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ON THE EULER-STIELTJES CONSTANTS FOR FUNCTIONS FROM THE GENERALIZED SELBERG CLASS

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Abstract. The class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ is a very broad class of L functions that contains the Selberg class, the class of all automorphic L functions and the Rankin–Selberg L functions, as well as products of suitable shifts of those functions. In this paper, we consider generalized Euler-Stieltjes constants $\gamma_n(F)$ attached to functions $F(s)$ from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$. These are coefficients in Laurent series expansion of function $F(s)$ at its pole. We derive an integral representation and an upper bound for these constants. The application of the obtained results in the case of product of suitable shifts of the Riemann zeta function is presented.

1. Introduction

L-functions are among the most significant objects studied in number theory. The most famous is the Riemann zeta function. Very well known examples include Dirichlet L, Dedekind, Hecke and Artin L functions, as well as automorphic and the Rankin-Selberg L functions. In recent research, there is a growing focus on classes of L functions rather than on individual functions. Typical examples include the Selberg class, class of functions with a given degree, extended or modified Selberg class. In this paper we consider L functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ introduced in [12]. It is a very broad class of L functions that contains the Selberg class, the class of all automorphic L functions and the Rankin–Selberg L functions, as well as products of suitable shifts of those functions.

We will investigate the coefficients appearing in the Laurent (Taylor) series representation of an L function from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$, referred to

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as generalized Euler-Stieltjes constants. Specifically, we will derive an upper bound for these coefficients.

L. Euler [13] in 1731 discovered and accurately computed, correct up to five decimal places, the constant term in the Laurent series expansion of the Riemann zeta function at $s = 1$, which is now known as the classical Euler constant γ.

T. J. Stieltjes [17] in 1885 proved the formula for all the coefficients in that expansion. Precisely, he proved

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{x\to\infty} \left(\sum_{n<x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \right), \tag{1.1}$$

where

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \gamma_k (s-1)^k = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k.$$

Thus, the constants γ_k $(k \geq 0)$ are named the Stieltjes constants, the generalized Euler constants or the Euler-Stieltjes constants. Some properties and a proof of equation (1.1) can be found in [3, 8].

Closely related to those coefficients are the constants η_k which appear in the Laurent series expansion of the logarithmic derivative of the Riemann zeta function at $s = 1$. Typically, these constants are referred as Euler-Stieltjes constants of the second kind, while γ_k are known as Euler-Stieltjes constants of the first kind (see e.g. [6]).

Both sets of these coefficients are important in theoretical and computational analytic number theory since they appear in various types of estimations and asymptotic analysis. Examples includes their use in determining a zero-free region of the Riemann zeta function near the real axis in the critical strip [1] and in Li positivity criterion for the Riemann hypothesis [6, 10, 11, 21]. Numerical evaluation and estimations of these coefficients are given in [23, 19], while some interesting formulas and bounds are derived in [30, 5].

This concept has been generalized in various settings. The coefficients arising in the Laurent (Taylor) series representation of an L function or its logarithmic derivative are called generalized Euler-Stieltjes constants of the first and the second kind, respectively.

Examples of results related to the Hurwitz zeta function are given in [7], for the harmonic Hurwitz zeta function in [20], those for the Dedekind zeta function in [16, 31], for the general setting of a non-co-compact Fuchsian group with unitary representation in [2], for a class of functions that have an Euler product representation in [15], for a subclass $\mathcal{S}^\flat$ of the Selberg class in [33], and for the Rankin-Selberg L functions in [26, 27]. Also, some investigations are done in the case of zeta functions with multiple variables, introducing multiple Stieltjes constants, for example, see [24, 4]. q-analogues of these coefficients are investigated in [9].

Special attention in research is given to derivation of bounds for (generalized) Euler-Stieltjes constants. Cases of the Riemann and Hurwitz zeta functions are extensively studied, see for example [3, 25, 37, 14, 29] and references therein. The bounds for Dirichlet L functions are given in [32], for the extended Selberg class in [18] and for the Rankin-Selberg L functions in [38].

