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MESON BAG STATES AND THE MOMENTUM EIGENSTATES

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Meson momentum-eigenstates are obtained by projecting from the boosted-bag model states. These momentum-eigenstates are then used to calculate the pion decay constant f_π, and the weak pion-kaon transition matrix elements $\langle \pi | H_\omega | K \rangle$. The transition matrix elements are parametrized to explain $K \to 2\pi$ decays and used to estimate the amplitude for rare decays $K \to \pi \gamma_Y$ involving a light vector particle γ_Y.

1. Introduction

It is well known[1-4] that the hadron static bag model (SBM) states are not momentum eigenstates. This statement is valid for any central potential model (CPM) static state. As a CPM we consider any model in which quarks move relative to some dynamic center, be it a central potential or the bag center. Examples can be found in Refs. 5—7.

By using methods developed in Refs. 8—9 one can boost static CPM states in such a way[10-13] that the result, known as the bola-model (BM)[8,14], is relativistically covariant. This assumes[8-13] that the potential can be written in a covariant way, which is usually the case. The resulting BM's are relativistic only in a kinematical sense, i. e. being invariant under Lorentz transformations. They are not relativistic in a full quantum field theory sense and their dynamical content corresponds to the quasi potential approximation[10,13,15,16] of the Bethe-Salpeter equation.

At first sight such states, described in detail in Refs. 8, 10—13 look like momentum eigenstates. However, closer examination reveals that these states contain a piece which is definitely not a momentum eigenstate, and which corresponds to a spurious center-of-mass (C. M.) motion similar to that for static CPM states[2]. In this paper these defects will be discussed and illustrated by attempting to calculate the pion decay constant f_π and the weak transition amplitude

$$\langle \pi | H_\omega | K \rangle$$

in the SBM.

A formal remedy to the above problems can be based[2-3] on the Peierls-Yoccoz projection[17], which for our purposes has to be made suitably relativistic. This formalism is to some extent related to the Peierls-Thouless method[18], however, our approach is formally, i. e. kinematicaly, explicitly covariant. As it is based on BM hadron wave functions, it cannot contain any deeper dynamical or physical information than the BM itself.

2. „Bola" — model states and matrix elements

In the »bola« version[8,10-13] of the boosted CPM a typical hadron wave function has the generic form

$$\chi (P; z_1^P, z_2^P, \ldots, z_k^P; \bar{z}_1^P, \ldots, \bar{z}_e^P, y) \equiv N_p e^{-iP\cdot y} \psi(z_1^P) \psi(z_2^P) \ldots$$
$$\ldots \psi(z_k^P) \bar{\psi}(z_1^P) \ldots \bar{\psi}(z_e^P). \tag{2.1a}$$

Here the factor $\exp(-iP \cdot y)$ describes the motion of the center-of-force, N_P is a suitable norm and $\psi(\bar{\psi})$ symbolize quark (antiquark) wave functions. In such a BM the coordinate x of a particular quark is sum of the center-of-force coordinate y and the coordinate z

$$x_i = y + z_i. \tag{2.1b}$$

The quark wave function has a following structure

$$\psi(z^P) = S(P)\, \eta(z_\perp(P))\, e^{-i\varepsilon||(P)z||\varepsilon}$$

$$S(P) = \frac{1}{\sqrt{2M(E+M)}} \begin{pmatrix} E+M & \vec{\sigma}\cdot\vec{P} \\ \vec{\sigma}\cdot\vec{P} & E+M \end{pmatrix} \tag{2.1c}$$

$$z_\perp(P)_\mu = z_\mu - \beta_\mu(\beta\cdot z)$$
$$z_{||}(P) = \beta \cdot z$$
$$\beta_\mu = P_\mu / M.$$

Here η is a solution of some CPM (this category includes the SBM), while P and M are the hadron momentum and mass. Additional details can be found in Refs. 8—13.

Any calculation involving the BM can be formulated in two ways. They are:

A) configurational space

B) n-representation.

In approach A), the wave function (2.1) is used to calculate the matrix element of some operator in coordinate space. This operator is a C-number from the quantum field theory point of view. The matrix elements of the currents in Refs. 12—13 were written in that way.

It was convenient to use the n-representation in the actual calculations (quarks being represented by operators) in order to evaluate the spin-flavour factors corresponding to hadron states. Those states were classified according to SU(6) spin-flavour symetry, as is usuali in SBM[19]. As an example we sketch the calculation of the pion decay constant f_π, which is defined by

$$\langle 0 | J_{\mu 5}(y) | \pi, \vec{P} \rangle = \mathrm{i} \frac{1}{\sqrt{2E(2\pi)^3}} f_\pi P_\mu \mathrm{e}^{-\mathrm{i} P \cdot y}. \tag{2.2}$$

Here meson states are normalized as follows

$$\langle \vec{P}_a | \vec{P}_b \rangle = \delta^{(3)} (\vec{P}_a - \vec{P}_b). \tag{2.3}$$

The dimension of a meson state is

$$\dim \{ | \vec{P} \rangle \} = \lambda^{-3/2}. \tag{2.4}$$

Here λ is the dimension of M. As the axial vector current $J_{\mu 5}(y)$ has the dimension λ^3, the dimension of f_π must be λ.

In the BM one can introduce quark field operators

$$\Psi(z^P) = \sum_k (a_k \psi_k (z^P) + b_k^* \bar{\psi}_k (z^P)) \tag{2.5}$$

and define the pseudoscalar meson state (with mass M) as

$$| M, P, y \rangle = \sum_s a_{k,P}^{*s} b_{k,P}^{*-s} | \hat{0} \rangle \, \mathrm{e}^{-\mathrm{i} P \cdot y}; \quad (\dim = \lambda^0). \tag{2.6}$$

The notation $|\hat{0}\rangle$ is to remind us that the BM (or CPM) vacuum is not the same as physical vacuum in (2.2).

In the BM the l. h. s. of (2.2) must be:

$$\langle \hat{0} | \bar{\Psi}(z^P) \gamma_\mu \gamma_5 \Psi(z^P) | M_\pi, P, y \rangle = \mathcal{N}_\mu \mathrm{e}^{-\mathrm{i} P \cdot y}. \tag{2.7}$$

Unfortunately this naive expression is quite unsuitable for any calculation as it has the wrong dimension

$$\dim\{\mathcal{N}_\mu\} = \lambda^3$$

while equation (2.2) has the dimension $\lambda^{3/2}$.

This problem is well known[2,3]. It has been encountered by us in the proton decay calculation[20]. As will be shown in section 3, one can remedy the situation by decomposing the state (2.1) into momentum eigenstates.

In the BM one has to keep in mind that quark creation (annihilation) operators depend on the momentum P (i. e. boost) of the hadron state. This is symbolized by the index $_P$ appearing in (2.6). The boost dependence of quark operators is important when one calculates the matrix element of a current sandwiched between meson states

$$\langle y, P_f, M_f| \bar{\Psi}(z^P)\, \Gamma_\mu \Psi(z^P)| M_i, P_i, y\rangle. \tag{2.8}$$

Here two quark (anti-quark) operators from the meson states are contracted with the quantum field operators Ψ. Two which are left and which correspond to a spectator quark cannot contract, as they do not belong to the same states. In the static case, $\vec{P}_f = \vec{P}_i = 0$, corresponding to some CPM (including SBM) they simply contract. In a general case ($\vec{P}_{i,f} \neq 0$) one can estimate the spectator quark propagator by looking at the configuration space formalism A). In that formalism the spectator quark wave functions from initial and final meson states overlap. One obtains the overlap factors

$$Z = \int \mathrm{d}^4 z \delta(L \cdot z)\, \bar{\psi}(z^{P_f})\, \hat{L} \psi(z^{P_i}); \quad \hat{L} = L^\mu \gamma_\mu. \tag{2.9a}$$

The corresponding expression in the n-formalism is

$$Z = \langle 0|\, a_{k,P_f} \int \mathrm{d}^4 z \delta(L \cdot z)\, \bar{\Psi}(z^P)\, \hat{L} \Psi(z^P)\, a^*_{k,P_i}\, |0\rangle. \tag{2.9b}$$

When $P_f = P_i$ the overlap factor Z becomes the normalization integral for quark wave functions. The four vector L has to be determined by some physical requirement. As shown in Refs. 12, 13 the conserved vector current (CVC) constraints can only be met with

$$L^\mu = \frac{\beta_i^\mu + \beta_f^\mu}{[(\beta_i + \beta_f)^2]^{1/2}}. \tag{2.10}$$

3. „Bola" — model state as a superposition of plane—wave eigenstates

The decomposition of a BM state into components $\varphi(\vec{l}, \omega_l)$ of momentum eigenstates, using the n-formalism, is

$$|M, P, 0\rangle = \int \mathrm{d}^4 l \delta(l^2 - m^2)\, \Theta(l_0)\, \varphi_P(l)\, |\vec{l}\rangle = \int \mathrm{d}^3 l \frac{1}{2\omega_l} \varphi_P(\vec{l}, \omega_l)\, |\vec{l}\rangle \quad (3.1)$$

$$\omega_l^2 = \vec{l}^2 + M^2.$$

Here M on the left-hand side denotes the mass M of a particular meson. On the right-hand side both φ and $|l\rangle$ depend also on M. From (3.1) one can find the normalization condition for φ

$$\langle 0, P, M | M, P, 0\rangle = 1 = \int \mathrm{d}^3 l\, \mathrm{d}^3 \tilde{l} \frac{1}{4\omega_l \tilde{\omega}_l} \varphi_P^* (\vec{\tilde{l}}, \tilde{\omega}_l)\, \varphi_P(\vec{l}, \omega_l)\, \delta^{(3)}(\vec{\tilde{l}} - \vec{l})$$

$$1 = \int \frac{\mathrm{d}^3 l}{4\omega_l^2} |\varphi_P(\vec{l}, \omega_l)|^2. \quad (3.2)$$

The states $|\vec{l}\rangle$ are by definition the physical meson states in the n-representation, which means explicitly:

$$|\vec{l}\rangle = A^*(\vec{l})\, |0\rangle. \quad (3.3)$$

Here A^* is a physical meson creation operator.

The component φ can be found[3] by using the wave functions (2.1).

