GLASNIK MATEMATIČKI
Vol. 60(80)(2025), 107 – 125

THE LAPLACE TRANSFORM ON THE CONES OF LATTICE-STRUCTURED BANACH SPACES

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Abstract. Characterizations of positive definite functions defined on convex cones using the Laplace transform of a measure are commonly referred to as Nussbaum-type theorems. This paper establishes a Nussbaum-type theorem in the context where the domain of a $B(\mathcal{H})$-valued positive definite function is a positive cone within a Banach space that is also a vector lattice, but not necessarily a Banach lattice. Such spaces include examples like Sobolev spaces $W^{1,p}(\Omega)$. Utilizing the Berg-Maserick theorem, we prove that the unique representing measure is Radon measure concentrated on a subset of the topological dual.

1. Introduction

A foundational result in harmonic analysis and representation theory is the Hausdorff-Bernstein-Widder theorem, established in 1928. It states that a function $\phi\colon [0,\infty\rangle \to \mathbb{R}$ is *completely monotone* on $[0,\infty\rangle$ if and only if it can be represented as the Laplace transform of a finite positive Borel measure μ on $[0,\infty\rangle$:

$$\phi(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\mu(t).$$

The notion of complete monotonicity, introduced by Hausdorff in 1921, implies that ϕ is continuous on $[0,\infty\rangle$, infinitely differentiable on $\langle 0,\infty\rangle$ and satisfies $(-1)^n\phi^{(n)}(x) \geq 0$, for all $n \in \mathbb{N}\cup\{0\}$ and $x > 0$.

Over time, the concept of complete monotonicity was replaced by the more modern framework of bounded positive definite functions. These functions,

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2020 *Mathematics Subject Classification.* 43A35, 44A10, 46A40.
Key words and phrases. Positive definite function, integral representation, Laplace transform, α-boundedness, Banach lattice.

defined as $\phi\colon [0,\infty\rangle \to \mathbb{R}$, satisfy:

$$\sum_{j,k=1}^{n} c_j c_k \phi(x_j + x_k) \geq 0, \quad \forall\, n \in \mathbb{N}, \{c_1, \ldots, c_n\} \subseteq \mathbb{R}, \{x_1, \ldots, x_n\} \subseteq [0,\infty\rangle\,.$$

The relationship between completely monotonic and positive definite functions, in a very general setting, is discussed in [10, Chapter 7].

Given that Laplace transforms serve as classic examples of positive definite functions, a natural question arises: can every positive definite function be represented as a Laplace transform, and under what conditions?

Significant progress on this question was made by Berg, Christensen, and Ressel in the 1980s [4], who developed a general theory for characterizing generalized Laplace transforms on commutative involutional semigroups with a neutral element $(S, \circ, *)$. In this abstract setting, the representing measure is defined on a broad space $\widehat{S}$, the space of all characters of S. A character is a non-zero homomorphisms of semigroups $\xi\colon S \to (\mathbb{C}, \cdot)$ such that $\xi(s^*) = \overline{\xi(s)}$ for all $s \in S$, and $\widehat{S}$ is equipped with the topology of pointwise convergence. Their work demonstrated that any exponentially bounded (bounded with respect to some absolute value) positive definite function $\phi\colon S \to \mathbb{C}$ admits an integral representation:

$$\phi(s) = \int_{\widehat{S}} \xi(s)\, \mathrm{d}\mu(\xi),$$

where μ is a unique positive Radon measure on $\widehat{S}$. This integral representation generalizes the concept of the Laplace transform, as the characters ξ are not necessarily of exponential type. To obtain the Laplace-like transform it was necessary to define the measure on a smaller set.

In a general framework, the problem involves identifying functions $\phi\colon S \to \mathbb{R}$ that can be expressed as Laplace transforms of some positive measure on $(V^*, \Sigma(V^*))$. Here, $S \subseteq V$ is a convex cone in the (not necesarilly finite-dimensional) real vector space V, V^* is the algebraic dual of V, and $\Sigma(V^*)$ is the smallest σ-algebra on V^* making all point evaluations $V^* \to \mathbb{R}$, $\lambda \mapsto \lambda(v)$ measurable. Specifically, the conditions on the function are examined to ensure it takes the form:

$$\phi(s) = \mathcal{L}(\mu)(s) = \int_{V^*} e^{\lambda(s)}\, \mathrm{d}\mu(\lambda), \quad \forall\, s \in S \tag{1.1}$$

Many authors have contributed to the study of this question. Positive definite functions on convex cones within finite-dimensional vector spaces were extensively examined by Bochner [6], Nussbaum [15] and Neeb [13, 12]. In contrast, the study of positive definite functions on infinite-dimensional convex cones has largely been motivated by concrete problems in probability theory and theoretical physics, leading to the consideration of specific cases for S. These cases were explored in the works of Dettweiler [7], Hoffmann-Jørgensen and Ressel [11], Ressel and Ricker [16], and Šikić [19].

This paper builds upon these foundations, focusing on lattice-structured Banach spaces which provide a promising environment for Nussbaum-type theorems. These spaces still form a large class of function spaces and include Sobolev spaces $W^{1,p}(\Omega)$. By leveraging the Berg-Maserick theorem, we establish a unique Radon measure representation under specific conditions.

Fitzsimmons utilized a similar framework and corresponding theorems for semigroups of bounded positive measurable functions in constructing superprocesses [8]. Šikić later generalized these results in [18], demonstrating that Fitzsimmons' construction represents a special case of a more general theory.

Glöckner conducted a particularly comprehensive study resolving problem (1.1) within the broadest framework so far in, proving that a positive definite function ϕ defined on a convex subset of a real vector space V needs to be continuous along line segments in order to satisfy (1.1) [10, Theorem 18.8]. The generalization of his work is manifested in the fact that $B(\mathcal{H})$-valued positive definite functions are observed (here $B(\mathcal{H})$ denotes the complex algebra of bounded operators on a Hilbert space $\mathcal{H}$). We will preserve this context in this paper, focusing on the case of $B(\mathcal{H})$-valued positive definite functions.

The concept of $B(\mathcal{H})$-valued positive definite functions was first introduced by B. Sz.-Nagy in 1960, where they were referred to as *positive type* functions. For a semigroup with involution $(S, \circ, *)$, these functions are mappings $\phi\colon S \circ S \to B(\mathcal{H})$ satisfying:

$$\sum_{j,k=1}^{n} \langle \phi(s_j \circ s_k^*) v_k, v_j \rangle \geq 0, \quad \forall\, n \in \mathbb{N}, \{v_1, \ldots, v_n\} \subseteq \mathcal{H}, \{s_1, \ldots, s_n\} \subseteq S.$$

If S contains a neutral element, then obviously $S \circ S = S$. In the more general case, the domain of ϕ is $S \circ S \subseteq S$.

The investigation of how different structures of convex cones impose specific conditions on positive definite functions, enabling their representation as Laplace transforms and determining when the corresponding measure is Radon, has been extensively studied by Glöckner in [10]. His research on this subject is both detailed and comprehensive.