In this paper, we derive bounds for generalized Euler-Stieltjes constants attached to functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. This class consists of all Dirichlet series converging in some half-plane, such that its meromorphic continuation is a meromorphic function of a finite order with at most finitely many poles, satisfying a functional equation of the Riemann type and such that its logarithmic derivative has a Dirichlet series representation. The class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ contains Selberg class as its subclass, but it also contains products of suitable shifts of functions from Selberg class as well as products of shifts of certain L functions possessing an Euler product representation that are not in Selberg class such as the Rankin–Selberg L functions.

The rest of the paper is organized as follows. In section 2 we give an overview of the setting we are dealing with. We introduce necessary notation, give a precise definition of the class of function under consideration, and recall some known results that will be used for the proofs. Section 3 contains some preliminary results about functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. Precisely, we derive some asymptotic bounds for $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and its functional equation factor. The main results are stated and proved in sections 4 and 5. In section 4 we introduce generalized Euler-Stieltjes constants for functions $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and prove an integral representation for these coefficients, while their bounds are proved in 5. In section 6 we apply derived result to function $\zeta(s-h)\zeta(s+h)$ (that belongs to the class $\mathcal{S}^{\sharp\flat}(h+1, 1)$).

2. Preliminaries and notations

The class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ is introduced in [12], it contains functions from the Selberg class, the class of all automorphic L functions, the Rankin-Selberg L functions and suitable products of these functions.

Let σ_0 and σ_1 be real numbers such that $\sigma_0 \geq \sigma_1 > 0$. The class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ is the class of functions F satisfying the following four axioms:

(i') (Dirichlet series representation) The function F possesses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}, \tag{2.1}$$

which converges absolutely for $\mathrm{Re} s > \sigma_0$.

(ii') (Meromorphic continuation) The function F can be continued to a meromorphic function on $\mathbb{C}$ possessing at most finitely many poles at points $s_1, \ldots, s_N$. There are finitely many numbers $m_1, \ldots, m_N$, smallest positive integers uniquely determined by the poles $s_1, \dots, s_N$ respectively, such that $\prod_{i=k}^{N}(s-s_k)^{m_k}F(s)$ is an entire function of finite order. [1]

(iii') (Functional equation) The function F satisfies the functional equation

$$\xi_F(s) = \omega\overline{\xi_F(\sigma_1 - \bar{s})}, \tag{2.2}$$

where the completed function ξ_F is defined as

$$\begin{aligned} \xi_F(s) = F(s)Q_F^s \prod_{j=1}^{r}\Gamma(\lambda_j s + \mu_j) \prod_{k=1}^{2M+\delta(\sigma_1)} (s-s_k)^{m_k} \\ \prod_{k=2M+1+\delta(\sigma_1)}^{N} (s-s_k)^{m_k}(\sigma_1 - s - \overline{s_k})^{m_k}, \end{aligned} \tag{2.3}$$

with $|\omega| = 1$, $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $\mu_j \in \mathbb{C}$, $j = 1, \dots, r$. The poles of the function F are arranged so that the first $2M + \delta(\sigma_1)$ poles ($0 \leq 2M + \delta(\sigma_1) \leq N$) are such that $s_{2j-1} + \bar{s}_{2j} = \sigma_1$, for $j = 1, \dots, M$, here $\delta(\sigma_1) = 1$ if $\sigma_1/2$ is a pole of F in which case $s_{2M+\delta(\sigma_1)} = \sigma_1/2$; otherwise $\delta(\sigma_1) = 0$. The number $d_F = 2\sum_{j=1}^{r}\lambda_j$ is called the degree of F.

(v') (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=2}^{\infty}\frac{c_F(n)}{n^s},$$

converging absolutely for $\mathrm{Re}s > \sigma_0$.