First let us remember some well-known formulae which connect field operators with wave functions. One can write in the BM framework:

$$\begin{aligned} \chi(P, z^P; \bar{z}^P, y) &= N_P \langle \hat{0}|\, \Psi(z^P)\, \bar{\Psi}(\bar{z}^P)\, |M, P, y\rangle = \\ &= N_P \langle \hat{0}|\, R(z^P, \bar{z}^P)\, |M, P, y\rangle = N_P \mathrm{e}^{-\mathrm{i}P\cdot y} h(z^P, \bar{z}^P). \end{aligned} \quad (3.4)$$

From (2.6) one obtains

$$|M, P, y\rangle = \mathrm{e}^{-\mathrm{i}P\cdot y} |M, P, 0\rangle. \quad (3.5)$$

For any momentum eigenstate one has

$$P_\mu |l\rangle = l_\mu |l\rangle. \quad (3.6)$$

When the BM state in (3.4) is decomposed into momentum eigenstates one obtains

$$\langle \hat{0}|\, R(z^P, \bar{z}^P)\, |M, P, 0\rangle = \int \mathrm{d}^3 l \frac{1}{2\omega_l} \varphi_P(\vec{l}, \omega_l)\, \langle 0|\, \hat{R}(z^P, \bar{z}^P)\, |\vec{l}\rangle. \quad (3.7)$$

In (3.7), as in (2.7), both the states $\langle 0|$, $|l\rangle$ and the operator $\hat{R}$ must be reinterpreted as »physical« quantities that have the required Lorentz properties. In that case the matrix element $\langle 0|\hat{R}|\vec{l}\rangle$ is just the Bethe-Salpeter definition[21] for a plane-wave momentum eigenstate. Obviously one must have

$$J(z)\,J(\bar{z})\,\langle\tilde{\vec{l}}|\,R^{*}(z^{P},\bar{z}^{P})\,|\hat{0}\rangle\,\hat{L}\hat{\bar{L}}\,\langle\hat{0}|\,R(z^{P},\bar{z}^{P})\,|\vec{l}\rangle \Rightarrow$$

$$\Rightarrow \int \mathrm{d}^{4}z\,\mathrm{d}^{4}\bar{z}\,\langle\tilde{\vec{l}}|\,\hat{R}^{*}\,|0\rangle\,Q\cdot\bar{Q}\,\langle 0|\,\hat{R}\,|\vec{l}\rangle = \delta(\tilde{\vec{l}}-\vec{l})\,\omega(l) \tag{3.8a}$$

where we have defined

$$J(z) = \int \mathrm{d}^{4}z\,\delta(L\cdot z). \tag{3.8b}$$

The factor $\omega(l)$ depends on the normalization of the eigenstates, and in our case $\omega = 1$. The operator Q in (3.8) is defined in Ref. 21. The »interpretation« (3.8) reveals a deeper physical meaning of this, not really exact[1,3], formalism, which will be further discussed below. From (3.6) one immediately finds

$$\langle 0|\,\hat{R}(z^{P}+\zeta_{\perp}^{P},\bar{z}^{P}+\zeta_{\perp}^{P})\,|\vec{l}\rangle = \mathrm{e}^{-\mathrm{i}l\cdot\zeta_{\perp}^{P}}\,\langle 0|\,\hat{R}(z^{P},\bar{z}^{P})\,|\vec{l}\rangle. \tag{3.9}$$

The component φ is found from the overlap integral

$$\begin{aligned}\mathcal{M}(P,\zeta_{\perp}^{P}) &= J(z)\,J(\bar{z})\,h^{*}(z^{P},\bar{z}^{P})\,h(z^{P}+\zeta_{\perp}^{P},\bar{z}^{P}+\zeta_{\perp}^{P}) = \\ &= \int \frac{\mathrm{d}^{3}l}{2\omega}\,\frac{\mathrm{d}^{3}\tilde{l}}{2\tilde{\omega}}\,\varphi_{P}^{*}(\tilde{\vec{l}},\tilde{\omega})\,\varphi_{P}(\vec{l},\omega)\,\mathrm{e}^{-\mathrm{i}l\cdot\zeta_{\perp}^{P}}\cdot\delta(\tilde{\vec{l}}-\vec{l}) = \\ &= \int \frac{\mathrm{d}^{3}l}{4\omega^{2}}\,\mathrm{e}^{-\mathrm{i}l\cdot\zeta_{\perp}^{P}}\,|\varphi_{P}(\vec{l},\omega)|^{2}.\end{aligned} \tag{3.10a}$$

The second line in (3.10a) was obtained by combining the expresions (3.8) and (3.9). In the BM[12,13] the integration over $\mathcal{M}(P,\zeta_{\perp}^{P})$ must be over the hyperplane $L\cdot\zeta = 0$. In the rest frame where

$$\zeta_{\perp}^{P} \xrightarrow{\vec{P}=0} (0,\vec{\zeta})$$

one obtains the earlier results[1-4].

By integrating (3.10a) one finds

$$J(\zeta)\,\mathcal{M}(P,\zeta_{\perp}^{P})\,\mathrm{e}^{\mathrm{i}l\cdot\zeta_{\perp}^{P}} = J(\zeta)\int \mathrm{d}^{3}l\,\mathrm{e}^{\mathrm{i}(l-k)\cdot\zeta_{\perp}^{P}}\,|\varphi_{P}(\vec{k},\omega_{k})|^{2}\,\frac{1}{4\omega_{k}^{2}}; \tag{3.10b}$$

$$k^{\mu} = (\omega_{k},\vec{k}).$$

Equation (2.10) gives

$$L^\mu = P^\mu / M$$

$$L \cdot \zeta = \frac{P \cdot \zeta}{M} = \frac{E}{M} \zeta^0 - \frac{\vec{P} \cdot \vec{\zeta}}{M} = 0; \quad \zeta^0 = \frac{\vec{P} \cdot \vec{\zeta}}{E} \tag{3.10c}$$

$$(k - l) \cdot \zeta_\perp^P \to (-) \vec{\zeta} \cdot \left(\vec{k} - \vec{l} - \frac{\vec{P}(\omega_k - \omega_l)}{E} \right).$$

Finally one obtains

$$\int \mathrm{d}^3 \zeta \mathcal{M} \left(P, \frac{\vec{\zeta} \cdot \vec{P}}{E}, \vec{\zeta} \right) \mathrm{e}^{-\mathrm{i}\vec{\zeta} \cdot \left(\vec{l} - \vec{P} \frac{\omega_k}{E} \right)} =$$

$$= (2\pi)^3 \int \mathrm{d}^3 k \delta \left(\vec{l} - \vec{k} - \vec{P} \frac{1}{E} (\omega_l - \omega_k) \right) |\varphi_P (\vec{k}, \omega_k)|^2 \frac{1}{4\omega_k^2} = \tag{3.10d}$$

$$= \frac{(2\pi)^3}{1 - \frac{\vec{P} \cdot \vec{l}}{E \omega_l}} |\varphi_P (\vec{l}, \omega_l)|^2 \frac{1}{4\omega_l^2}.$$

For the meson rest system $\vec{P} = 0$, this formula goes over into the one quoted in Refs. 1—3,

$$|\varphi_M (\vec{l}, \omega_l)|^2 = \frac{4\omega_l^2}{(2\pi)^3} \int \mathrm{d}^3 \zeta \mathcal{M} (M, 0, \vec{\zeta}) \, \mathrm{e}^{-\mathrm{i}\vec{\zeta} \cdot \vec{l}} \tag{3.11}$$

Examination of expressions (3.10) and (3.11) elaborates the meaning of the momentum eigenstates $|l\rangle$ and of the operator $\hat{R}$. Both are determined by the quasi-potential formalism of the BM which is used to calculate the components φ. $\hat{R}$ and $|l\rangle$ transform correctly under Lorentz transformations, but they contain only the physical dynamics or information that can be found in the BM framework. Therefore they are obviously »mock-meson« quantities, in a sense used in Ref. 22 for example. All calculations in which they will be employed must necessarily be »mock-meson« calculations.

4. Calculation of matrix elements

It is instructive to re-examine the relation between (2.2) and (2.7). One has

$$\int \mathrm{d}^4 y J(z) \langle 0 | \bar{\Psi}(z^P) \gamma_\mu \gamma_5 \Psi(z^P) | M, P, y \rangle \, \mathrm{e}^{\mathrm{i}q \cdot x} =$$

$$= \int \mathrm{d}^4 y \, \mathrm{e}^{\mathrm{i}(q - P) \cdot y} N_P J(z) \bar{\psi}(z^P) \gamma_\mu \gamma_5 \psi(z^P) \, \mathrm{e}^{\mathrm{i}q \cdot z} =$$

$$= (2\pi)^4 \delta^{(4)}(q - P) N_P \int \mathrm{d}^4 z \delta(L \cdot z) \bar{\psi}(z^P) \gamma_\mu \gamma_5 \psi(z_P) \, \mathrm{e}^{\mathrm{i}P \cdot z} =$$

$$= (2\pi)^4 \delta^{(4)}(q - P) \cdot Z_\mu(P). \tag{4.1}$$

The BM state $|M, P, y\rangle$ in (4.1) can be decomposed as in (3.1) and (3.5) and one obtains

$$\int d^4y J(z) \int d^3l \frac{\varphi_P(\vec{l}, \omega)}{2\omega} \langle 0| J_{\mu 5}(z) |\vec{l}\rangle \, e^{i(q-P)y} e^{iq\cdot z} =$$

$$= (2\pi)^4 \delta^{(4)}(q-P) \int d^3l \frac{\varphi_P(\vec{l}, \omega)}{2\omega} i \frac{1}{\sqrt{2\omega}} l_\mu J(z) e^{-i(l-P)\cdot z} f_\pi(M^2)(2\pi)^{-3/2}. \tag{4.2a}$$

The integral over z in (4.2a) can be carried out explicitly, giving

$$J(z) e^{-i(l-P)\cdot z} = \frac{M}{E} (2\pi)^3 \delta\left(\vec{l} - \frac{\omega}{E}\vec{P}\right). \tag{4.2b}$$

Inserting this into (4.2a) one finds:

$$(2\pi)^4 \delta^{(4)}(q-P)(2\pi)^3 \frac{M}{E} \frac{1}{1-\frac{\vec{P}^2}{E^2}} \frac{\varphi_P(\vec{P},E)}{2E\sqrt{2E}} iP_\mu f_\pi (M^2) \frac{1}{(2\pi)^{3/2}} =$$

$$= (2\pi)^4 \delta^{(4)}(q-P)(2\pi)^3 \frac{1}{2M\sqrt{2E}} \varphi_P(\vec{P}, E)\, i P_\mu f_\pi(M^2) \frac{1}{(2\pi)^{3/2}}. \tag{4.2c}$$

Comparing (4.1) and (4.2) leads to the relation which determines the pion decay constant,

$$Z_\mu(P) = \frac{(2\pi)^3}{2M\sqrt{2E}} \varphi_P(\vec{P}, E)\, i P_\mu f_\pi(M^2) \frac{1}{(2\pi)^{3/2}}. \tag{4.3}$$

Since both sides in (4.3) are dimensionless, the problems encountered in section 2 are resolved.