Glöckner demonstrated that in the case of an empty interior there may not exist a *Radon* representative measure on V^* nor any representative measure on the topological dual $(V', \Sigma(V'))$. This was shown through a concrete example of a 1-bounded positive definite function on the cone $S = \ell^1_+$ of all nonnegative absolutely summable sequences [10, Proposition 20.9]. The cone S is obviously closed, generating and has empty interior. For the continuous, 1-bounded positive definite function

$$\phi\colon (x_n)_{n\in\mathbb{N}} \mapsto \prod_{n\in\mathbb{N}} (1 + x_n)^{-1}$$

on ℓ^1_+ no representing measure exists on $((\ell^1)', \Sigma((\ell^1)'))$ nor does any Radon measure exist on $(V^*, \Sigma(V^*))$. Glöckner's example highlights the challenges in generalizing Laplace transform representations to broader settings. Despite these challenges, lattice-structured Banach spaces provide a promising environment for Nussbaum-type theorems.

In this paper we examine the conditions on the initial space, convex cone, and the function itself to obtain an integral representation using the Laplace transform of a measure defined on the topological dual space. The α-boundedness condition ensures the growth of positive definite function ϕ is controlled by the absolute value α, which itself is a function that measures "size" in a way compatible with the semigroup structure. For the exact definition of α-boundedness, see Remark 2.2.

Our results are focused on lattice-structured Banach spaces $(B, \|\cdot\|, \leq)$ with a positive cone B_+ that is closed and generating. In spaces with an order unit, the positive cone also has a non-empty interior (Lemma 3.2), which guarantees the existence of a *Radon* representative measure. By refining techniques from [10] and adapting them to this specific setting, we establish that the measure is concentrated on a smaller subset of continuous α-bounded characters, which is homeomorphic to the set of α-bounded continuous linear functionals (Remark 4.6). The main result of this paper is stated here, with the proof to be presented later in Section 4.

THEOREM 1.1. *Let $(B, \|\cdot\|, \leq)$ be a lattice-structured Banach space with an order unit, and α a locally bounded absolute value on B_+. For every α-bounded positive definite function $\phi\colon B_+ \to \mathrm{Herm}^+(\mathcal{H})$ if there exists an order unit* u *and a sequence (r_n) of positive real numbers converging to zero such that*

$$\lim_{n\to\infty} \phi(r_n u) = \phi(0) \quad \textit{in the ultraweak topology},$$

then there exists a unique Radon $\mathrm{Herm}^+(\mathcal{H})$-valued measure μ on B' with support in C_α, such that

$$\phi(x) = \int_{B'} e^{\lambda(x)}\,\mathrm{d}\mu(\lambda), \quad \forall\, x \in B_+.$$

2. Vector valued positive definite functions and measures

2.1. *Commutative semigroups with involution.*

DEFINITION 2.1. *A function $\alpha\colon S \to [0, \infty\rangle$ on an involutive semigroup $(S, \circ, *)$ is called an* absolute value *if:*

(i) $\alpha \not\equiv 0$,
(ii) $\alpha(s^*) = \alpha(s)$ *for all $s \in S$,*
(iii) $\alpha(s \circ t) \leq \alpha(s)\alpha(t)$ *for all $s, t \in S$.*

If S contains a neutral element e, condition (i) *is equivalent to $\alpha(e) \geq 1$.*

REMARK 2.2. Let α be an absolute value on an involutive semigroup $(S, \circ, *)$. The exact definition of α-boundedness for $B(\mathcal{H})$-valued positive definite functions is provided in [10, Chapter 7]. There, a $B(\mathcal{H})$-valued positive definite function ϕ is α-bounded if the associated invariant positive definite kernel K_ϕ is α-bounded, i.e., K_ϕ is exponentially bounded and the associated $*$-representation is α-bounded. This leads to a more practical characterization of α-boundedness for semigroups S with a neutral element:

- $\phi\colon S \to B(\mathcal{H})$ is α-bounded if and only if $\|\phi(s)\| \leq C\alpha(s)$ for all $s \in S$ and some $C > 0$, see [14],
- $\phi\colon S \to \mathbb{C}$ is α-bounded if $|\phi(s)| \leq C\alpha(s)$, see [4].

In particular, a positive definite function $\phi\colon S \to \mathbb{C}$ is bounded if and only if it is 1-bounded.

REMARK 2.3. Let S be a convex cone in a real vector space V. If ϕ is a $B(\mathcal{H})$-valued positive definite function on S, then for all $s \in S, v \in \mathcal{H}$:

$$\left\langle \phi(\tfrac{s}{2} + \tfrac{s}{2})v, v \right\rangle \geq 0 \quad \Longrightarrow \quad \phi(s) \in \mathrm{Herm}^+(\mathcal{H}).$$

For scalar-valued positive definite functions on S, it is evident that they are real and non-negative.

REMARK 2.4. The set of $B(\mathcal{H})$-valued positive definite functions on an involutive semigroup S forms a convex cone closed in the weak operator topology on $B(\mathcal{H})$, while the α-bounded subset forms a closed subcone. These assertions are formalized in [10, Corollary 7.8] from which we extract a statement that will later be useful in proofs.

Let S be an involutive semigroup acting on the set X on the right, $\mathcal{H}$ be a complex Hilbert space, and α be an absolute value on S. If $K\colon X \times X \to B(\mathcal{H})$ is an S-invariant positive definite kernel on X and $A \in \mathrm{Herm}_1^+(\mathcal{H})$ a positive trace class operator, we define $K_A\colon X \times X \to \mathbb{C}$ by $K_A(x,y) := \mathrm{tr}(K(x,y)A)$. Then K_A is an S-invariant positive definite kernel on X and K is α-bounded if and only if so are all kernels K_A.

DEFINITION 2.5. *For a commutative semigroup with involution $(S, \circ, *)$, a* character *of S is a non-zero homomorphism $\xi\colon S \to \mathbb{C}$ of semigroups S and $(\mathbb{C}, \cdot, ^{-})$, i.e., it holds:*

(i) *$\xi(s \circ t) = \xi(s)\xi(t)$ for all $s, t \in S$,*
(ii) *$\xi(s^*) = \overline{\xi(s)}$ for all $s \in S$.*

If S has a neutral element e, the condition $\xi \not\equiv 0$ is equivalent to $\xi(e) = 1$.

Note that each character restricted to $S \circ S$ is a scalar positive definite function.

REMARK 2.6. If the involution on the semigroup S is the identity then $\overline{\xi(s)} = \xi(s)$, meaning that every character on S is real. Additionally, if S is

2-divisible (i.e., every element $s \in S$ can be expressed as $s = t \circ t$ for some $t \in S$), then every character on S is non-negative since:

$$\xi(s) = \xi(t \circ t) = (\xi(t))^2 \geq 0.$$

These conditions are clearly satisfied for convex cones.

