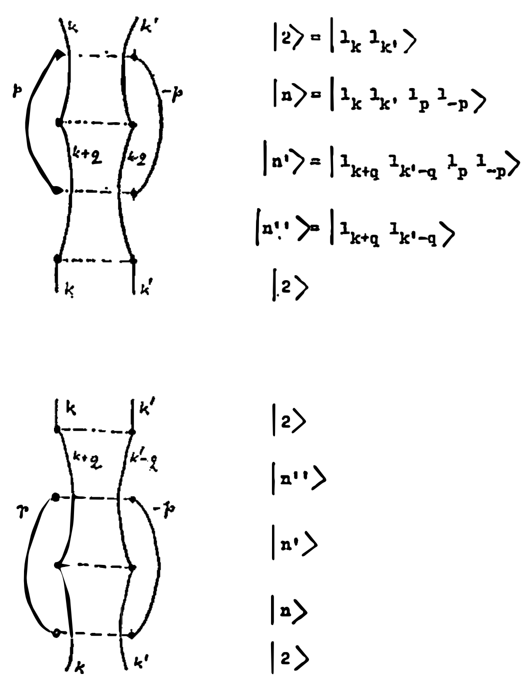

Fig. 2. Non-ladder diagrams of the fourth order.

3. *Analysis of non-ladder interaction terms*

We study non-ladder interaction diagrams although it is believed[8] they should be cancelled by some specific terms from the lower orders. Namelly they include much physics of the other interaction diagrams and allow us evaluate some approximations. We think that the influence of Bogoliubov approximation on the interaction spectrum is studied for the first time.

Let us consider the behaviour of the expression V_t, Eq. (7), against the change in density. Our analysis has been done for a relatively simple potential that includes the main characteristics of some real potentials

$$V_q = \begin{cases} V(-q^2 + g^2), & q < g, \quad V > 0 \\ 0 & q > g. \end{cases} \tag{8}$$

The corresponding potential in the r-space has the form

$$V(r) = \pi^{-2}\, V g^3\, r^{-2}\, j_2(gr), \tag{9}$$

where $j_2(r)$ is the spherical Bessel function. To get some impression about the parameters, V and g were derived by comparision of the Eq. (9) with Bruch-McGee potential for helium atoms

$$V(r) = \begin{cases} \varepsilon\,[B^2\, e^{-2r/a} - 2B\, e^{-r/a}], & r < r_0 \\ -C_6/r^6 - C_8/r^8, & r > r_0 \end{cases} \tag{10}$$

where

$\varepsilon = 12.7687\ 10^{-23}$ J $\qquad C_6 = 9444.6969\ 10^{-83}$ Jm^6

$a = 0.49413\ 10^{-10}$ m $\qquad C_8 = 37174.177\ 10^{-103}$ Jm^8

$r_0 = 3.6828\ 10^{-10}$ m $\qquad B = 455.674.$

That has been done by imposing the equality between first zeros of the expressions (9) and (10), and minima as well. It has been found $V = 2.87037\ 10^{-71}$ Jm^5 and $g = 2.39917\ 10^{10}$ m^{-1}.

In order to compare our result with experiment[3)*], we consider the case of antiparallel momenta $\vec{k}_1 = -\vec{k}_2\ (= \vec{k})$. The expression (7) in this case becomes

$$\begin{aligned} V_t = -m^3\hbar^{-6}\,\varrho^2\,(2\pi)^{-4} \int_0^g \mathrm{d}q\, q^2 \int_0^\infty \mathrm{d}p\, p^2 \int_{-1}^{1} \mathrm{d}t\, V_q^2\,(V_p + V_{p+2k})^2 \cdot \\ \cdot \{[p^2 + 2pkt - 2m\varrho\,(V_k - V_{p+k})/\hbar^2 \\ [p^2 + q^2 + 2pkt - 2m\varrho\,(V_k - V_{k+p} - V_q)/\hbar^2] \\ [q^2 + 2m\varrho\, V_q/\hbar^2]\}^{-1}. \end{aligned} \tag{11}$$

* We again emphasize our goal described in Sec. 1.