In our model $Z_\mu(P)$ (4.1) can be calculated explicitly. One finds

$$Z_\mu(P) = -\frac{3}{\sqrt{6}} a \left(-\frac{\vec{P}}{E}\right)^0_\mu \cdot [\int d^4z \delta(P\cdot z/M)\, \overline{\psi}(z^P) \gamma_0\gamma_5 \psi(z^P) e^{iP\cdot z}]_{\vec{P}=0} =$$

$$= +\sqrt{6}\frac{P_\mu}{M} K; \quad N_P = 1 \tag{4.4}$$

$$K = 4\pi \int_0^R r^2\, dr\, (u_d^2(r) - v_d^2(r)).$$

Here $a\left(-\frac{P}{E}\right)^{0}_{\mu}$ is a Lorentz transformation coefficient, and $(-3)/\sqrt{6}$ is the SU(6) spin-flavour factor. The functions u_d and v_d are defined in (A11).

As a first estimate we calculate

$$\frac{\varphi_P(P, E)}{\sqrt{2E}} = \sqrt{2M \frac{R_0^3}{(2\pi)^{3/2}} [1 - 6c + 15c^2]} \tag{4.5}$$

in the approximation employed in Ref. 2. (See Appendix for the values of the parameters.) Inserting (4.4) and (4.5) in (4.3) one finds:

$$f_\pi(M^2) = \left[\frac{12}{M(2\pi)^{3/2} R_0^3 (1 - 6c + 15c^2)}\right]^{1/2} \cdot K. \tag{4.6}$$

This expression is analogous to the expression for $F_\pi^{bag}(0)$* of Ref. 2. (As our calculation was based on a different theoretical scheme, no $\vec{P}$ dependent factors are produced.)

With convential bag-model parameters

$$m_u = m_d = 0; \qquad R = 3.26 \text{ GeV}^{-1} \qquad (R_0 = 1.79 \text{ GeV}^{-1}), \tag{4.7a}$$

and with

$$K = 0.48 \tag{4.7b}$$

one obtains the value

$$f_\pi = 0.534 \text{ GeV} \tag{4.7c}$$

which is almost five times larger then the experimental result[27)]

$$f_\pi(\text{exp}) = 0.944\, m_\pi = 0.132 \text{ GeV}.$$

This result shows how badly a model which uses the valence quarks only, works for the pion. As the pion is probably mostly a Goldstone boson, the lowest Fock-state seems a very poor approximation for the physical pion.

With a much larger radius R, the theoretical expression (4.7) leads to a better result. One finds[26)]:

$$f_\pi \cong f_\pi(\text{exp.}) \qquad R = 8.3 \text{ GeV}^{-1} \qquad (R_0 = 4.55 \text{ GeV}^{-1}). \tag{4.8}$$

These numbers agree with Ref. 2. The radius which was used there

$$R_0 = 3.1 \text{ GeV}^{-1},$$

* $f_\pi = \sqrt{2}\, F_\pi$

lies between our two choices (4.7) and (4.8), and gives

$$f_\pi = 0.311 \text{ GeV}.$$

It is important to use momentum eigenstates in BM based calculations. For example, $K \to \pi$ amplitudes (See Section 5 for details) calculated in either SBM or BM model do not satisfy the well-known SU(3) (flavour) based constraint[23,24]. It is difficult to impose this constraint in the SBM where no momenta are explicitly exhibited. Even in the pure BM this procedure is ill-defined.

One calculates the matrix element containing a four-quark operator sandwiched between two meson states. Such an operator would naturally appear in an estimate of the $K \to 2\pi$ decay amplitudes, when one uses current-algebra (CA) and partial conservation of axial vector current (PCAC). All technical details can be found in Refs. 23, 24. Some discussion related to our approach, but for $\vec{P} = 0$, can be found in Refs. 3, 4.

A four quark operator of the form

$$\begin{gathered} \mathcal{O}_{abcd}(z^P) = \overline{\Psi}_a(z^P)\,\Gamma_\mu \Psi_b(z^P)\,\overline{\Psi}_c(z^P)\,\Gamma^\mu \Psi_d(z^P) \\ \Gamma_\mu = \gamma_\mu(1-\gamma_5) \end{gathered} \tag{4.9}$$

appears in an effective weak Hamiltonian H_w[23,24], where a, b, c, d denote quark flavours. This is the form which this operator has in the BM.

The matrix element is

$$\begin{aligned} &\int \mathrm{d}^3 y J(z)\,\langle y, P_2, M_2|\mathcal{O}_{abcd}(z^P)|\,M_1, P_1, y\rangle = \\ &= (2\pi)^3\,\delta^{(3)}(\vec{P}_2 - \vec{P}_1)\,\mathrm{e}^{\mathrm{i}(E_2-E_1)y_0}\,J(z)\,\langle 0, P_2, M_2\,|\mathcal{O}_{abcd}|\,M_1, P_1, 0\rangle = \\ &= (2\pi)^3\,\delta^{(3)}(\vec{P}_2 - \vec{P}_1)\,\mathrm{e}^{\mathrm{i}(E_2-E_1)\lambda_0}\,\hat{W}(P_2, P_1). \end{aligned} \tag{4.10}$$

The decomposition (3.1) leads to

$$\int \mathrm{d}^3 y J(z)\,\mathrm{e}^{\mathrm{i}(P_2-P_1)y} \int \mathrm{d}^3 l\,\mathrm{d}^3 \tilde{l}\cdot \frac{\varphi^*_{P_2}(\vec{\tilde{l}}, \tilde{\omega})}{2\tilde{\omega}}\cdot\frac{\varphi_{P_1}(\vec{l}, \omega)}{2\omega}\cdot\langle\vec{\tilde{l}}|\,\mathcal{O}_{abcd}(z)|\vec{l}\rangle. \tag{4.11}$$

According to a well known theorem[25] the $K \to 2\pi$ decay amplitude must vanish in the SU(3) (flavour) symmetry limit. The matrix elements in (4.10) and (4.11) are related to the $K \to 2\pi$ decay amplitude through PCAC and CA[23,24], which involves a reduction of one pion field whose four momentum P_μ^π goes to zero. This means that the theoretical amplitude which results after the reduction, should vanish when all four momenta (i. e. those belonging to the kaon and to the remaining pion), go to zero. Thus if one decomposes[23] the $K \to 2\pi$ decay amplitude into the leading orders in meson momenta, the leading terms should at least be bilinear in those momenta.

Additional details concerning this question and the matrix elements in (4.10) and (4.11) are given in Section 5. Here we use the fact[23] that the matrix element in (4.11) can be written as

$$\langle \tilde{\vec{l}} | \hat{\mathcal{O}}_{abcd}(z) | \vec{l} \rangle = e^{i(\tilde{l}-l)\cdot z} \frac{1}{(2\pi)^3 \sqrt{2\omega}\sqrt{2\tilde{\omega}}} \hat{a}\tilde{l} \cdot l; \quad l^0 = \omega. \tag{4.12}$$

In general $\hat{a}$ must be a scalar function of $\tilde{l}$ and l.

Clearly one must have

$$\hat{a} \xrightarrow[\tilde{\vec{l}} \to 0]{\vec{l} \to 0} \text{const.}$$

In the following procedure $\hat{a}$ will be treated as a constant. The meaning of this last assumption will be discussed below.

The physical amplitude (See (5.10) below) is determined by the linear combination of four-quark operators[23]. In an actual calculation one has to make the replacements

$$\mathcal{O}_{abcd} \to \sqrt{2}\,\tilde{G}_F \sum_i c_i\, \mathcal{O}_i$$

$$\hat{W} \to W$$

$$\hat{a} \to a.$$

The integration over d^3y in (4.11) was selected in order to avoid the embarrassing condition

$$\vec{P}_1 = \vec{P}_2, \qquad E_1 = E_2. \tag{4.13}$$

This condition leads to an entirely unphysical $W(P_1, P_2)$ which corresponds to a situation in which both mesons, K and π, have equal masses. In the usual SBM approaches[3,23,27] the calculated quantity has been

$$\vec{P}_2 = \vec{P}_1 = 0$$

$$W(M_2, M_1), \qquad M_2 \neq M_1.$$

After the integration over y, both (4.10) and (4.11) can be divided by the same factor $(2\pi)^3\, \delta^{(3)}(\vec{P}_2 - \vec{P}_1) \cdot \exp[i(E_2 - E_1)y^0]$. One ends with an equality

$$W(P_2, P_1)|_{\vec{P}_1=\vec{P}_2} = \int \mathrm{d}^3 l\, \mathrm{d}^3 \tilde{l} \int \mathrm{d}^4 z \delta (L \cdot z)\, \mathrm{e}^{\mathrm{i}(\tilde{l}-l)\cdot z} \cdot$$

$$\cdot \frac{\varphi^*_{\tilde{P}_2}(\vec{\tilde{l}}, \tilde{\omega})\, \varphi_{P_1}(\vec{l}, \omega)}{(2\tilde{\omega}\, 2\omega)^{3/2}\, (2\pi)^3}$$

$$\cdot\, a \tilde{l} \cdot l|_{\vec{P}_1=\vec{P}_2} = aI(\vec{P}_1). \tag{4.14}$$

The condition $\vec{P}_1 = \vec{P}_2$ in (4.14) can be dropped if necessary by omitting the Dirac δ-function $\delta(\vec{P}_2 - \vec{P}_1)$. In that case $W(P_2, P_1)$ can be understood as a suitable definition for the physical quantity which one studies. One can immediately draw parallels with applications of CA[29] where one usually does not impose any constraint on the momenta of the initial and final hadrons.

In the rest frame one has

$$J(z)\, \mathrm{e}^{\mathrm{i}(\tilde{l}-l)\cdot z} = \int \mathrm{d}^3 z\, \mathrm{e}^{\mathrm{i}(\vec{\tilde{l}}-\vec{l})\cdot \vec{z}} = (2\pi)^3\, \delta^{(3)}(\vec{\tilde{l}} - \vec{l}). \tag{4.15}$$

From (4.14) one obtains the relation

$$W(M_2, M_1) = (2\pi)^3 \int \mathrm{d}^3 l \frac{\varphi^*_{M_2}(\vec{l}, \tilde{\omega})\, \varphi_{M_1}(\vec{l}, \omega)}{(2\tilde{\omega}\, 2\omega)^{3/2}\, (2\pi)} \cdot a \cdot (\omega\tilde{\omega} - \vec{l}^2)$$

$$\tilde{\omega}^2 = \vec{l}^2 + M_2^2; \qquad \omega^2 = \vec{l}^2 + M_1^2 \tag{4.16}$$

which connects a SBM calculation with a decomposition of the »mock physical« decay amplitude. The adjective »mock« is earned here by the fact that the amplitude a cannot be closer to the »real physical« world than the quantities ω and φ_M are.