Definition 2.7. *Let S be a commutative semigroup with involution, and let $\alpha\colon S \to [0,\infty\rangle$ be an absolute value on S. Define $\widehat{S}$ as the set of all characters on S, and $\widehat{S}_\alpha$ as the set of all α-bounded characters on S, i.e., characters $\xi\colon S \to \mathbb{C}$ such that $|\xi(s)| \leq \alpha(s)$ for all $s \in S$. Sets $\widehat{S}$ and $\widehat{S}_\alpha$ are equipped with the topology of pointwise convergence, i.e., the topology inherited from the product topology on $\mathbb{C}^S$.*

2.2. *Cone-valued measures.* Let us recall some standard definitions. If C is a convex cone in a real vector space V, the dual cone of C is

$$C^* := \{\lambda \in V^* : \lambda(C) \subseteq [0,\infty\rangle\}.$$

For convex cone $T \subseteq V^*$ we define

$$^*T := \{x \in V : \lambda(x) \geq 0, \, \forall\, \lambda \in T\}.$$

The proof of the following lemma can be found in [13, Lemma I.5].

Lemma 2.8. *If S is a generating closed convex cone in a Banach space V, then $S^* \subseteq V'$.*

The following definition of a cone-valued measure, introduced by Neeb in [13], extends the notion of a positive measure $\mu\colon \Sigma \to [0,\infty]$ on a measurable space (X,Σ), where $[0,\infty]$ is considered as a compactification of the convex cone $[0,\infty\rangle$ which is dense in $[0,\infty]$. Due to their extensiveness, the definitions for the integration of scalar and vector-valued functions with respect to cone-valued measures are omitted here but can be found in [13] or [10].

Definition 2.9. *A* range data *is a triple $(W, W^\sharp, C)$ where W is a real vector space, $W^\sharp$ a vector subspace of W^* and C a convex cone in W, satisfying the following conditions.*

(i) *C is pointed and generating, meaning $C \cap (-C) \subseteq \{0\}$ and $C - C = W$.*

(ii) *$C = {}^*(C^*) = \{x \in W : \lambda(x) \geq 0, \forall\, \lambda \in C^*\}$ holds where*

$$C^* := \left\{\lambda \in W^\sharp : \lambda(C) \subseteq [0,\infty\rangle\right\}.[1]$$

Additionally, C^* is generating, i.e., $C^* - C^* = W^\sharp$.

(iii) *Let*

$$C_\infty := \hom\left(C^*, [0,\infty]\right)$$

[1] The set C^* defined here is not the full dual cone, but only its intersection with $W^\sharp$.

denote the compact topological monoid of monoid homomorphisms from C^* into the additive monoid $[0,\infty]$, equipped with the topology of pointwise convergence. The mapping

$$k\colon C \to C_\infty, \quad x \mapsto (\lambda \mapsto \lambda(x))$$

is injective by (i) and (ii). We require that $k(C) = \hom(C^*, [0,\infty\rangle)$.

The mapping k becomes a topological embedding when W is equipped with the $\sigma(W, W^\sharp)$ topology, the coarsest topology in which all linear functionals from $W^\sharp$ are continuous. Moreover, $k(C)$ is dense in C_∞, making C_∞ a compactification of C [13, Proposition I.4].

Definition 2.10. *Let $(W, W^\sharp, C)$ be range data and (X,Σ) a measurable space. A C-*valued measure *on (X,Σ) is a σ-additive function $\mu\colon \Sigma \to C_\infty$ such that $\mu(\emptyset) = 0$. The C-valued measure μ is:*

- finite *if $\mu(X) \in \hom(C^*, [0,\infty\rangle) \sim C$,*
- σ-finite *if there exists a sequence of sets $(X_n), X_n \in \Sigma$ such that $X = \bigcup_{n\in\mathbb{N}} X_n$ and $\mu(X_n) \in C$ for all $n \in \mathbb{N}$.*

Remark 2.11. For $\lambda \in C^*$, a positive measure $\mu_\lambda\colon \Sigma \to [0,\infty]$ can be defined by:

$$\mu_\lambda(A) = \mu(A)(\lambda), \;\; A \in \Sigma.$$

Note that $\mu_{\lambda_1+\lambda_2} = \mu_{\lambda_1} + \mu_{\lambda_2}$ hence the mapping $\lambda \mapsto \mu_\lambda$ is a monoid homomorphism $C^* \to [0,\infty]^\Sigma$. Conversely, if $(\nu^\lambda)_{\lambda\in C^*}$ is a family of positive measures on (X,Σ) such that $\lambda \mapsto \nu^\lambda$ is a monoid homomorphism $C^* \to [0,\infty]^\Sigma$, there exists a unique C-valued measure ν on (X,Σ) such that $\nu_\lambda = \nu^\lambda$, $\forall\, \lambda \in C^*$.

If X is a Hausdorff space with Borel σ-algebra $\mathcal{B}(X)$, we say that a C-valued measure on $(X, \mathcal{B}(X))$ with range data $(W, W^\sharp, C)$ is *Radon* if the associated positive measure μ_λ (defined as above) is Radon for all $\lambda \in C^*$.

Remark 2.12. If μ is a finite cone-valued measure on a measurable space (X,Σ) with range data $(W, W^\sharp, C)$, it can also be regarded as a σ-additive map $\mu\colon \Sigma \to W$ where W is equipped with the $\sigma(W, W^\sharp)$ topology. Thus, μ can be interpreted as a vector measure, taking values in the convex cone C. Definition 2.10 generalizes this notion by allowing “infinite” values outside W.

We introduce notation concerning bounded operators, Hermitian operators and certain topologies. Let $\mathcal{H}$ be a complex Hilbert space. We define the following spaces and notations:

- $B(\mathcal{H})$: the complex algebra of bounded operators on $\mathcal{H}$,
- $\mathrm{Herm}(\mathcal{H})$: the real vector subspace of Hermitian operators,
- $\mathrm{Herm}^+(\mathcal{H})$: the convex cone of positive semidefinite operators,

- $B_1(\mathcal{H})$: the subalgebra of trace class operators, consisting of operators $T \in B(\mathcal{H})$ that satisfy $\|T\|_1 := \mathrm{tr}\sqrt{T^*T} < \infty$, where the trace functional for a positive bounded linear operator A is defined as $\mathrm{tr}(A) = \sum \langle Ae_k, e_k\rangle$.

