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POSITIVE ENERGY BOUND STATES AT HIGHER PARTIAL WAVES

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Positive energy bound states (PEBS) at higher order (than the 0^{th}) partial waves have been studied using a particular form factor. The phase shifts have been calculated in detail for the cases of $l = 0, 1, 2$. It is found that for a rank-1 separable potential the expected behaviour, at the PEBS energy, is obtained. Finally, some important conclusions have been drawn regarding problems previously studied in connection with s-wave positive energy bound states.

1. Introduction

The question of positive energy bound states (PEBS) is fundamental in theoretical nuclear physics and the subject has been of interest for a long time[1,2], where it is found that for a certain class of non-local potentials the s-wave Schrödinger equation:

$$\frac{d^2\psi}{dr^2} + k^2\psi = \int_0^\infty k(r, r')\,\psi(r')\,dr' \tag{1}$$

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has solution $\psi(k_0, r)$ such that

$$\psi(k_0, r) \to 0 \text{ as } r \to \infty \tag{2}$$

at the energy $k^2 = k_0^2$. Such a solution is called a positive energy bound state.

These PEBS have been interpreted as zero-width resonances[3,4], however, recently[5,6] and for a rank-n separable potential, some general conditions have been given and discussed for the existence of zero-width resonance or scattering phase shifts at PEBS and hence it was concluded that a new interpretation was needed.

Moreover, all studies concerning PEBS made use of the s-wave Schrödinger equation and no attempt has been made to study cases with higher order l-waves.

Therefore, we think it is wor'hwhile to generalize our study of PEBS to cases of any l-wave ($l = 0, 1, 2, \ldots$). In this case we take the l-wave Schrödinger equation with the non-local separable potentia

$$K_l(r, r') = \lambda_l g_l(r) g_l(r') \tag{3}$$

as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \frac{l(l+1)}{r^2}\right) \psi_l(r) = \lambda_l g_l(r) \int_0^\infty g_l(r') \psi_l(r') \, \mathrm{d}r'. \tag{4}$$

The solution of the above equation can be seen as

$$\psi_l(k, r) = N \varphi_0(k, r), \tag{5}$$

where N is a normalization constant and $\varphi_l(r)$ satisfies the equations

$$\frac{\mathrm{d}^2 \varphi_l}{\mathrm{d}r^2} + \left(k^2 - \frac{l(l+1)}{r^2}\right) \varphi_l(r) = g_l(r) \tag{6}$$

$$\varphi_l(k, 0) = 0 \tag{7}$$

and λ_l is given by

$$\frac{1}{\lambda_l} = \int_0^\infty g_l(r) \varphi_l(k, r) \, \mathrm{d}r. \tag{8}$$

$\varphi_l(k, r)$ can be determined from Eqs. (5) and (6) as

$$\varphi_l(k, r) = \int_0^\infty G_l(r, r') g_l(r') \, \mathrm{d}r' \tag{9}$$

with the Green function $G_l(r, r')$ given by

$$G_l(r, r') = k r_< r_> j_l(k r_<) \eta_l(k r_>), \tag{10}$$

where $j_l(kr)$ and $\eta_l(kr)$ are the spherical Bessel and Neumann functions, and we can get a PEBS at the energy $k^2 = k_l^2$ when $\varphi(k_l, r) \to 0$ as $r \to \infty$.

2. A form factor

To demonstrate how to obtain PEBS for any l-wave we consider the form factor

$$g_l(r) = r^{l+2}\,\mathrm{e}^{-ar}. \tag{11}$$

This is to be compared with the modified Yukawa form factor used by Cattapan, Pisent and Vansani[7] as a suitable, physically reasonable, potential in their study of nucleon-nucleus scattering with the exact treatment of Coulomb interactions. Moreover, the choice of this particular form factor will lead to exact solutions as will be seen later.