5. *Kaon — pion weak amplitudes*

The weak amplitude $W(P_f, P_i)$ for a $K \to \pi$ transition is needed in the calculation of unusual kaon decays such as $K \to \pi\gamma_Y$. The exotic massive vector particle γ_Y couples to new flavours; in this case to strangeness[28]. The amplitude $W(P_f, P_i)$ also appears in some theoretical descriptions[23,24,26] of $K \to 2\pi$ decays. A very simplified theoretical picture[23,24], based on current algebra (CA), contains only W amplitudes multiplied by so called continuation factors, which depend on meson masses Obviously, if one wants to have some acceptable estimate for W, which is to be used in the calculation of the $K \to \pi\gamma_Y$ transition, one has to produce W's which fit experimental $K \to 2\pi$ data.

In a CA-based procedure one uses a general momentum expansion[23)]

$$A\,(K\,(P)) \to \pi\,(P_1)\,\pi\,(P_2)) = L_0 + P_1\,(P_2 + P)\,L_1 + \\ + P_2\,(P_1 + P)\,L_2 + P\,(P_2 - P_1)\,L_3 + \dots \tag{5.1}$$

In general L_i's are some scalar functions of all meson momenta. This decomposition can be also understood[23,24)] as a decomposition in the leading order of meson momenta. In that case the parameters L_i are constants which are calculable in the vacuum-saturation approximation[24,27)] or in the chiral perturbation theory[24)]. In both cases one obtains explicitly $L_0 = 0$. This is also a consequence of the Cabibbo—Gell-Mann theorem[25)]. In the strict SU(3) flavour symmetry the amplitude (5.1) must vanish. (The strict symmetry means that all meson have same masses and momenta.) In that limit only the first term in (5.1) survives, i. e.

$$A\,(M(P)) \to M\,(P)\,M\,(P)) = \widetilde{L}_0. \tag{5.2}$$

If $\widetilde{L}_0$ is a constant, it must be zero. However, in the general case, one might have

$$L_0\,(P_2, P_1, P) \neq 0; \qquad L_0\,(P, P, P) = 0. \tag{5.3}$$

Direct calculations, based on BM (see (5.28) below) show that the L_i's are not constants.

All earlier approaches[23,24,26)] assumed $L_0 \equiv 0$ even when using SBM states.

The CA-based procedure[29)] connects the W amplitude to the decay amplitude (5.1) in the limits $P_1 = 0$ or $P_2 = 0$. One obtains

$$\begin{aligned} A\,(P, P_1, P_2) &\xrightarrow{P_1=0} L_0\,(P, 0, P_2) + P \cdot P_2\,[L_2\,(P, 0, P_2) + L_3\,(P, 0, P_2)] \\ A\,(P, P_1, P_2) &\xrightarrow{P_2=0} L_0\,(P, P_1, 0) + P \cdot P_1\,[L_1\,(P, P_1, 0) - L_3\,(P, P_1, 0)]. \end{aligned} \tag{5.4}$$

The theorem[25)] applies to the r. h. s. of (5.4) only when all four-momenta are equal. As one of them is always fixed to be zero, this means:

$$L_0\,(0, 0, 0) \equiv 0. \tag{5.5}$$

In the BM-based calculations (or even SBM-based ones) this is not a trivial statement. The limit (5.5) is not calculable in BM or better to say SBM. In the bag model one does not know how to construct states corresponding to a massles-pion or kaon. One can only assume that (5.5) is correct.

This influences the extrapolation (or conrtinuation)[23,24)] from the CA-based result to the physical amplitude. By decomposing the BM state (3.1) one has to introduce momentum eigenstates. Only such a procedure ensures that the momentum dependence of L_i's can be studied properly. Otherwise everything is messed up by hidden P dependence of meson wave functions, which exists even in SBM[1-5].

Before going into that, let us briefly review the approximations made in earlier[23,24,27] theoretical investigations. There the limiting procedure has been carried out by taking into account the overall Dirac δ-function

$$\delta(P_1 + P_2 - P). \tag{5.6}$$

This gives[23,24]

$$\begin{aligned} \lim_{P_1 \to 0} (A\delta) &= (L_2 + L_3)\,\varkappa^2 = A_1^{CA} \\ \lim_{P_2 \to 0} (A\delta) &= (L_1 - L_3)\,\varkappa^2 = A_2^{CA}. \end{aligned} \tag{5.7}$$

Here the L_i's are treated as constants and $\varkappa$ is the common four-momentum of the kaon and pion in the $K \to \pi$ transition.

From expression (5.7) one finds

$$\delta(P_1 + P_2 - P) \to \delta(P_2 - P); \qquad P_2 = P = \varkappa \tag{5.8a}$$

and

$$P_1 = P = \varkappa. \tag{5.8b}$$

If the L_i's are assumed to be constants the physical amplitude is approximated by[23,24]:

$$A(K \to 2\pi)_{\rm phys} \cong \frac{m_K^2 - m_\pi^2}{\varkappa^2}\,(A_1^{CA} + A_2^{CA}). \tag{5.9}$$

Such an approximation means that the expression (4.16) can be calculated at

$$M_1 = M_2 = m; \qquad \varkappa^2 = m^2, \qquad \varkappa^\mu = (m, 0)$$

$$W_i(m, m) = a_i \varkappa^2 \int \mathrm{d}^3 l\, \frac{|\varphi_m(\vec{l}, \omega)|^2}{(2\omega)^3} = a_i \varkappa^2 S \tag{5.10}$$

$$\frac{1}{f_\pi}\, a_i \varkappa^2 = \frac{1}{f_\pi}\,\frac{W_i(m, m)}{S} = A_i^{CA}(m, m)$$

$$A(K \to 2\pi)_{\rm phys} \cong \frac{1}{f_\pi}\,\frac{(m_K^2 - m_\pi^2)}{\varkappa^2}\,(a_1 + a_2)\,\varkappa^2.$$

The last line in the approximation (5.8) was usually[23,27] estimated as follows

$$S \cong (\sqrt{2} m_K)^{-1}$$

$$\varkappa^2 = m_K^2; \quad \text{or} \quad \varkappa^2 = \frac{1}{2}(m_K^2 + m_\pi^2) \tag{5.11}$$

$$A(K \to 2\pi)_{\text{phys}} \cong \frac{(m_K^2 - m_\pi^2)}{\varkappa^2} \frac{1}{f_\pi} \sum_i W_i(m_K, m_\pi)(\sqrt{2} m_K).$$

However the constraints (5.6) are not absolutely necessary. One can remove one pion momentum and keep the physical momenta of other mesons. It seems that such point of view was taken in other CA applications[29].

The calculation of $W(P_f, P_i)$ (4.10) is simplified when both mesons have equal momenta $\vec{P}_f = \vec{P}_i$. For the physical $K \to 2\pi$ decay, one can choose

$$\vec{P}_1 = 0$$

$$\vec{P}_2 = \vec{P} \qquad E = m_\pi + E_2 \tag{5.12a}$$

which gives

$$\vec{P}_D = 0.73917 \text{ GeV}. \tag{5.12b}$$

One finds for example

$$\hat{W}^A(\vec{P}, m_K, m_\pi) = \langle K^- | (\bar{s}u)(\bar{u}d)_{AA} | \pi^- \rangle (\vec{P}) =$$

$$= \frac{3\pi}{8 m_i m_f a_i a_f L^0} \int_0^R r^2 \, \mathrm{d}r \left\{ \frac{j_0(\varrho)}{\sqrt{1+\gamma}} [[(a_i^2 - \vec{P}^2)(a_f^2 - \vec{P}^2) + \right.$$

$$+ 2(a_i a_f - \vec{P}^2)](-4)(u_1 v_3 + u_3 v_1) u_1 v_1 +$$

$$+ [(a_i^2 + \vec{P}^2)(a_f^2 + \vec{P}^2) - 4 a_i a_f \vec{P}^2] \, 4 (u_1^3 u_3 + v_1^3 v_3)] +$$

$$+ \frac{j_1(\varrho)}{1+\gamma} |\vec{P}| (1 + \beta \vec{P}^2) [a_i (a_f^2 + \vec{P}^2) - a_f (a_i^2 + \vec{P}^2)] \, 8 [v_1^3 u_3 - \tag{5.13}$$

$$- u_1^3 v_3 + 3(v_1 v_3 - u_1 u_3) u_1 v_1] +$$

$$\left. + \frac{\frac{1}{2} j_0(\varrho) - \frac{2}{3} j_2(\varrho)}{(1+\gamma)^{3/2}} \vec{P}^2 (1 + \beta \vec{P}^2)^2 (a_i - a_f)^2 (-16)(u_1 v_3 + u_3 v_1) u_1 v_1] \right\}$$

$$(1, 2, 3 = u, d, s).$$

Here:

$$a_N = (E_N + M_N); \qquad N = i, f; \qquad i = \pi, \qquad f = K$$

$$\beta = \frac{(E_i/m_i) + (E_f/m_f) - (E_i + m_i)(1/M_i + 1/M_f)}{M_i (E_i + M_i)(E_i/M_i + E_f/M_f)}$$

$$\gamma = 2\beta \vec{P}^2 + \beta^2 \vec{P}^4$$

$$\varrho = \frac{ar}{\sqrt{1 + \gamma}} \tag{5.14}$$

$$a = (3\varepsilon_1 + \varepsilon_3) \cdot \frac{E_f - E_i}{M_i M_f \left(\frac{E_i}{M_i} + \frac{E_f}{M_f}\right)} |\vec{P}|$$

$$L^0 = \frac{E_i/M_i + E_f/M_f}{[(E_i/M_i + E_f/M_f)^2 - \vec{P}^2 (1/M_i + 1/M_f)^2]^{1/2}}.$$

Calculation was carried out for the product of axial vector curents (AA). (Here L^0 is the time component of vector L^μ (2.10).)