Recall that the operator $A \in B(\mathcal{H})$ is hermitian if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$ and positive semidefinite if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Operators $A \in B(\mathcal{H})$, $B \in B_1(\mathcal{H})$ satisfy AB, $BA \in B_1(\mathcal{H})$ with $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Using the mapping

$$B(\mathcal{H}) \ni A \mapsto \phi_A \in B_1(\mathcal{H})', \quad \phi_A(B) = \mathrm{tr}(AB),\ B \in B_1(\mathcal{H})$$

we identify the Banach space $(B(\mathcal{H}), \|\cdot\|)$ with the dual of a space $(B_1(\mathcal{H}), \|\cdot\|_1)$. This mapping can also be expressed as follows:

$$B(\mathcal{H}) \times B_1(\mathcal{H}) \to \mathbb{C}, \quad (A, B) \mapsto \mathrm{tr}(AB).$$

Define:

$$\mathrm{Herm}_1(\mathcal{H}) := \mathrm{Herm}(\mathcal{H}) \cap B_1(\mathcal{H}), \quad \mathrm{Herm}_1^+(\mathcal{H}) := \mathrm{Herm}^+(\mathcal{H}) \cap \mathrm{Herm}_1(\mathcal{H}).$$

Similar as above, using the mapping

$$\mathrm{Herm}_1(\mathcal{H}) \times \mathrm{Herm}(\mathcal{H}) \to \mathbb{R}, \quad (A, B) \mapsto \mathrm{tr}(AB),$$

we identify $\mathrm{Herm}_1(\mathcal{H})$ with the subspace of $\mathrm{Herm}(\mathcal{H})'$. Let us recollect that *ultraweak operator topology* on $B(\mathcal{H})$ is the initial topology on $B(\mathcal{H})$ with respect to the family of linear functionals $B(\mathcal{H}) \to \mathbb{C}, A \mapsto \mathrm{tr}(AB)$ where $B \in B_1(\mathcal{H})$.

Next result was proved in [13, Proposition I.7].

PROPOSITION 2.13. *The triple*

$$(W, W^\sharp, C) = (\mathrm{Herm}(\mathcal{H}), \mathrm{Herm}_1(\mathcal{H}), \mathrm{Herm}^+(\mathcal{H}))$$

satisfies the conditions of Definition 2.9*. The dual cone is* $C^* = \mathrm{Herm}_1^+(\mathcal{H})$.

Measures of interest in this context are $\mathrm{Herm}^+(\mathcal{H})$-valued measures with range data $(W, W^\sharp, C) = (\mathrm{Herm}(\mathcal{H}), \mathrm{Herm}_1(\mathcal{H}), \mathrm{Herm}^+(\mathcal{H}))$, i.e., σ-additive functions

$$\mu \colon \Sigma \to \mathrm{hom}(\ \mathrm{Herm}_1^+(\mathcal{H}), [0, \infty]),$$

such that $\mu(\emptyset) = 0$. The measure μ is finite if $\mu(X) \in \mathrm{hom}(\mathrm{Herm}_1^+(\mathcal{H}), [0, \infty\rangle)$ $\sim \mathrm{Herm}^+(\mathcal{H})$ where the homeomorphism '$\sim$' is induced by

$$k \colon \mathrm{Herm}^+(\mathcal{H}) \to \mathrm{hom}(\mathrm{Herm}_1^+(\mathcal{H}), [0, \infty\rangle), \quad A \mapsto (B \mapsto \mathrm{tr}(BA)).$$

For the range data $(W, W^\sharp, C) = (\mathbb{R}, \mathbb{R}, [0, \infty\rangle)$, a C-valued measure μ corresponds to a standard positive measure $\mu \colon \Sigma \to [0, \infty]$.

2.3. *Laplace transform of cone-valued measures.*

DEFINITION 2.14. *If V is a real vector space and $x \in V$, we define $e_x \colon V^* \to \mathbb{R}$ by $e_x(\lambda) = e^{\lambda(x)}$ for all $\lambda \in V^*$. The algebraic dual V^* is equipped with the weak-$*$ topology.*

DEFINITION 2.15. *Let V be a real vector space and let E be a vector subspace of V^*. Define $\Sigma(E)$ as the smallest σ-algebra on E making all evaluations $ev_x \colon E \to \mathbb{R}$, $\lambda \mapsto \lambda(x)$ measurable for every $x \in V$. A σ-algebra Σ on E is said to be* admissible *if $\Sigma(E) \subseteq \Sigma$.*

The proof of the following lemma can be found in [10, Lemma 14.3].

LEMMA 2.16. *Let V be a real vector space and let $\mathcal{B}(V^*)$ denote the Borel σ-algebra on V^*. Then $\mathcal{B}(V^*)$ is admissible, and $\Sigma(V^*) = \mathcal{B}(V^*)$ if and only if $\dim V \leq \aleph_0$ (countable dimension). In the latter case, every σ-finite measure μ on $(V^*, \mathcal{B}(V^*))$, which is finite on compacts, is a Radon measure.*

DEFINITION 2.17. *Let V be a real vector space, Σ an admissible σ-algebra on V^*, and μ a C-valued measure on (V^*, Σ) with range data $(W, W^\sharp, C)$. The* Laplace transform of measure *μ is defined as:*

$$\mathcal{L}(\mu) \colon V \to C_\infty, \quad x \mapsto \int_{V^*} e_x \, \mathrm{d}\mu = \int_{V^*} e^{\lambda(x)} \, \mathrm{d}\mu(\lambda).$$

We call $\mathcal{D}(\mu) := \{x \in V : \mathcal{L}(\mu)(x) \in C\}$ the domain *of $\mathcal{L}(\mu)$ and say that μ is* admissible *if $\mathcal{D}(\mu) \neq \emptyset$.*

Nussbaum-type theorems are always based on a generalized Laplace transform. Glöckner developed a version of such a theorem tailored to his particularly general conditions, utilizing results from the theory of C^*-algebras. In contrast, we rely on the well-known Berg-Maserick theorem, published in [5], which is derived from the integral version of the Krein-Milman theorem. This theorem was proven for commutative semigroups with involution $(S, \circ, *)$ that possess a neutral element. The notation $M_+(\widehat{S})$ denotes the set of nonnegative Borel measures on $\widehat{S}$, the set of all characters on S. We conclude this subsection with a statement of that theorem.

THEOREM 2.18 (Berg and Maserick). *If $\phi \colon S \to \mathbb{C}$ is a positive definite and α-bounded function, then there exists a unique Radon measure $\mu \in M_+(\widehat{S})$ such that*

$$\phi(s) = \int_{\widehat{S}} \xi(s) \, \mathrm{d}\mu(\xi), \quad \forall\, s \in S.$$

Furthermore, the measure μ is concentrated on $\widehat{S}_\alpha$, the compact set of α-bounded characters of S.

3. Lattice-structured Banach spaces

3.1. *Definitions and properties.* A *lattice-structured Banach space*, is a real ordered Banach space $(B, \|\cdot\|, \leq)$ that is also a vector lattice. An ordered Banach space is a Banach space that is also an ordered vector space and the positive cone $B_+ = \{x \in B : x \geq 0\}$ is closed in the norm topology. More precisely, partial order '$\leq$' is transitive, reflexive, antisymmetric relation and it satisfies the following compatibility conditions with algebraic operations:

(i) $x \leq y \implies x + z \leq y + z \quad$ for all $x, y, z \in B$,

(ii) $x \leq y \implies \lambda x \leq \lambda y \quad$ for all $x, y, z \in B, \lambda \geq 0$.

Assertion $x < y$ indicates that $x \leq y$ and $x \neq y$, and $x \geq y$ is equivalent to $y \leq x$.