Taking now this form factor into account we can solve Eq. (4) for PEBS at

$$k_l^2 = (2l+3)\,a^2 \tag{12}$$

and with a wave function

$$\varphi_l(k_l, r) = N(ar^{l+2} + r^{l+1})\,\mathrm{e}^{-ar} \tag{13}$$

provided that

$$\lambda_l = \frac{1}{4}\cdot\frac{(l+2)}{(l+1)}\frac{(2a)^{2l+2}}{(2l+3)!}. \tag{14}$$

3. The formal scattering solution

Adopting a similar procedure to that of Husain and Ali[8], we see that the asymptotic form for $\varphi_l(k, r)$, as $r \to \infty$, is

$$\varphi_l(k, r) \to \sin(kr - l\pi/2) - \cos\left(kr - \frac{l\pi}{2}\right) A_l \lambda_l \int_0^\infty g_l(r')\, r' j_l(kr')\,\mathrm{d}r' \tag{15}$$

with

$$A_l = k \int_0^\infty g_l(r')\, r' j_l(kr')\,\mathrm{d}r' / [1 - \lambda_l \int_0^\infty \int_0^\infty G_l(r, r')\, g_l(r)\, g_l(r')\,\mathrm{d}r\mathrm{d}r'] \tag{16}$$

and therefore

$$\tan\delta_l = -\lambda_l A_l \int_0^\infty g_l(r')\, r' j_l(kr')\,\mathrm{d}r' =$$

$$= -\lambda_l k \left(\int_0^\infty g_l(r')\, r' j_l(kr')\,\mathrm{d}r'\right)^2 / [1 - \lambda_l \int_0^\infty \int_0^\infty G_l(r, r')\, g_l(r)\, g_l(r')\,\mathrm{d}r\mathrm{d}r']. \tag{17}$$

Now with the appropriate form factor $g_l(r)$ we can calculate these phase shifts using Eq. (17) as will be seen in the next section.

4. Examples

Working with the form factor given by Eq. (11) we can calculate an δ_l for $l = 0, 1, 2, \ldots$ Below we show the calculational results for $l = 0, 1, 2$ as examples. Note that in all cases we assumed a rank-1 potential.

Case 1 (*s-wave*): In this case $g_0 = r^2 e^{-ar}$, $\lambda_0 = \frac{32}{3} a^7$, and if we put $x = k^2/a^2$ then the PEBS is at the energy $x_0 = 3$. Moreover, the phase shift is given by

$$\tan \delta_0 = \frac{-128 \sqrt{x}(x-3)^2}{(x-3)\{3(x-3)(x+1)^4 + 8[2(x+1)^3 + 48x - 16]\}}. \tag{18}$$

Fig. 1 shows the phase shift δ_0 as a function of energy x, where we see that we get the expected jump of π at $x = x_0$. However, it is to be noted that the phase shift does not tend to zero as x tends to infinity but instead it attains a value of π.

Case 2 (*p-wave*): $g_1 = r^3 e^{-ar}$, $\lambda_1 = \frac{8}{5} a^9$, and the PEBS is at $x_1 = k^2/a^2 = 5$ while $\tan \delta_1$ has the form

$$\tan \delta_1 = \frac{-512 x^{3/2} (x-5)^2/(x-5)}{(x+1)^4 (5x^3 + 9x + 6) - 128(9x^2 - 14x + 1)}. \tag{19}$$

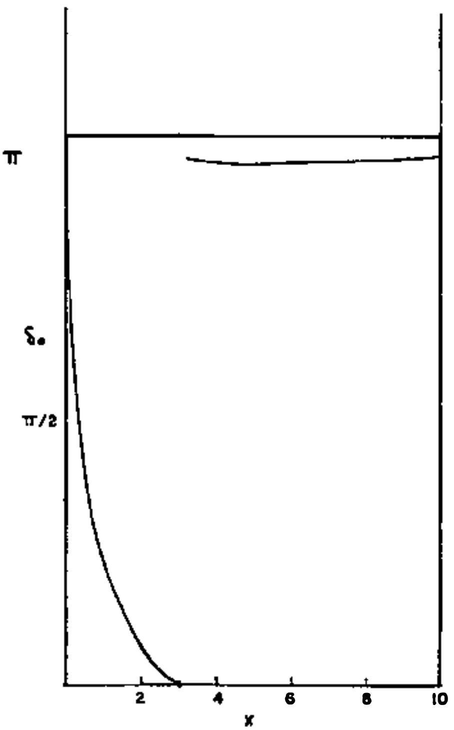

Fig. 1. A plot of the phase shift δ_0 (*s*-wave) versus the energy x with $g_0 = r^2 e^{-ar}$.