In the limit $\vec{P} = 0$ this expression goes into the SBM values which were used earlier[23/24/27]. Thus for example

$$\langle K^- | (\bar{s}u)(\bar{u}d)_{AA} | \pi^- \rangle (\vec{P} = 0) = \frac{1}{4} 6 (a - 3b). \tag{5.15}$$

Here a and b are well known notations for the integrals over the SBM wave functions

$$a = 4\pi \int_0^R r^2 \mathrm{d}r \, [u_1^2 u_3 + v_1^2 v_3]$$
$$\tag{5.16}$$
$$b = 4\pi \int_0^R r^2 \mathrm{d}r \, [u_1^2 v_1 v_3 + u_1 u_3 v_1^2].$$

The subscript show quark flavours (1, 2, 3 = u, d, s). In the numerical evaluations one can use[23] for the quark masses

$$m_u = m_d = 0; \qquad m_s = 0.279 \text{ GeV}. \tag{5.17}$$

Thus

$$u_1 = u_2; \qquad v_1 = v_2; \qquad \varepsilon_1 = \varepsilon_2.$$

The expression (5.15) is one of the terms which were included in the expression (5.11).

There are many dubious points in the derivation of the expression (5.11), which has been just described. First: the value $S = (\sqrt{2}m_K)^{-1}$, which was determined by dimensional arguments[23], might be too large. A dimension was missing for the reasons which became apparent in formulae (2.7). The remedy (4.3) was found by the expansion into momentum eigenstates (3.1). Thus for the $K \to 2\pi$ transition one should use the expression (5.10) and calculate (5.15) and/or the analogous contributions for $m = m_K = \varkappa$. The mass correction is not that important. One finds

$$m_\pi \neq m_K; \qquad a = 4.78 \cdot 10^{-3}\ \mathrm{GeV}^3; \qquad b = 1.94 \cdot 10^{-3}\ \mathrm{GeV}^3$$

$$R = 3.385\ \mathrm{GeV}^{-1} \tag{5.18}$$

$$m_\pi = m_K = m; \qquad a = 4.51 \cdot 10^{-3}\ \mathrm{GeV}^3; \qquad b = 2.07 \cdot 10^{-3}\ \mathrm{GeV}^3.$$

The first values were obtained by using (5.16). In the second case one makes the replacements $u_3, v_3 \to u_1, v_1$ in (5.16).

The direct calculation of integral S in (5.10) gives:

$$S(m_\pi) = 0.649\ \mathrm{GeV}^{-1}$$

$$S(m_K) = 0.503\ \mathrm{GeV}^{-1} \tag{5.19}$$

$$[(\sqrt{2}m_K)^{-1} = 1.43\ \mathrm{GeV}^{-1}].$$

The mass dependence enters in the expression (5.10) for S through the relation $\omega = \sqrt{\vec{l}^2 + \varkappa^2}$; $\varkappa = m_\pi, m_K$.

Formulae (5.10) can be rewritten (and recalculated) by using systematically $\varkappa^\mu = (m, 0)$ for example

$$A(K \to 2\pi)_{\mathrm{phys}} \cong \frac{m_K^2 - m_\pi^2}{m^2} \frac{1}{f_\pi} \sum_i W_i(m, m)\, S^{-1}(m). \tag{5.20}$$

If all quarks are massless, the SBM value of m is

$$m = 2 \cdot \frac{2.04}{R} + F(R). \tag{5.21}$$

Here R is the bag radius and $F(R)$ are bag-model terms which combine with the quark energies $(2.04)/R$ to produce the meson mass.

It might be better to use (5.20) instead of (5.11). However, this might not be worthwhile. The derivation of (5.20) is beset by a serious ambiguity, concerning L_0.

The most important ingredient in the derivation of either (5.11) or (5.12) was the relation (5.3) or (5.5) .In order to illustrate that somewhat better, let us study the expression (5.4) in the limit $\vec{P} = \vec{P}_1 = \vec{P}_2 = 0$, where one obtains:

$$\begin{aligned} A_1^{CA} &= L_0^1 + (m_K m_\pi)(L_2^1 + L_3^1) \\ A_2^{CA} &= L_0^2 + (m_K m_\pi)(L_1^2 - L_3^2). \end{aligned} \tag{5.22}$$

Here

$$L_i^1 = L_i(m_K, 0, m_\pi); \qquad L_i^2 = L_i(m_K, m_\pi, 0) \qquad i = 0, 1, 2.$$

The physical decay amplitude (5.1) with all mesons on the mass shell and with overall four-momentum conservation is

$$A_{phys} = L_0 + (m_K^2 - m_\pi^2)(L_1 + L_2). \tag{5.23}$$

Here we have neglected terms proportional to $\Delta m_\pi^2 = m_{\pi^+}^2 - m_{\pi^0}^2$ which can appear in the $K^+ \to \pi^+\pi^0$ transition. Only with

$$L_0 = L_0^1 = L_0^2 = 0; \quad L_1 \cong L_1^2; \quad L_2 \cong L_2^1; \quad L_3^1 = L_3^2 \tag{5.24}$$

can one combine (5.23) and (5.24) into something similar to (5.11) or (5.20) (See (5.33) below).

If one does not use

$$L_0^{1,2} \equiv 0 \tag{5.25}$$

one obtains from (5.22)

$$\begin{aligned} (A_1^{CA} + A_2^{CA})\frac{m_K^2 - m_\pi^2}{m_K m_\pi} &= \frac{m_K^2 - m_\pi^2}{m_K m_\pi}(L_0^1 + L_0^2) + \\ &+ (m_K^2 - m_\pi^2)(L_1^2 + L_2^1 + L_3^1 - L_3^2). \end{aligned} \tag{5.26}$$

The first term in (5.26) can hardly be approximately equal to L_0 in (5.23).

However one can find some general, more or less convincing, arguments that $K \to 2\pi$ amplitude must be at least bilinear in meson momenta. They are summarised in the Appendix.

In our language that means

$$\begin{aligned} L_0 &= L_0^0 + \sum_{i,j} p_i \cdot p_j L_{ij} + \ldots \\ L_0^0 &\equiv 0. \end{aligned} \tag{5.27}$$

The terms L_{ij} are included in L_1,L_2 and L_3. One obtains the form (5.1) with L_0=0.

In the following we will understand CA-based expressions for $K \to 2\pi$ amplitudes as suitable parametrisations for experimental values. The parameters, determined by fitting experimental $K \to 2\pi$ amplitudes will then be used to evaluate $K \to \pi\gamma_Y$ amplitudes.

Using BM one can calculate CA-inspired amplitudes, in the frame (5.12a), as some functions of $\vec{P}$. Their ingredients are W^N matrix elements, one of which, (5.13), is shown above. For the BM amplitudes the notation a_{BM} will be used. These quantities are proportional to the A^{CA} amplitudes in (5.26), i. e. $A^{CA} \sim \sim a_{BM} I^{-1}$. One obtains

$$a_{BM}(K^+ \to \pi^+\pi^0) = \frac{1}{f_\pi} \tilde{G}_F 6 c_4 D_1$$

$$a_{BM}(K^0 \to \pi^+\pi^0) = \frac{\sqrt{2}}{f_\pi} \tilde{G}_F \{[c_1 - 2(c_2 + c_3 + c_4)] D_1 + + \left(c_6 + \frac{16}{3} c_5\right) D_2 - \eta A\} \tag{5.28a}$$

$$a_{BM}(K^0 \to \pi^0\pi^0) = \frac{\sqrt{2}}{f_\pi} \tilde{G}_F \{[c_1 - 2(c_2 + c_3 - 2c_4)] D_1 + + \left(c_6 + \frac{16}{3} c_5\right) D_2 - \eta A\}$$

$$\tilde{G}_F = G_F \sin\Theta_c \cos\Theta_c.$$

Here c_i are the Wilson coefficients defined and given in Refs. 23, 24, for example. Furthermore

$$D_1 = \frac{4\pi}{L^0} \int_0^R r^2\, \mathrm{d}r \left\{ \frac{j_0(\varrho)}{\sqrt{1+\gamma}} \left[\frac{E_i E_f - \vec{P}^2}{M_i M_f} (u_1^3 u_3 + v_1^3 v_3) - - \frac{E_i E_f - \vec{P}^2 + 2 M_i M_f}{M_i M_f} (u_1 v_3 + u_3 v_1) u_1 v_1 \right] + + \frac{j_1(\varrho)}{1+\gamma} |\vec{P}| (1 + \beta \vec{P}^2) \frac{E_f - E_i}{M_i M_f} [v_1^3 u_3 - u_1^3 v_3 + 3(v_1 v_3 - u_1 u_3) u_1 v_1] - - \frac{\frac{1}{3} j_0(\varrho) - \frac{2}{3} j_2(\varrho)}{(1+\gamma)^{3/2}} \cdot \frac{\vec{P}^2 (1 + \beta \vec{P}^2)^2 (a_i - a_f)^2}{M_i M_f a_i a_f} (u_1 v_3 + u_3 v_1) u_1 v_1 \right\} \tag{5.28b}$$

$$D_2 = \frac{4\pi}{L^0} \int_0^R r^2\, \mathrm{d}r . \frac{j_0(\varrho)}{\sqrt{1+\gamma}} [u_1^3 u_3 + v_1^3 v_3 + (u_1 v_3 + u_3 v_1) u_1 v_1]$$

$$A = \frac{1}{L^0} \frac{1}{\sqrt{4M_i M_f a_i a_f}} \int_0^R r^2 \, \mathrm{d}r \left[\frac{4\pi}{\sqrt{1+\gamma}} j_0(\varrho') (a_i a_f - \vec{P}^2) \cdot (u_1 u_3 - v_1 v_3) + \right.$$

$$\left. + \frac{4\pi}{1+\gamma} j_1(\varrho') |\vec{P}| (1 + \beta \vec{P}^2)(a_i - a_f)(u_1 v_3 + u_3 v_1) \right] \cdot Z$$

$$Z = \frac{1}{\sqrt{4M_i M_f a_i a_f}} \left[(a_i a_f + \vec{P}^2) - \frac{\vec{L} \cdot \vec{P}}{L^0} (a_i + a_f) \right] \int_0^R r^2 \mathrm{d}r \cdot$$

$$\cdot \frac{4\pi}{\sqrt{1+\gamma}} j_0(\varrho'') (u_1^2 + v_1^2)$$

Here

$$\varrho' = \varrho \frac{\varepsilon_1 + \varepsilon_3}{3\varepsilon_1 + \varepsilon_3} \qquad \varrho'' = \varrho \frac{2\varepsilon_1}{3\varepsilon_1 + \varepsilon_3}. \tag{5.28c}$$

All other notations are explained in (5.14) and (5.16).

In the static limit $\vec{P} = 0$ one obtains

$$D_1 = a - 3b \qquad D_2 = (a + b). \tag{5.29}$$

With $\eta = 0$ the formulae (5.27) go into the expressions (3.3) of Ref. 23.