In addition to its algebraic and order structure, B is a *vector lattice* (or Riesz space), meaning that for any pair of elements $x, y \in B$ the supremum $x \vee y$ and infimum $x \wedge y$ exist in B. The lattice operations enable the decomposition of x as

$$x = x^+ - x^-, \quad |x| = x^+ + x^-,$$

where

$$x^+ := x \vee 0 \quad \text{and} \quad x^- := -x \vee 0.$$

Here, $|x|$ denotes the *absolute value* of x, and the cone B_+ is generating because $B = B_+ - B_+$.

Remark 3.1. A lattice-structured Banach space $(B, \|\cdot\|, \leq)$ is not necessarily a Banach lattice. In Banach lattices, the norm satisfies an additional monotonicity condition:

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \quad \text{for all } x, y \in B.$$

This property does not hold in general lattice-structured Banach spaces, such as certain Sobolev spaces.

In Banach lattices, the positive cone is closed since the mappings $x \mapsto x^+$, $x \mapsto x^-$ and $x \mapsto x \vee y$ are uniformly continuous. Properties regarding partially ordered sets and vector lattices, can be found in [2, 3, 17].

3.2. *Order units and order dual.* The following definitions and results highlight the interplay between the topological and order structures of lattice-structured Banach spaces, showcasing their rich mathematical properties.

A subset $A \subseteq X$ of an ordered vector space X is called *order bounded* if there exist elements $u, v \in X$ such that $u \leq a \leq v$ for all $a \in A$; in other words, A is bounded from above and below. *An order interval* is any set of the form $[x, y] = \{z : x \leq z \leq y\}$. If x and y are incomparable, then $[x, y] = \emptyset$. Notice that a set is order bounded if and only if it fits within an order interval. If X is an ordered Banach space, every order interval is a closed set, as it can be expressed as $[a, b] = (a + X_+) \cap (b - X_+)$. In Banach lattices, the notions of order boundedness and norm boundedness are equivalent.

An order unit is a particularly important element in such spaces. An element $u \in X_+$ in an ordered vector space X is an *order unit* if, for every $x \in X$, there exists a scalar $\lambda > 0$ such that $x \leq \lambda u$. Observe that an order unit is not unique, as $u + X_+$ and λu are also order units for any given order unit u and $\lambda > 0$. In a vector lattice, the inequality $x \leq \lambda u$ is equivalent to $|x| \leq \lambda u$.

The existence of an order unit implies that the positive cone X_+ has a nonempty interior relative to the topology of X. This property not only ensures that X_+ is generating but also facilitates the approximation of elements of X using elements of X_+. The following lemma formalizes this statement and can be found in [3] or [2].

LEMMA 3.2. *Let X be an ordered Banach space, $u \in X_+$, and $B_0 := \{x \in X : \|x\| \leq 1\}$ the unit ball in X. The following statements are equivalent:*
(a) *u is an order unit,*
(b) *$u \in \operatorname{Int} X_+$,*
(c) *$\lambda B_0 \subseteq [-u, u]$ for some $\lambda > 0$.*

Consequently, if an ordered vector space X does not have an order unit, then its positive cone X_+ has an empty interior in any vector topology. Property (c) implies that the order interval $[-u, u]$ is a neighborhood of zero, meaning $\operatorname{Int} X_+ \neq \emptyset$ if and only if every norm bounded interval is order bounded.

Let $(B, B_+, \|\cdot\|)$ be a lattice-structured Banach space. A linear functional $f\colon B \to \mathbb{R}$ is *positive* if $f(x) \geq 0$ for all $x \in B_+$. Note that every positive linear functional on B is continuous, as guaranteed by Lemma 2.8. The topological dual of B, denoted by B', is an ordered Banach space with the operator norm and the ordering induced by the dual cone of B_+:

$$B'_+ := \{f \in B' : f(x) \geq 0, \forall\, x \in B_+\}.$$

Therefore, B'_+ is a positive cone in B' which consists exactly of positive linear functionals and it is closed in the weak-$*$ topology on B'. Since B_+ is closed, it follows that

$$(B'_+)' = \{x \in B : f(x) \geq 0, \forall\, f \in B'_+\} = B_+.$$

A linear functional f on an ordered vector space X is *order bounded* if $f([x, y])$ is a bounded subset of $\mathbb{R}$ for every order interval $[x, y]$. For any positive linear functional f, the inequality $x - y \geq 0$ implies $f(x - y) \geq 0$, i.e., $x \geq y \Rightarrow f(x) \geq f(y)$. Thus, every positive linear functional is both monotone and order bounded. It follows that in the space B, every positive linear functional is both norm and order bounded. The set of all order bounded linear functionals on X forms a vector space, denoted by $X^\sim$, called the *order dual* of X. Riesz demonstrated that the order dual of a vector lattice is itself a vector lattice, with the order $f \leq g$ if $f(x) \leq g(x)$, $\forall\, x \in X_+$.

For lattice-structured Banach spaces $(B, B_+, \|\cdot\|)$, it holds that $B^\sim \leq B'$ and $B^\sim = B'_+ - B'_+$ ([3, Corollary 2.50]). In Banach lattices, these notions coincide, so $B' = B^\sim$.

3.3. *Examples.* Examples of lattice-structured Banach spaces include:

1. Finite-dimensional spaces: $\mathbb{R}$ and $\mathbb{R}^n$ with Euclidean norm where $x \leq y$ if $x_i \leq y_i$ for each $i = 1, \ldots, n$ and lattice operations are given by:

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}),$$
$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

   Order units are elements 1 and $(1, \ldots, 1)$.

2. Function spaces: Space of continuous real functions $C(K)$ on compact topological space K is a Banach lattice with pointwise ordering and lattice operations:

$$(f \vee g)(x) = \max\{f(x), g(x)\},$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

   In this scenario, any strictly positive function serves as a strong order unit. Furthermore, every lattice-structured Banach space $(B, \|\cdot\|, \leq)$ with $\operatorname{Int} B_+ \neq \emptyset$ is topologically isomorphic to $(C(K), C(K)_+, \|\cdot\|_\infty)$ for some compact Hausdorff space K, see [2, Theorem 9.32].

   Spaces L^∞ are also Banach lattices with a constant function $\mathbb{1}$ as an order unit and almost everywhere pointwise ordering. Lattice operations are given as in the example of $C(K)$. In a sequence space ℓ^∞ an order unit is every sequence (u_n) for which $\inf_{i \in \mathbb{N}}\{u_i\} > 0$. Banach lattices ℓ^p and L^p for $1 \leq p < \infty$ as well as c_0 and $M(X)$ (space of finite real measures) do not have an order unit.