We took $k = 1.95\,10^{10}\,\mathrm{m}^{-1}$, the wave vector of a roton in liquid ^{4}He. Numerical integration of the expression (11) has been performed and the results are presented in Table 1 (column a). We see that the interaction changes its behaviour in vicinity of the experimental density and becomes repulsive for greater densities.

TABLE 1.

V_t \ ϱ	a	b
0.2186	0.00343	0.0327
2.186	−16.37	−0.286
5.465	−0.0406	−0.0514
21.86	0.0703	0.0315

Interaction energy as function of the density; energy is given in 10^{-20} J and density in $10^{28}\,\mathrm{m}^{-3}$.

The expression (7) allows further important analysis. Diagrams in Fig. 2 are unconnected. The unconnected part appears from the interaction between particles from the condensate. Namely two particles from the condensate interact and go again in the condensate, no matter what is happening with excited particles with momenta $\hbar\vec{k}_1$ and $\hbar\vec{k}_2$. Regarding our approach to the Bogoliubov theory of the weakly interacting bose gas[4)], we are led to consider the summation of all such possibilities. After Bogoliubov summing the expression (7) reads

$$V_t^B = \frac{1}{2\Omega^2} \sum_q \sum_p (V_p + V_{p+k_1-k_2})^2 \left(\sqrt{e_q^2 + 2\varrho e_q V_q} - e_q - \varrho V_q\right) \{[e_{k_1} + e_{k_2} - e_{k_1+p} - e_{k_2-p} + \varrho(V_{k_1} + V_{k_2} - V_{k_1+p} - V_{k_2-p})]\,[e_{k_1} + e_{k_2} - 2e_q - e_{k_1+p} - e_{k_2-p} + \varrho(V_{k_1} + V_{k_2} - V_{k_1+p} - V_{k_2-p} - 2V_q)]\}^{-1}. \quad (12)$$

Again we integrated for the case $\vec{k}_1 = -\vec{k}_2$ and the same parameters. The results are presented in Table 1 (column b). Bogoliubov summation does not change the spectrum qualitatively; it has a strong influence on the intensity of the interaction.

To achieve a satisfactory accuracy the integration region was divided into smaller areas and 32-point Gauss integration was applied to each area. For one value of the interaction energy the computing time was about four hours on IBM-4341.

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APPENDIX

1.

We list here the matrix elements used in the calculation.

Diagonal:

$$\langle 0; N | V | N; 0 \rangle = \frac{1}{2\Omega} N (N-1) V_0 = V_{00}$$

$$\langle 1_k 1_{k'}, n_p n_{-p} | V | 1_k 1_{k'}, n_p n_{-p} \rangle = V_{00} + \frac{N - 2(n+1)}{\Omega} (V_k + V_{k'} + 2n V_p) +$$
$$+ \Omega^{-1} [V_{k-k'} + n (V_{k-p} + V_{k+p} + V_{k'-p} + V_{k'+p} + n V_{2p})], \qquad n = 0\ 1\ 2\ \dots$$

$$\langle 1_{k+p} 1_{k'-p} n_p n_{-p} | V | 1_{k+p} 1_{k'-p} n_p n_{-p} \rangle = V_{00} + \frac{N - 2(n+1)}{\Omega} (V_{k+p} +$$
$$+ V_{k'-p} + 2n V_p) + \Omega^{-1} [V_{k-k'+2p} + n (V_k + V_{k+2p} +$$
$$+ V_{k'-2p} + V_k + n V_{2p})], \qquad n = 0, 1, 2, \dots$$

Nondiagonal:

$$\langle 1_k 1_{k'} n_p n_{-p} | V | 1_{k+q} 1_{k'-q} n_p n_{-p} \rangle = \Omega^{-1} (V_q + V_{k-k'+q}),$$
$$n = 0, 1, \dots$$

$$\langle 1_k 1_{k'} (n-1)_p (n-1)_{-p} | V | 1_k 1_{k'} n_p n_{-p} \rangle = \Omega^{-1} \sqrt{(N-2n)(N-2n-1)}\, n V_p,$$
$$n = 1, 2\ \dots$$

2.