The term multiplied by η corresponds to a $\Delta I = 1/2$, SU(3) octet, operator

$$\eta h_\omega \left(\Delta I = \frac{1}{2} \right) = \sqrt{2} \eta \, \tilde{G}_F \int \mathrm{d}\tau \, \bar{s}(x) \, \mathrm{d}(x) \tag{5.30}$$

which was introduced »ad hoc« in order to fit experimental decay amplitudes. It is added to the standard H_ω, as for example

$$\langle \pi^+ | H_\omega | K^+ \rangle \to \langle \pi^+ | (H_\omega + \eta h_\omega) | K^+ \rangle. \tag{5.31}$$

The parameter η is determined by fitting experimental data, see (5.34) below. The operator (5.30) would not contribute to the physical $K \to 2\pi$ amplitude[24] as it is proportional to the *s-d* self energy term. Here this operator is used as a compensation for any ambiguity associated with the CA procedure (5.26) and as rough parametrization of the soft-gluon nonperturbative effects (See (5.5) in Ref. 24).

Experimentaly $|A\,(K^0 \to \pi^+\pi^-)|$ is always larger than $|A\,(K^0 \to \pi^0\pi^0)|$. Unfortunately formulae (5.28) with $\eta = 0$ give, either in the static limit ($\vec{P} = 0$), or with $\vec{P}$ given by (5.12), always $|A\,(K^0 \to \pi^0\pi^0)| > |A\,(K^0 \to \pi^+\pi^-)|$. This is reversed in the chiral-bag model[30]. In the BM the »ad hoc« introduction of the term (5.29) seems to be the only remedy.

The expressions a_{BM}, calculated in the BM at $\vec{P}_D$ given by (5.12b), are used as l. h. s. in equation (4.14). This determines the corresponding α's (5.10) which are supposed to be identical to the combinations $L_2 + L_3$ and $L_1 - L_3$ (5.7). If $L_0 = 0$ those combinations determine the on-mass-shell (o. m. s) value of a general amplitude (5.1)

$$A\,(K \to 2\pi)_{o \cdot m \cdot s} = L_0 + (m_K^2 - m_\pi^2)\,L_1 + (m_K^2 - m_\pi^2)\,L_2 + \dots$$
$$\cong (m_K^2 - m_\pi^2)\,[(L_1 - L_3) + (L_2 + L_3)]. \tag{5.32}$$

The final theoretical approximation has the structure

$$A\,(K \to 2\pi)_{th} = a_{BM}\,(\vec{P}_D)\,(m_K^2 - m_\pi^2)\,I^{-1}\,(\vec{P}_D). \tag{5.33}$$

By changing parameters R (bag radius) and η the experimental amplitudes can be fitted. With

$$R = 3.385\ \text{GeV}^{-1}$$
$$\eta = -18.18927 \cdot 10^{-3}\ \text{GeV}^3 \tag{5.34}$$

one obtains an almost perfect fit, as shown in Table 1.

TABLE 1.

Decay	$\lvert A_{th}\rvert \cdot 10^8$ GeV^{-1}	$\lvert A_{exp}\rvert \cdot 10^8$ GeV^{-1}
$K^+ \to \pi^+\pi^0$	1.8338	1.83
$K^0 \to \pi^+\pi^-$	27.959	27.96
$K^0 \to \pi^0\pi^0$	25.367	25.36

Theoretical $K \to 2\pi$ amplitudes corresponding to formula (5.33).

However the α which emerges from (4.14) is not a constant, but a function of $\vec{P}$, i. e. $\alpha = \alpha\,(\vec{P})$. This is immediately obvious from expressions (5.28) which determine the l. h. s. of (4.14). In those expressions one can find terms proportional to $\vec{P}^4$ and to higher powers (in Bessel functions). This means that BM based calculations include higher powers of the meson momenta than are explicitly displayed in (5.1).

This departure from a constant α is to some extent corrected by recalculating α for each $\vec{P}$ which is needed in some process. A more exact way would be to substitute under the integral in the r. h. s. of (5.14) the power series expansion

$$\alpha_1 l \cdot \tilde{l} + \alpha_2 l^2 \tilde{l}^2 + \alpha_3 (l \cdot \tilde{l})^2 + \ldots \tag{5.35}$$

Obviously it would be prohibitively difficult to determine α_i's from (4.14).

This $\vec{P}$ dependence is quite strong as shown in Fig. 1, below.

6. Hyperphoton channel

The amplitudes a_{BM} (5.28), with a suitable continuation to off-mass-shell kaons, are used as an input in γ_Y bremstrahlung diagrams, which where used by Ref. 31. Our method differes from other estimates of $K \to \pi\gamma_Y$ transition amplitudes[32-35] either in details or in general assumptions. The estimate given here does not include a (direct) γ_Y emission from the weak vertex.

The decay rate is given by formula (13) of Ref. 31

$$\Gamma(K^+ \to \pi^+\gamma_Y) = \frac{|\vec{P}|}{2m_K^2} |a(K^+ \to \pi^+)|^2 \frac{f^2}{m_Y^2} \tag{6.1}$$

$$a(K^+ \to \pi^+) = f_\pi \cdot a_{BM}(K^+ \to \pi^+\pi^0)\, m_\pi^2 I^{-1}(\vec{P}).$$

Here $\vec{P}$ is the momentum of the outgoing pion in the frame in which the initial kaon is at rest. By neglecting the hyperphoton mass ($m_Y \cong 0$) one obtains

$$|\vec{P}| \cong \frac{m_K^2 - m_\pi^2}{2m_K}. \tag{6.2}$$

(Note that one must retain $m_Y \neq 0$ in the factor f^2/m_Y^2.)

The product $a_{BM} I^{-1}$ is evaluated using the parameters (5.34) evaluated at the momentum (6.2) with pion on the mass shell and kaon off mass shell, i. e.

$$\begin{aligned} E_K' &= E_\pi \\ E_\pi^2 &= \vec{P}^2 + m_\pi^2. \end{aligned} \tag{6.3}$$

One finds

$$\begin{aligned} a(K^+ \to \pi^+) &= 1.18139 \cdot 10^{-9}\ \mathrm{GeV}^2 \\ I(\vec{P}) &= 0.0251868\ \mathrm{GeV} \end{aligned} \tag{6.4}$$

and

$$BR = \frac{\Gamma(K^{\pm} \to \pi^{\pm} \gamma_Y)}{\Gamma(K^{\pm} \to \text{all})} = 2.533 \cdot 10^{16}\ \text{eV}^2 f^2 \lambda^2. \tag{6.5}$$

Here $f^2/m_Y^2 = f^2\lambda^2$. Other existing theoretical values for BR (6.5), in the same units ($10^{16}\ \text{eV}^2 f^2 \lambda^2$), are given in Table 2.

TABLE 2.

Ref.	BR
31	6.163
32	403.0
33	380.0
34	320.0 33.0
35	56.0 444.0
this work	2.533

Branching ratios for $K^{\pm} \to \pi^{\pm} \gamma_Y$, in units of $10^{16}\ \text{eV}^2 f^2 \lambda^2$. See text and Eq. (6.5) for further details.

Our result is comparable only with that of Ref. 31. It is smaller than the older estimate[31)] because it includes corrections for the off-mass-shell kaon and for the difference of physical momenta $\vec{P}_D$ and $\vec{P}$. Part of that correction is in a_{BM} which is calculated at $\vec{P} \neq \vec{P}_D$, and part is in factor $I^{-1}(\vec{P})$. Our result is about 2.4 times smaller than the BR of Ref. 31. Part of the correction, (if one forgets the part which is in a_{BM}) is given by the ratio

$$\left| \frac{(I\vec{P})}{I(P_D)} \right| \cong (1.75)^2 = 3.06 \tag{6.6}$$

$$I(P_D) = 0.04407\ \text{GeV}.$$

This is close to the more exact result which was mentioned above.

The amplitude $a(K \to \pi)$ in (6.1) can be calculated for any $\vec{P}$. It is useful to define the quantities $F_i(|\vec{P}|)$ by

$$\begin{aligned} a(K^+ \to \pi^+) &= F_1(|\vec{P}|)\, m_\pi^2 \\ a(K^0 \to \pi^0) &= F_2(|\vec{P}|)\, m_\pi^2. \end{aligned} \tag{6.7}$$

Here the kaon is off mass-shell, i. e.

$$E'_K = E_\pi = (\vec{P}^2 + m_\pi^2)^{1/2}.$$

Our calculation gives $F_i(|\vec{P}|)$'s which are strongly $\vec{P}$ dependent, as shown in Fig. 1. Refs. 24 and 31 used constant F_i's

$$F_1 \sim 1.52 \cdot 10^{-7}$$

$$F_2 \sim 1.052 \cdot 10^{-7}.$$

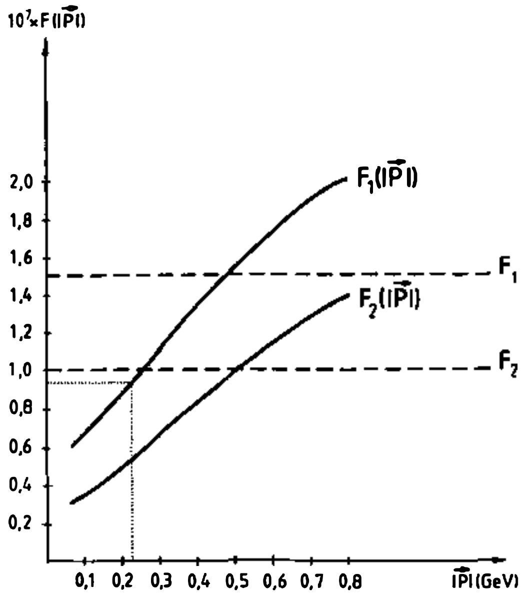

Fig. 1. Momentum $|\vec{P}|$ dependence of functions $F_i(|\vec{P}|)$ defined by (6.7). The physical value of F_1 (6.8) is shown by dotted lines. Horizontal dashed lines show F_1 and F_2 values from Ref. 24.

Our value for the momentum (6.2)

$$F_1\left(\frac{m_K^2 - m_\pi^2}{2m_K}\right) = 0.9521$$

is about 60% smaller.

7. *Conclusion*

The numerical results obtained in this paper, show that the projection to momentum eigenstates plays an important role in the BM model. As shown in Section 2 the pion decay constant f_π cannot be reasonably calculated in a pure BM or SBM model.