3. Sobolev spaces: Certain Sobolev spaces, such as $W^{1,p}(\Omega), \Omega \subseteq \mathbb{R}^n$, are vector lattices under the almost everywhere pointwise ordering inherited from $L^p(\Omega)$ spaces, but fail to satisfy the monotonicity condition for Banach lattices. All Sobolev spaces $W^{m,p}(\Omega)$ are ordered Banach spaces with closed positive cone in (m,p)-norm, but only $W^{1,p}(\Omega)$ are vector lattices (more precisely, sublattices of $L^p(\Omega)$). As an example let $f \in W^{1,1}([0,1])$, $f(x) = x$, $x \in [0,1]$. It follows that $0 \leq f \leq \mathbb{1}$, but $\|f\| > \|\mathbb{1}\|$, $\mathbb{1}$ denoting constant function $x \mapsto 1$, $\forall\, x \in [0,1]$. Sobolev spaces have non-empty interior if $p \geq n$. More information on Sobolev spaces can be found in [1] and [9].

4. Main results

This entire section is devoted to the proof of Theorem 1.1. All propositions and lemmas presented here serve as auxiliary results that support the proof.

Let $(B, B_+, \|\cdot\|)$ be a lattice-structured Banach space, B' its topological dual space, and B_+ a positive cone in B. Let Σ be an admissible σ-algebra on B', i.e., $\Sigma \supseteq \Sigma(B')$. Let μ be a $\mathrm{Herm}^+(\mathcal{H})$-valued measure on (B', Σ) with range data $(\mathrm{Herm}(\mathcal{H}), \mathrm{Herm}_1(\mathcal{H}), \mathrm{Herm}^+(\mathcal{H}))$. Since we identify $\mathrm{Herm}_1(\mathcal{H}) \cong \mathrm{Herm}(\mathcal{H})'$, the dual cone of $\mathrm{Herm}^+(\mathcal{H})$ is $\mathrm{Herm}_1^+(\mathcal{H})$. Following notation from Definition 2.9, by $C_\infty = \hom(\, \mathrm{Herm}_1^+(\mathcal{H}), [0, \infty])$ we denote the topological compactification of $C = \mathrm{Herm}^+(\mathcal{H})$ that represents the "infinite" values outside the cone.

The mapping $e_x \colon B' \to \mathbb{R}$ is defined as $e_x(\lambda) = e^{\lambda(x)}$. The Laplace transform of measure μ is defined as:

$$\mathcal{L}(\mu) \colon B \to C_\infty, \quad x \mapsto \int_{B'} e_x \,\mathrm{d}\mu = \int_{B'} e^{\lambda(x)} \,\mathrm{d}\mu(\lambda),$$

where $\mathcal{D}(\mu) \coloneqq \{x \in B : \mathcal{L}(\mu)(x) \in C\}$ is the *domain* of $\mathcal{L}(\mu)$.

REMARK 4.1. If a positive definite function $\phi \colon B_+ \to \mathrm{Herm}^+(\mathcal{H})$ can be represented as the Laplace transform $\phi = \mathcal{L}(\mu)|_{B_+}$, then obviously $B_+ \subseteq \mathcal{D}(\mu) = \{x \in B : \mathcal{L}(\mu)(x) \in C\}$. Since $0 \in B_+$, it follows that $\phi(0) = \mathcal{L}(\mu)(0) = \mu(B') \in C$, meaning that μ is finite.

Given that a Radon measure may not exist if the positive cone B_+ has an empty interior (or any representing measure on the topological dual), we restrict our study to spaces B with order units, or equivalently $\mathrm{Int}\, B_+ \neq \emptyset$. Since in our setting we consider a topological dual instead of an algebraic one, real continuity instead of continuity on line segments is achieved (in contrast to [10]).

LEMMA 4.2. *The Laplace transform $\mathcal{L}(\mu)$ of a* $\mathrm{Herm}^+(\mathcal{H})$*-valued measure is a $B(\mathcal{H})$-valued positive definite function which is ultraweakly continuous for finite measures μ.*

PROOF. Positive definiteness is already proved in [10, Theorem 14.12]. To prove continuity, (x_n) be a sequence in B such that $\|x_n - x\| \to 0$. For each $A \in \mathrm{Herm}_1^+(\mathcal{H})$, define a positive measure μ_A via $\mu_A(B) \coloneqq \mu(B)(A)$, $B \in \Sigma$ (Remark 2.11). Since the ultraweak topology coincides with a pointwise convergence topology given the identification $\mathrm{Herm}_1(\mathcal{H}) \cong \mathrm{Herm}(\mathcal{H})'$, it suffices to show that $\mathcal{L}(\mu_A)(x_n) \to \mathcal{L}(\mu_A)(x)$, for all A.

Let $A \in \mathrm{Herm}_1^+(\mathcal{H})$. For the positive measure μ_A, it is evident that $\mu_A(B') = \mu(B')(A) < \infty$. For $\lambda \in B'$, the function e^λ is a continuous, and thus $e_{x_n} \to e_x$ pointwise. Since λ is a continuous linear functional there exists a neighborhood U of x on which λ is bounded, meaning $e^\lambda|_U \leq c$ for some constant $c > 0$. Without loss of generality, we can assume the sequence $(x_n) \in U$, which ensures that $\int e_{x_n} \,\mathrm{d}\mu_A \leq c\mu_A(B') < \infty$. From Lebesgue's dominated convergence theorem it follows:

$$\int_{B'} e_{x_n}\,\mathrm{d}\mu_A \to \int_{B'} e_x\,\mathrm{d}\mu_A, \;\; \forall\, A \in \mathrm{Herm}_1^+(\mathcal{H}),$$

which shows that the Laplace transform is a continuous function in the ultra-weak topology on $B(\mathcal{H})$. ☐

Since 1-boundedness (i.e., boundedness) is a rather restrictive condition for a positive definite function, we consider a more general class of α-bounded positive definite functions, where α is at least locally bounded absolute value. In this case, linear functionals that induce α-bounded characters of the exponential type are continuous, making it natural to observe measures defined on B' rather than on the entire algebraic dual B^*.

Let C_α denote the set of linear functionals that induce α-bounded characters:

$$C_\alpha := \{\lambda \in B^* : e^{\lambda(x)} \le \alpha(x), \forall\, x \in B_+\},$$

where C_α is equipped with a weak-$*$ topology, and α is a locally bounded absolute value. The set C_α is a closed subset of B^*, and, as shown in the proof of the following lemma, $C_\alpha \subseteq B'$. Therefore, C_α consists of continuous linear functionals λ such that $\lambda(x) \le \ln(\alpha(x))$ for each $x \in B_+$.

Lemma 4.3. *Let $\xi\colon B_+ \to [0,\infty\rangle$ be a character of B_+, with $\mathrm{Int}\, B_+ \neq \emptyset$, and let α be a locally bounded absolute value on B_+. A character ξ is continuous if and only if $\xi > 0$ and ξ is α-bounded. In that case, there exists a unique linear functional $\lambda \in B'$ such that $\xi(x) = e^{\lambda(x)}$, $\forall x \in B_+$.*

Proof. Let ξ be a continuous character of B_+. We first show that $\xi > 0$. Assume $\xi(x_0) = 0$ for some $x_0 \ge 0$, and let $x \in \mathrm{Int}\, B_+$. Then there exists $n \in \mathbb{N}$ such that $z := nx - x_0 \ge 0$. Using the properties of characters, we have:

$$\xi(nx) = \xi(x)^n = \xi(z)\xi(x_0) = 0.$$

Thus, $\xi(x) = 0$. If ξ had a zero point on B_+, this would imply $\xi|_{\mathrm{Int}\, B_+} = 0$. By continuity, ξ would then be identically zero on all of B_+, contradicting the definition of a character. Therefore, it must hold that $\xi > 0$.