Again, in Fig. 2, we give the phase shift δ_1 as a function of energy x. In this case the phase shift drops from x to a minimum at about $x = 0.1$; then it rises to a maximum at $x = 1.0$, approximately, and gradually goes down to zero as $x \to 5$. For $x < 5$, $\delta_1 \cong \pi$.

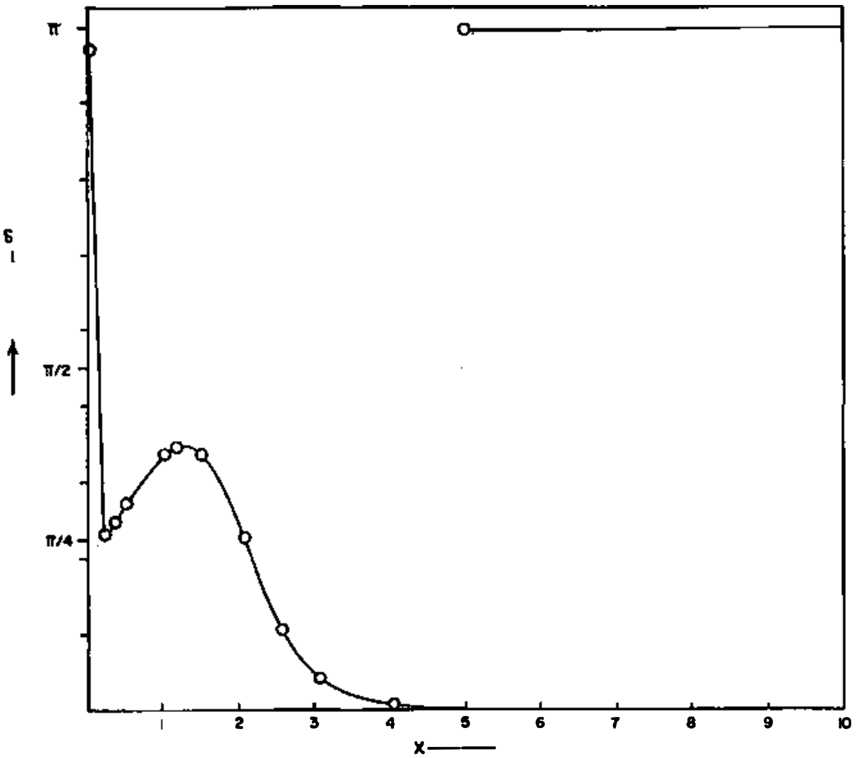

Fig. 2. A plot of the phase shift δ_1 (p-wave) versus the energy x with $g_1 = r^3 e^{-ar}$.

Case 3 (d-wave): In this case $g_2 = r^4 e^{-ar}$, $\lambda_2 = \frac{128}{945} a^{11}$, and the PEBS is at $x_2 = 7$, with $\tan \delta_2$ given by

$$\tan \delta_2 = \frac{-2^{15} x^{5/2} (x-7)^2/(x-7)}{35 (x+1)^3 (3x-5) - 32 D(x)} \tag{20}$$

with

$$D(x) = (x+1)^5 (21 - 15x^2 - 2x) + \\ + 128 (1 - 17x + 35x^2 - 11x^3). \tag{21}$$

The results are shown in Fig. 3 where the same sort of picture, as in Fig. 2, emerges.