As an another example we calculate $K \rightarrow 2\pi$ decay amplitudes by using the formalism of Ref. 23 with parameters (5.34). The results:

$$
\begin{aligned}
A(K^+ \rightarrow \pi^+\pi^0) &= 3.58 \cdot 10^{-8}\ \mathrm{GeV} \\
A(K^0 \rightarrow \pi^+\pi^-) &= 21.29 \cdot 10^{-8}\ \mathrm{GeV} \qquad (7.1) \\
A(K^0 \rightarrow \pi^0\pi^0) &= 16.23 \cdot 10^{-8}\ \mathrm{GeV}
\end{aligned}
$$

should be compared with the ones in Table 1.

The projection to the momentum eigenstates is essential for the continuation from $K \rightarrow 2\pi$ decay amplitudes (where all particles are on the mass shell) to $K \rightarrow \rightarrow \pi$ amplitude (6.1) where kaon is off the mass-shell. Some idea about the order of correction is given by the ratio $I(\vec{P})/I(\vec{P}_D)$ (6.6).

One should mention some uncertainities concerning our result (6.5). We have choosen to fit $K^+ \rightarrow \pi^+\pi^0$ decay by changing the bag radius R. The fit could have been also made by changing c_4, i. e. by introducing a suitable multiplication factor[23,24] and by using standard radii.

The results are summarized in the Table 3.

TABLE 3.

Amplitude*)	A	B	C
$A(K^+ \rightarrow \pi^+\pi^0)$	1.834	1.83	1.83
$A(K^0 \rightarrow \pi^+\pi^-)$	27.959	27.96	27.96
$A(K^0 \rightarrow \pi^0\pi^0)$	25.367	25.37	25.37
$a(K^+ \rightarrow \pi^+)$	1.814	1.792	1.787

Decay amplitudes.

A) R, η, c_4 as used in chapter 6.
B) $R = 3.26\ \mathrm{GeV}^{-1}$; $\zeta c_4 = 0.33623$; $\eta = -18.005363$
C) $R = 3.3\ \mathrm{GeV}^{-1}$; $\zeta c_4 = 0.3585$; $\eta = -18.065847$
*) $A(K \rightarrow 2\pi)$ is in units 10^{-8} GeV and $a(K^+ \rightarrow \pi^+)$ is in units $10^{-9}\ \mathrm{GeV}^2$.

Another uncertainty concerns final state $\pi - \pi$ interaction in the $K \rightarrow 2\pi$ decays which has been often discussed[34,36–38]. In order to have some rough estimate of that effect we have corrected our theoretical amplitudes by multiplicative factors taken from Ref. 38. The QCD correction (renormalization) factors c_i were multiplied by the appropriate final state interaction factors[38] as follows:

$$\begin{aligned} c_i &\to \sqrt{d_0} c_i & i &= 1, 2, 3 \\ c_4 &\to \sqrt{d_2} c_4 & \sqrt{d_0} &= 2.2 \\ \eta &\to \sqrt{d_0} \eta & \sqrt{d_2} &= 0.51. \end{aligned} \tag{7.2}$$

It turns out that with d_i ($i = 0, 2$) factors alone and with $\eta = 0$ one cannot reproduce the experimental values. The introduction of an »ad hoc« term proportional to η is still needed. Some interesting results are shown in Table 4. Only the last column in which all amplitudes could be succesfully parametrized was used to estimate the probability for γ_Y emision.

TABLE 4.

Amplitude*)	A	B	C
$A(K^+ \to \pi^+ \pi^0)$	1.8283	1.141	1.8283
$A(K^0 \to \pi^+ \pi^-)$	5.96	5.491	27.96
$A(K^0 \to \pi^0 \pi^0)$	8.546	7.104	25.3737
$a(K^+ \to \pi^+)$	—	—	1.1508

Decay amplitudes with final state interactions.

A) $R = 2.97\ \mathrm{GeV}^{-1}$; $\eta = 0$
B) $R = 3.26\ \mathrm{GeV}^{-1}$; $\eta = 0$
C) $R = 2.97\ \mathrm{GeV}^{-1}$; $\eta = -8.9297 \cdot 10^{-3}\ \mathrm{GeV}^3$
*) $A(K \to 2\pi)$ is in units 10^{-8} GeV and $a(K^+ \to \pi^+)$ is in units $10^{-9}\ \mathrm{GeV}^2$.

All of our theoretical estimates lie within the limits

$$1.020 < \mathrm{BR}(K^+ \to \pi^+ \gamma_Y) < 2.533 \qquad \text{(in units } 10^{16}\ \mathrm{eV}^2 f^2 \lambda^2\text{)}. \tag{7.3}$$

We should also mention that results (7.3) do not include possible contributions from the direct emission terms, and these contributions presumably account for the large discrepancies among the different estimates given in Table 2. They appear because γ_Y can be emitted from a quark line inside the weak vertex. Such terms can be roughly estimated in a chiral Lagrangian by introducing γ_Y-field by a suitable gauge transformation[24,28]. The fact that our present results give further evidence of the wide variation of estimates for BR ($K^+ \to \pi^+ \gamma_Y$), lends support to the observation made in Ref. 39 that one must look to the decays $K \to \to \pi\pi\gamma_Y$ for results which are (essentially) model-independent.

All these simplifications and uncertainties (See Tables and (7.3)) justify the use of approximate SBM functions (A14) in order to simplify the projection to momentum eigenstates.

It is certain that the use of momentum eigenstates should be an essential part of theoretical evaluations.

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Appendix

One can understand (5.1) as a display of the leading terms in the power series expansion in meson momenta. From that point of view L_0 in (5.1) is a constant, i. e. L_0^0 in (5.27), and that constant must be zero. That seems to hold for any calculation based on CA. As already stated in Section 5, the $K \to 2\pi$ amplitude must be at least bilinear in meson momenta.

Let us first remember that the relation (5.3) was obtained by constructing simple SU(3) covariants containing three meson fields and transforming as 8 or 27[25]. For three identical meson fields those covariants vanish. »Identical meson fields« means that they belong to the same octet and have identical momenta, thus (5.2) and (5.3). In the limit in which all momenta are equal, the vanishing of the matrix element

$$\langle \pi (P)\, \pi (P) |\, H_w \,| K (P) \rangle \tag{A1}$$

is a consequence of SU(3) flavour group properties.

After a CA-based procedure, one ends with the expressions (5.22). They also have to vanish in the exact SU(3) symmetry limit, i. e. when $m_\pi = m_K = 0$. In that limit only L_0^1 and L_0^2 survive in (5.22), and thus $L_0 (0, 0, 0) = 0$ (5.5) is required. In BM or SBM the amplitudes A_i^{CA} are proportional to the matrix element

$$\langle \pi |\, H_w \,| K \rangle. \tag{A2}$$

Contrary to the (A1) matrix element, the vanishing of this matrix element is not a consequence of the SU(3) group properties but has a dynamical origin.

With identical meson fields Φ_β^α (α, $\beta = 1, 2, 3$) transforming as members of an octet, and assuming that H_w transforms as an octet, one can build two covariants, which transform as the (2, 3) or (3, 2) member of an octet

$$A (\Phi \to 2\Phi) = a_1 (\Phi_\alpha^2 \Phi_\delta^\alpha \Phi_3^\delta + \Phi_\alpha^3 \Phi_\delta^\alpha \Phi_2^\delta) + a_2 (\Phi_3^2 \Phi_\delta^\alpha \Phi_\alpha^\delta + \Phi_2^3 \Phi_\delta^\alpha \Phi_\alpha^\delta). \tag{A3}$$

Here 2 and 3 refer to row and column in a 3×3 meson matrix. The repeated indices are summed, over. CP invariance, which can be taken as valid for our purposes, requires

$$A (\Phi \to 2\Phi)^{CP} = (-)\, a_1 (\Phi_2^\alpha \Phi_\alpha^\delta \Phi_\delta^3 + \Phi_3^\alpha \Phi_\alpha^\delta \Phi_\delta^2) + (-)\, a_2 (\Phi_2^3 \Phi_\alpha^\delta \Phi_\delta^\alpha + \Phi_3^2 \Phi_\alpha^\delta \Phi_\delta^\alpha). \tag{A4a}$$

From (A4) one finds

$$A\,(\Phi \to 2\Phi)^{CP} = -A\,(\Phi \to 2\Phi) \tag{A4b}$$

while CP invariance requires positive sign. Thus

$$A\,(\Phi \to 2\Phi) = -A\,(\Phi \to 2\Phi) = 0.$$

For there different meson fields, say B^{α}_{β}, C^{α}_{β} and D^{α}_{β}, a CP invariant amplitude is

$$\begin{aligned} A\,(B \to DC) &= a_1\,[B^2_{\alpha}C^{\alpha}_{\delta}D^{\delta}_3 - D^2_{\alpha}C^{\alpha}_{\delta}B^{\delta}_3 + (2 \longleftrightarrow 3)] + \\ &+ a_2\,[B^2_{\alpha}D^{\alpha}_{\delta}C^{\delta}_3 - C^2_{\alpha}D^{\alpha}_{\delta}B^{\delta}_3 + (2 \longleftrightarrow 3)] + \\ &+ a_3\,[C_{\alpha}B^{\alpha}_{\delta}D^{\delta}_3 - D^2_{\alpha}B^{\alpha}_{\delta}C^{\delta}_3 + (2 \longleftrightarrow 3)]. \end{aligned} \tag{A5}$$

This is zero for identical meson fields

$$A\,(B \to DC)|_{B=D=C=\Phi} = A\,(\Phi \to 2\Phi) = 0. \tag{A6}$$

Simmilar, but lengthier, deductions can be made for the 27 pice of H_W. The meson to meson transition matrix element (A2) has the following SU(3) structure:

$$\begin{aligned} M\,(B \to C) &= h_1\,(B^2_{\alpha}C^{\alpha}_3 + B^3_{\alpha}C^{\alpha}_2) + h_2\,(B^{\alpha}_3\,C^2_{\alpha} + B^{\alpha}_2C^3_{\alpha}) \xrightarrow{CP} h_1\,(B^{\alpha}_2C^3_{\alpha} + \\ &+ B^{\alpha}_3C^2_{\alpha}) + h_2\,(B^3_{\alpha}C^{\alpha}_2 + B^2_{\alpha}C^{\alpha}_3). \end{aligned} \tag{A7}$$

CP invariance requires the $h_1 = h_2 = h$

$$M\,(B \to C) = h\,[B^2_{\alpha}C^{\alpha}_3 + B^{\alpha}_3C^2_{\alpha} + (2 \longleftrightarrow 3)]. \tag{A8}$$

This does not vanish for identical meson fields

$$M\,(\Phi \to \Phi) = 2h\,[\Phi^2_{\alpha}\Phi^{\alpha}_3 + (2 \longleftrightarrow 3)]. \tag{A9}$$