Since ξ is a non-negative, submultiplicative function, every character of B_+ is also an absolute value on B_+. By continuity, ξ is locally bounded, so we set $\alpha := \xi$. Applying [10, Lemma 13.2] to this setting, we conclude that $\xi(s) = e^{\lambda(s)}$ for some $\lambda \in B'$.

Conversely, suppose $\xi > 0$ and ξ is α bounded. By the same lemma, we have $\xi(s) = e^{\lambda(s)}$ for some $\lambda \in B^*$. The condition $\xi \le \alpha$ implies $\lambda \in C_\alpha$, i.e., $\lambda(x) \le \ln(\alpha(x))$ so λ is locally bounded on B_+. Since $0 \in B_+$, there exists a neighborhood U of zero such that λ is bounded on $U \cap B_+$. It follows that λ is also bounded on $(U \cap B_+) - (U \cap B_+)$, which is itself a neighborhood of zero.

Thus, for locally bounded α, $C_\alpha \subseteq B'$ holds. We have shown that $\xi(s) = e^{\lambda(s)}$ for some $\lambda \in B'$, thereby proving that ξ is continuous. □

REMARK 4.4. It is evident from the proof that, instead of requiring $\xi > 0$, it suffices to assume $\xi(u) > 0$ for some order unit u. This implies that if a character ξ has a zero point, then ξ must vanish on the entire interior of B_+.

COROLLARY 4.5. *Let $\widehat{B}_+$ be the topological semigroup of characters of B_+, equipped with the topology of pointwise convergence, and let $\widehat{B}_+^c$ denote the subgroup of continuous characters of B_+. Let the topology on B' be a weak-$*$ topology. The mapping $\beta\colon B' \to \widehat{B}_+^c$, $\lambda \mapsto \exp \circ \lambda|_{B_+}$ is a homeomorphism. Moreover, the set of discontinuous characters $\widehat{B}_+ \setminus \widehat{B}_+^c$ is nonempty.*

PROOF. A continuous functional $\lambda \in B'$ clearly induces the continuous character $\xi = \exp\circ\lambda|_{B_+}$. Conversely, by the previous lemma, every continuous character arises in this manner, establishing a bijection between B' and $\widehat{B}_+^c$. Note that each $\lambda \in B'$ is determined by its definition on the generating cone B_+, as $\lambda(x) = \lambda(x^+) - \lambda(x^-)$. Therefore, the weak-$*$ topology on B' is the coarsest topology that makes point evaluations on B_+ continuous. The topology on $\widehat{B}_+^c$ is the relative topology induced by the pointwise convergence topology on $\widehat{B}_+$. Hence, the two topologies coincide.

The positive cone B_+ contains the neutral element 0, allowing us to define a discontinuous character $\xi := 1_{\{0\}}$. □

REMARK 4.6. Let $(\widehat{B}_+)^{\alpha,c}$ denote the set of α-bounded continuous characters of B_+, where α is a locally bounded absolute value on B_+. From the previous corollary and Lemma 4.3, it follows that $\beta(C_\alpha) = (\widehat{B}_+)^{\alpha,c}$, and therefore:

$$(\widehat{B}_+)^{\alpha,c} \sim C_\alpha \subseteq B'.$$

Notice that for $\alpha \equiv 1$, we have:

$$C_1 = \{\lambda \in B^* : e^{\lambda(x)} \leq 1, \forall\, x \in B_+\} = -(B'_+).$$

In this case, the relation becomes:

$$(\widehat{B}_+)^{1,c} \sim C_1 = -(B'_+) \subseteq B^\sim \subseteq B'.$$

The assertion that the dual cone of a generating, closed convex cone in a Banach space consists of continuous linear functionals is a result in its own right (Lemma 2.8).

PROPOSITION 4.7. *Let $(B, \|\cdot\|, \leq)$ be a lattice-structured Banach space with an order unit, and let α be a locally bounded absolute value on B_+. Let $\phi\colon B_+ \to \mathrm{Herm}^+(\mathcal{H})$ be an α-bounded positive definite function. Then the following conditions are equivalent:*

(i) ϕ is ultraweakly continuous,

(ii) there exists an order unit u *and a sequence of positive real numbers* (r_n) *converging to zero such that* $\lim_{n\to\infty} \phi(r_n u) = \phi(0)$ *in the ultraweak topology.*

Proof. If ϕ is continuous on B_+, then it is also continuous on the ray through $u \in \operatorname{Int} B_+$, satisfying condition (ii). We now show that the converse is true.

Let u be an order unit, and let (r_n) be a sequence of positive real numbers converging to zero such that $\lim_{n\to\infty} \phi(r_n u) = \phi(0)$, in the ultraweak topology. Let (x_n) be a sequence in B_+ with $x_n \to x$. Then $\phi(x_n) \to \phi(x)$ ultraweakly if and only if $\phi_A(x_n) \to \phi_A(x)$ for all $A \in \mathrm{Herm}_1^+(\mathcal{H})$, where $\phi_A(x) = \mathrm{tr}(A\phi(x))$. Therefore, without loss of generality, we can assume $\mathcal{H} = \mathbb{C}$.

By the Berg-Maserick theorem, there exists a unique positive Radon measure μ on $\widehat{B}_+$ such that $\phi(r_n u) = \int_{(\widehat{B}_+)^\alpha} \xi(r_n u)\, \mathrm{d}\mu(\xi)$. From Remark 4.4, it holds that $(\widehat{B}_+)^{\alpha,c} = \{\xi \in (\widehat{B}_+)^\alpha : \xi(u) > 0\}$, and this set is Borel measurable since it is open. Observe that ϕ can be written as:

$$\phi(r_n u) = \int_{(\widehat{B}_+)^{\alpha,c}} \xi(r_n u)\, \mathrm{d}\mu(\xi) + \int_{(\widehat{B}_+)^\alpha \setminus (\widehat{B}_+)^{\alpha,c}} \xi(r_n u)\, \mathrm{d}\mu(\xi).$$

Since $(\widehat{B}_+)^\alpha \setminus (\widehat{B}_+)^{\alpha,c} = (\widehat{B}_+)^\alpha \setminus (\widehat{B}_+)^{\alpha,t}$ and $r_n u \in \operatorname{Int} B_+$, the second integral evaluates to zero because discontinuous characters vanish on the interior of the positive cone (Lemma 4.3, Remark 4.4). Thus we have:

$$\phi(r_n u) = \int_{(\widehat{B}_+)^{\alpha,c}} \xi(r_n u)\, \mathrm{d}\mu(\xi).$$