In this partition we give the meaning of the symbols used in Eq (6).

$$E_2^{(0)} = e_2 = \frac{\hbar^2}{2m} (k_1^2 + k_2^2)$$

$$\varepsilon_m^{F(0)} = e_m$$

$$\varepsilon F_2^{(1)} = e_2 + V_{22}$$

$$\varepsilon_m^{F(1)} = e_m + V_{mm}$$

$$E_2^{(2)} = E_2^{(1)} + \sum_n \frac{V_{2n} V_{n2}}{E_2^{(0)} - \varepsilon_n^{F(0)}}$$

$$\varepsilon_m^{F(2)} = \varepsilon_m^{F(1)} + \sum_{\substack{m' \\ \neq a}} \frac{V_{mm'} V_{m'm}}{E_2^{(0)} - \varepsilon_{m'}^{F(0)}} + \sum_{\substack{m' \\ = a}} \frac{V_{mm'} V_{m'm}}{\varepsilon_m^{F(0)} - \varepsilon_{m'}^{F(0)}}$$

$$E_2^{(3)} = E_2^{(1)} + \sum_n \frac{V_{2n} V_{n2}}{E_2^{(1)} - \varepsilon_n^{F(1)}} + \sum_{nn'} \frac{V_{2n} V_{nn'} V_{n'2}}{(E_2^{(0)} - \varepsilon_n^{F(0)})(E_2^{(0)} - \varepsilon_{n'}^{F(0)})}$$

$$\varepsilon_m^{F(3)} = \varepsilon_m^{F(1)} + \sum_{\substack{m' \\ \neq a}} \frac{V_{mm'} V_{m'm}}{E_2^{(1)} - \varepsilon_{m'}^{F(1)}} + \sum_{\substack{m' \\ = a}} \frac{V_{mm'} V_{m'm}}{\varepsilon_m^{F(1)} - \varepsilon_{m'}^{F(1)}}$$

$$+ \sum_{\substack{m'm'' \\ \neq a}} \frac{V_{mm'} V_{m'm''} V_{m''m}}{(E_2^{(0)} - \varepsilon_{m'}^{F(0)})(E_2^{(0)} - \varepsilon_{m''}^{F(0)})} + \sum_{\substack{m'm'' \\ = a}} \frac{V_{mm'} V_{m'm''} V_{m''m}}{(\varepsilon_m^{F(0)} - \varepsilon_{m'}^{F(0)})(\varepsilon_m^{F(0)} - \varepsilon_{m''}^{F(0)})}$$

The letter a denotes states which lead to a total term with N as a factor.

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MEĐUDJELOVANJE IZMEĐU ELEMENTARNIH POBUĐENJA U VIŠEČESTIČNOM BOZONSKOM SISTEMU

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Koristili smo rearanžiranu Feenbergovu formulu koja je bila prilagođena za opisivanje stanja sa dvije kvazičestice izvan kondenzata. Bilo je zapaženo da dva dijagrama stanja, čiji je red veličine N^0 (N je broj čestica u sistemu) u četvrtom redu perturbacione formule sadrže interakciju između čestica koje su izvan kondenzata. Energija međudjelovanja opisana ovim dijagramima, računata je za potencijal $V_q = V(-q^2 + g^2)$, koji sadrži glavne odlike realnog potencijala između dva atoma helija, i za antiparalelne impulse $\hbar\vec{k}_2 = -\hbar\vec{k}_1$. Pokazano je da spektar međudjelovanja znatno ovisi o gustoći te da interakcija mijenja predznak i postaje odbojna kada gustoća raste. Zahvaljujući formi interakcionih dijagrama, bilo je moguće izvršiti Bogoljubovljevo sumiranje. Nakon ponovne numeričke integracije nađeno je da ova aproksimacija ne mijenja oblik spektra te da znatno utječe na intenzitet interakcije.