Thus the vanishing of L^i_0 in (5.22) must have dynamical origin. This coefficient or function, must be proportional to product of meson masses $m_{\pi}m_K$, i. e. it must be bilinear in meson momenta. Such a contribution is already included in the $p \cdot pL$ terms, so that one can omit L^1_0 and L^2_0, i. e. $L_0 \equiv 0$. This bilinearity in meson momenta is supported by the vacuum saturation approximation

$$\langle \pi\,(p_{\pi})|\,H_w|\,K\,(p_K)\rangle \cong \langle \pi\,(p_{\pi})|J_{\mu 5}\,|0\rangle\,\langle 0|\,J^{\mu}_5|\,K\,(p_K)\rangle = p_{\pi} \cdot p_K\ \text{const.} \tag{A10}$$

The BM quark wave functions, which enter (4.1) are given by

$$\psi(z^P) = S(P)\,\eta(z^{\perp}(P))\,e^{-i\varepsilon z_{\parallel}(P)}. \tag{A11a}$$

Here

$$S\left(\frac{-\vec{P}}{E}\right) = \frac{1}{\sqrt{2M(E+M)}}\begin{pmatrix} E+M & \vec{\sigma}\cdot\vec{P} \\ \vec{\sigma}\cdot\vec{P} & E+M \end{pmatrix} \tag{A11b}$$

$$P = (E, \vec{P}).$$

In the rest frame

$$\eta(z^{\perp}(P)) \xrightarrow{\vec{P}=0} \begin{pmatrix} u(r)\,\chi \\ i\vec{\sigma}\cdot\hat{r}v(r)\,\chi \end{pmatrix}$$

$$u(r) = \frac{N}{\sqrt{4\pi}} j_0\left(r\sqrt{\varepsilon^2 - m_q^2}\right) \tag{A11c}$$

$$v(r) = \frac{N}{\sqrt{4\pi}}\sqrt{\frac{\varepsilon - m_q}{\varepsilon + m_q}}\, j_1\left(r\sqrt{\varepsilon^2 - m_q^2}\right).$$

The following quark masses m_q [23] have been used

$$m_u = m_d = 0; \qquad m_s = 0.279 \text{ GeV}. \tag{A12}$$

One also has for u, d quarks

$$\omega_{1,-1} = 2.0428 \qquad \varepsilon = \omega R^{-1}. \tag{A13}$$

Here R is the bag radius.

In order to calculate $\varphi_P(\vec{l}, \omega_l)$ (3.1) the quark wave functions appearing in (3.10) and (3.11) where approximated[2] by

$$\psi_w(\vec{r}) = \frac{e^{-r^2/2R_0^2}}{\sqrt{R_0^3\pi^{3/2}(1+3/2\beta^2)}}\begin{pmatrix} \chi \\ i\beta\frac{\vec{\sigma}\cdot\vec{r}}{R_0}\chi \end{pmatrix}. \tag{A14}$$

The parameters R_0 ($R_0 \neq R$, but it has the same order of magnitude) and β were fixed as in Ref. 2.

For meson states containing quark flavours a, b one finds from (3.4), (3.8) and (3.10)

$$\mathcal{M}_{a,b}(P, \zeta_P^{\perp}) = \int d^4 z_a \delta(\hat{P} z_a) \overline{\psi_{wa}(z_a)_P^{\perp}} S^{-1}\left(-\frac{\vec{P}}{E}\right) \hat{P}\gamma \cdot S\left(-\frac{\vec{P}}{E}\right) \psi_{wa}((z_a - \zeta)_P^{\perp}) \cdot$$

$$\cdot \exp[i\varepsilon_a(z_a - (z_a - \zeta_P^{\perp}))_P^{\parallel}] \cdot$$

$$\cdot \int d^4 z_b \delta(\hat{P} z_b) \overline{\psi_{W_b}^c((z_b - \zeta)_P^{\perp})} S^{-1}\left(-\frac{\vec{P}}{E}\right) \hat{P} \cdot \gamma S\left(-\frac{\vec{P}}{E}\right) \psi_{W_b}^c(z_b)_P^{\perp} \cdot$$

$$\cdot \exp[i\varepsilon_b(z_b - (z_b - \zeta_P^{\perp}))_P^{\parallel}]. \tag{A15a}$$

Here $\psi_{W_b}^c$ correspond to an anti-quark, and

$$\hat{P}_\mu = P_\mu/|P|; \tag{A15b}$$

With the approximate quark wave functions ψ_W the integration can be carried out explicitly

$$\mathcal{M}_{ab}(P, \zeta_\perp^P) = e^{(\zeta_\perp^P)^2 \frac{1}{4}\left(\frac{1}{R_{0a}^2} + \frac{1}{R_{0b}^2}\right)} \cdot \left(1 + C_a \frac{(\zeta_\perp^P)^2}{R_{0a}^2}\right)\left(1 + C_b \frac{(\zeta_\perp^P)^2}{R_{0b}^2}\right); \tag{A16}$$

$$C_i = \frac{\beta_i^2}{4 + 6\beta_i^2} \qquad i = a, b.$$

The expression (A16) is obviously a Lorentz-scalar. Another useful Lorentz-scalar is the expression

$$\frac{|\varphi_P(\vec{l}, \omega_l)|^2}{2\omega_l} = \frac{2(l \cdot P)}{(2\pi)^3 M} \int d^4 \zeta \delta(\hat{P} \cdot \zeta) \mathcal{M}_{a,b}(P, \zeta_\perp^P) \cdot e^{il \cdot \zeta_\perp^P} \tag{A17}$$

which in the rest frame ($P = (M, 0)$; $l \cdot P = \omega_l \cdot M$) goes into the expression (3.11). The l. h. s. in (3.11) can be integrated, giving

$$\frac{|\varphi_P(\vec{l}, \omega_l)|^2}{2\omega_l} = \frac{2(l \cdot P)}{(2\pi)^{3/2} M} \cdot R_{ab}^3 \, e^{-\tilde{\vec{l}}^2 R_{ab}^2/2} \cdot [C_{ab}^{(1)} + C_{ab}^{(2)} \tilde{\vec{l}}^2 + C_{ab}^{(3)} \tilde{\vec{l}}^2] \tag{A18}$$

$$\tilde{\vec{l}} = \vec{l} + \frac{\vec{P}(\vec{P} \cdot \vec{l})}{M(E + M)} - \frac{\vec{P}}{M}\omega_l$$

$$C_{ab}^{(1)} = 1 - 3\left(\frac{C_a}{R_{0a}^2} + \frac{C_b}{R_{0b}^2}\right) R_{ab}^2 + 15\,\frac{C_a C_b}{R_{0a}^2\, R_{0b}^2}\, R_{ab}^4$$

$$C_{ab}^{(2)} = \left(\frac{C_a}{R_{0a}^2} + \frac{C_b}{R_{0b}^2}\right) R_{ab}^4 - 10\,\frac{C_a C_b}{R_{0a}^2\, R_{0b}^2}\, R_{ab}^6$$

$$C_{ab}^{(3)} = \frac{C_a C_b}{R_{0a}^2\, R_{0b}^2} \cdot R_{ab}^8$$

$$R_{ab}^2 = \frac{2R_{0a}^2\, R_{0b}^2}{R_{0a}^2 + R_{0b}^2}$$

The result (A18) goes into the formulae (5) of Ref. 2 when $a = b$ and $\vec{P} = 0$. The rest frame ($\vec{P} = 0$) form of (A17) differs from Refs. 2 result only by a normalisation factor. This difference is caused by our normalisation (2.3) of momentum eigenstates.

Our numerical results were obtained by using

$$\begin{aligned} \widetilde{G}_F &= G_F \sin\Theta_c \cos\Theta_c = 0.261 \cdot 10^{-5}\ \mathrm{GeV}^{-2} \\ f_\pi &= 130.43\ \mathrm{MeV} \\ m_K &= (m_{K^+} + m_{K^0})/2 = 0.49567\ \mathrm{GeV} \\ m_\pi &= (2m_{\pi^+} + m_{\pi^0})/3 = 0.13803\ \mathrm{GeV}. \end{aligned} \tag{A19}$$

In the theoretical calculation of f_π we have used in (4.6)

$$\begin{aligned} R &= 3.26\ \mathrm{GeV}^{-1} \\ R_0 &= 0.54887R = 1.79\ \mathrm{GeV}^{-1} \\ K &= 0.4797 \\ C_{ud}^{(1)} &= 1 - 6C_u + 15C_u^2 = 0.76796 = 1 - 6c + 15c^2 \text{ in (4.6).} \end{aligned} \tag{A20}$$

In the calculation of the $K \to 2\pi$ amplitudes, formulae (5.28—5.33) with $\vec{P}_D$ (5.12) we have used

$$\begin{aligned} R &= 3.385\ \mathrm{GeV}^{-1} \\ \omega_{1,-1}(s) &= 2.562133 \\ \varepsilon_u = 0.60348\ \mathrm{GeV} &\qquad \varepsilon_s = 0.7569\ \mathrm{GeV} \\ D_1 &= -0.7223 \cdot 10^{-4}\ \mathrm{GeV}^3 \\ D_2 &= 0.8596 \cdot 10^{-3}\ \mathrm{GeV}^3 \\ A &= 0.11216 \\ Z &= 0.5418493 \\ I &= 0.044071548\ \mathrm{GeV}. \end{aligned} \tag{A21}$$

The parameters used in the projecting of momentum eigenstates are shown in Table A.

TABLE A.

$M(ab)$	$C^{(1)}_{ab}$	$C^{(2)}_{ab}$	$C^{(3)}_{ab}$	$R^2_{ab} \cdot \mathrm{GeV}^2$
$\pi(ud)$	0.76796	0.23499	0.022090	3.4589
$K(us)$	0.79599	0.20301	0.015313	3.3240

Parameters which determine $\varphi(\vec{l}, \omega)$.

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MEZONSKA STANJA VREĆA-MODELA I VLASTITA STANJA LINEARNOG IMPULSA

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Mezonska stanja, koja su vlastita stanja linearnog impulsa, projicirana su iz ubrzanih stanja vreća-modela. Ta vlastita stanja impulsa upotrebljena su za proračun pionske konstante raspada f_π i slabog pion-kaon prijelaznog matričnog elementa $\langle\pi| H_\omega |K\rangle$. Prijelazni matrični elementi su parametrizirani tako da objašnjavaju $K \to 2\pi$ raspade, te su zatim upotrebljeni pri procjeni amplitude za rijetke raspade $K \to \pi\gamma_Y$ u kojima se javlja vektorska čestica γ_Y.