Letting $n \to \infty$, it follows that:

$$\phi(0) = \lim_{n\to\infty} \int_{(\widehat{B}_+)^{\alpha,c}} \xi(r_n u)\, \mathrm{d}\mu(\xi).$$

Since $\xi \in \widehat{B}_+^{\alpha,c}$ is α-bounded with respect to the locally bounded α, we can assume $\xi(r_n u) \le c$ for all $n \in \mathbb{N}$ and some constant $c > 0$. By continuity, $\xi(r_n u) \to \xi(0) = 1$. Using the Lebesgue's dominated convergence theorem, it follows that:

$$\phi(0) = \mu((\widehat{B}_+)^{\alpha,c}).$$

On the other hand,

$$\phi(0) = \int_{(\widehat{B}_+)^\alpha} \xi(0)\, \mathrm{d}\mu(\xi) = \mu((\widehat{B}_+)^\alpha).$$

Thus, $\mu((\widehat{B}_+)^\alpha \setminus (\widehat{B}_+^{\alpha,c})) = 0$, and we conclude that:

$$\phi(x) = \int_{(\widehat{B}_+)^{\alpha,c}} \xi(x)\, \mathrm{d}\mu(\xi).$$

From $\|x_n - x\| \to 0$, it follows that:

$$\phi(x_n) = \int_{(\widehat{B}_+)^{\alpha,c}} \xi(x_n)\,\mathrm{d}\mu(\xi) \to \int_{(\widehat{B}_+)^{\alpha,c}} \xi(x)\,\mathrm{d}\mu(\xi) = \phi(x).$$

□

Finally, we present the proof of Theorem 1.1.

Proof. Define a sequence of functions $\phi_A \colon B_+ \to [0,\infty\rangle$ by

$$\phi_A(x) := \mathrm{tr}(A\phi(x)), \quad A \in \mathrm{Herm}_1^+(\mathcal{H}).$$

By Remark 2.4, the functions (ϕ_A) form a sequence of α-bounded positive definite functions. Therefore by the Berg-Maserick theorem, for each $A \in \mathrm{Herm}_1^+(\mathcal{H})$ there exists a unique positive Radon measure ν_A on $\widehat{B}_+$ such that:

$$\phi_A(x) = \int_{(\widehat{B}_+)^{\alpha}} \xi(x)\,\mathrm{d}\nu_A(\xi), \quad \forall\, x \in B_+.$$

The measure ν_A depends additively on A, as shown below ($\hat{x}$ is point evaluation on the set of characters):

$$\begin{aligned}\int \widehat{x}\,\mathrm{d}\nu_{A+B} &= \phi_{A+B}(x) = \mathrm{tr}((A+B)\phi(x)) = \mathrm{tr}(A\phi(x)) + \mathrm{tr}(B\phi(x)) \\ &= \phi_A(x) + \phi_B(x) = \int \widehat{x}\,\mathrm{d}\nu_A + \int \widehat{x}\,\mathrm{d}\nu_B = \int \widehat{x}\mathrm{d}(\nu_A + \nu_B),\end{aligned}$$

which implies $\nu_{A+B} = \nu_A + \nu_B$. This demonstrates that the mapping $A \mapsto \nu_A$ is a homomorphism of monoids. Therefore, a unique $\mathrm{Herm}^+(\mathcal{H})$-valued measure ν exists such that

$$\nu(E)(A) = \nu_A(E), \quad \forall A \in \mathrm{Herm}_1^+(\mathcal{H}), E \in \mathcal{B}(\widehat{B}_+),$$

as per [13, Theorem I.10]. By definition, ν is Radon measure. It follows that:

$$\phi_A(x) = \int \widehat{x}\,\mathrm{d}\nu_A = \left(\int \widehat{x}\,\mathrm{d}\nu\right)(A),$$

and consequently:

$$\phi(x) = \int_{(\widehat{B}_+)^{\alpha}} \xi(x)\,\mathrm{d}\nu(\xi).$$

As in the proof of the previous proposition, we decompose ϕ as:

$$\phi(x) = \int_{(\widehat{B}_+)^{\alpha,c}} \xi(x)\,\mathrm{d}\nu(\xi) + \int_{(\widehat{B}_+)^{\alpha}\setminus(\widehat{B}_+)^{\alpha,c}} \xi(x)\,\mathrm{d}\nu(\xi), \quad x \in B_+.$$

Let β be a homeomorphism from Remark 4.6:

$$\beta \colon C_\alpha \to (\widehat{B}_+)^{\alpha,c}, \quad \lambda \mapsto e^{\lambda}|_{B_+}$$

and define $\mu := \beta^{-1}(\nu)$. Then μ is a Radon measure on B' [10, Lemma 3.3] with support in C_α. We deduce that for each $x \in B_+$:

$$\begin{aligned}\phi(x) &= \int_{C_\alpha} e^{\lambda(x)}\, \mathrm{d}\mu(\xi) + \int_{(\widehat{B}_+)^\alpha \setminus (\widehat{B}_+)^{\alpha,c}} \xi(x)\, \mathrm{d}\nu(\xi) \\ &= \mathcal{L}(\mu)(x) + \int_{(\widehat{B}_+)^\alpha \setminus (\widehat{B}_+)^{\alpha,c}} \xi(x)\, \mathrm{d}\nu(\xi).\end{aligned}$$

By the previous proposition, ϕ is ultraweakly continuous, so the function

$$\int_{(\widehat{B}_+)^\alpha \setminus (\widehat{B}_+)^{\alpha,c}} \xi(x)\, \mathrm{d}\nu(\xi) = \phi(x) - \mathcal{L}(\mu)(x)$$

is also ultraweakly continuous. Since the integral on the left vanishes on $\operatorname{Int} B_+$, it must be identically zero on B_+. Thus:

$$\phi(x) = \mathcal{L}(\mu)(x), \quad \forall x \in B_+.$$

□

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Received: 12.7.2024.
Revised: 9.12.2024.

LAPLACEOVA TRANSFORMACIJA NA STOŠCIMA REŠETKASTO STRUKTURIRANIH BANACHOVIH PROSTORA

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Sažetak. Karakterizacije pozitivno definitnih funkcija definiranih na konveksnim stošcima korištenjem Laplaceove transformacije mjere obično se nazivaju teoremima Nussbaumovog tipa. Ovaj rad postavlja teorem Nussbaumovog tipa u kontekstu gdje je domena pozitivno definitne funkcije s vrijednostima u $B(\mathcal{H})$ pozitivni stožac unutar Banachovog prostora koji je ujedno i vektorska rešetka, ali ne nužno Banachova rešetka. Takvi prostori uključuju primjere poput Soboljevljevih prostora $W^{1,p}(\Omega)$. Koristeći Berg-Maserickov teorem, dokazujemo da je jedinstvena reprezentativna mjera Radonova mjera koncentrirana na podskupu topološkog duala.