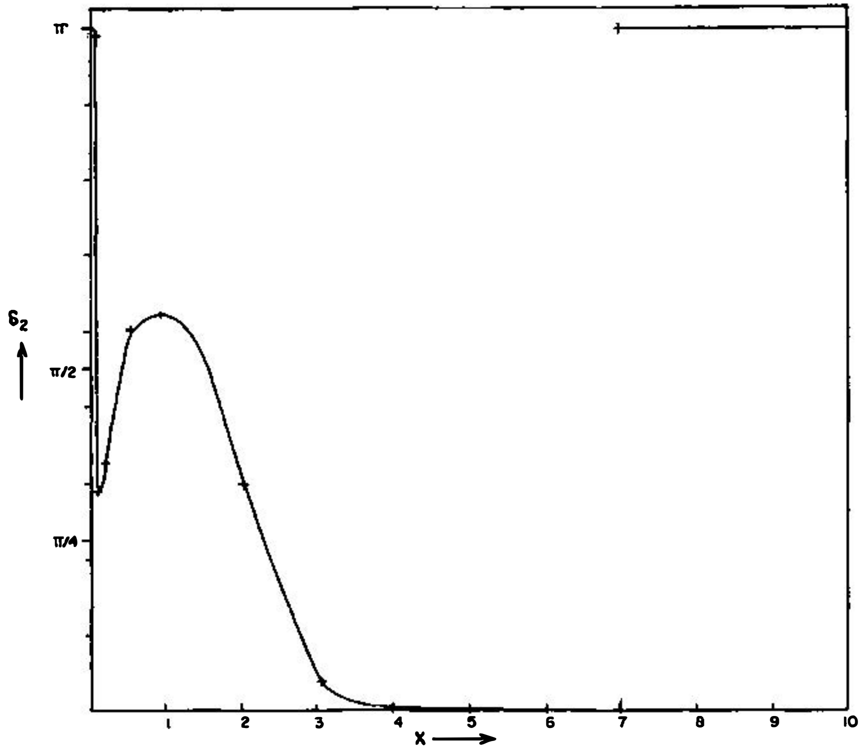

Fig. 3. A plot of the phase shift δ_2 (d-wave) versus the energy x with $g_2 = r^4 e^{-ar}$.

5. Concluding discussion

From the results obtained for the particular form factor $r^{l+2}e^{-r}$ and for the rank-1 separable potential, it is clear that the phase shift exhibits the expected behaviour at the PEBS energy for all partial waves studied and, of course, this corresponds to the situation used to be interpreted as zero-width resonances.

Moreover, as noted before, for $x < x_l$ (x_l is the PEBS energy) the phase shift attains a constant value of π as $x \to \infty$, which is peculiar to the normal behaviour of the PEBS phase shift in that region. This might be owed to a characteristic of the form factor chosen.

Accordingly, many problems studied in connection with PEBS (s-wave) need to be re-investigated for the case of any partial wave; to mention in particular the generalization to the Levinson theorem.

However, one also expects that some of the conclusions drawn before about PEBS, still hold for the case of any l-wave, e. g. one expects that when a rank-2 potential is used then scattering phase shifts may be obtained at the PEBS energy. Therefore, a thorough revision of all these problems, for any l-wave, is necessary.

Finally, one notes that for our form factor and for the l^{th}-partial wave one gets the expression $2^{2l^3+7}x^{l+1/2}(x - x_l)^2$ in the numerator for $\tan\delta_l$.

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VEZANA STANJA POZITIVNE ENERGIJE KOD VIŠIH PARCIJALNIH VALOVA

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Studirana su vezana stanja pozitivne energije (PEBS) kod viših (većih od 0) parcijalnih valova za dani faktor oblika. Fazni pomaci izračunati su detaljno za slučajeve $l = 0, 1, 2$. Nađeno je da se za rank-1 separabilni potencijal dobiva očekivano PEBS ponašanje. Na kraju su povučeni neki važni zaključci povezani s ranije razmatranim problemima u vezi s-valnih vezanih stanja pozitivne energije.