The notion of trivial and non-trivial zeros of the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ naturally arises from definition of the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. The zeros of $\xi_F(s)$ are called the non-trivial zeros of $F(s)$. All the other zeros of $F(s)$ are called trivial zeros, and they arise from the poles of the gamma functions appearing in (2.3). We will denote the set of non-trivial zeros of $F(s)$ by $Z(F)$. By the functional equation and the Euler sum representation, follows that all the non-trivial zeros of $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ lie in the critical strip $\sigma_1 - \sigma_0 \leq \mathrm{Re}s \leq \sigma_0$.

It is easy to see that functional equation (2.2) can be written in the following form

$$F(s) = \Psi_F(s)\overline{F(\sigma_1 - \bar{s})}, \tag{2.4}$$

where $\Psi_F(s)$ is defined by

[1] If the continuation of F is analytic we put $\prod_{i=k}^{N}(s-s_k)^{m_k} = 1$.

$$\Psi_F(s) = \omega Q_F^{\sigma_1 - 2s} \prod_{j=1}^{r} \frac{\Gamma\left(\lambda_j(\sigma_1 - s) + \overline{\mu_j}\right)}{\Gamma\left(\lambda_j s + \mu_j\right)} \prod_{k=1}^{2M+\delta(\sigma_1)} \left(\frac{\sigma_1 - s - \overline{s_k}}{s - s_k}\right)^{m_k}. \tag{2.5}$$

Function Ψ_F is called the functional equation factor for $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$.

3. Asymptotic bounds for functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and its functional equation factor

In the following lemmas we give some asymptotic bounds for functions $F(s)$ from the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and its factor $\Psi_F(s)$ of the functional equation as $|\,t\,| \to \infty$, where $s = \sigma + it$. The following notation is used in the sequel. We write $f(t) \sim g(t)$ as $t \to \infty$ if $\lim\limits_{t\to\infty} \frac{f(t)}{g(t)} = 1$, and $f(t) = O(g(t))$ as $t \to \infty$ if there exist constants $C > 0$ and t_0 such that $|\,f(t)\,| \le C\,|\,g(t)\,|$ for all $|\,t\,| \ge t_0$. If constant C appearing in O notation depends on parameter(s) p we write O_p instead of O to emphasize the dependence.

Lemma 3.1. *Let Ψ_F be functional equation factor for a function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, then*

$$|\,\Psi_F(\sigma + it)\,| \sim Q_F^{\sigma_1 - 2\sigma}\,|\,t\,|^{d_F\left(\frac{\sigma_1}{2} - \sigma\right)} \prod_{j=1}^{r} \lambda_j^{\lambda_j(\sigma_1 - 2\sigma)}, \tag{3.1}$$

as $|\,t\,| \to +\infty$.

Proof. From (2.5) follows that $\Psi_F(\sigma + it)$ can be written as

$$\Psi_F(\sigma + it) = \omega Q_F^{\sigma_1 - 2(\sigma + it)} \prod_{k=1}^{2M+\delta(\sigma_1)} \left(\frac{\sigma_1 - \sigma - it - \overline{s_k}}{\sigma + it - s_k}\right)^{m_k}$$
$$\times \exp\left(\sum_{j=1}^{r} (\log\Gamma\left(\lambda_j(\sigma_1 - \sigma - it) + \overline{\mu_j}\right) - \log\Gamma\left(\lambda_j(\sigma + it) + \mu_j\right))\right).$$

It is easy to see that $\left|\left(\frac{\sigma_1 - \sigma - it - \overline{s_k}}{\sigma + it - s_k}\right)^{m_k}\right| \to 1$, as $|\,t\,| \to +\infty$, for all $k = 1, \ldots, 2M + \delta(\sigma_1)$, so

$$\prod_{k=1}^{2M+\delta(\sigma_1)} \left|\left(\frac{\sigma_1 - \sigma - it - \overline{s_k}}{\sigma + it - s_k}\right)^{m_k}\right| \to 1, \quad \text{as} \quad |\,t\,| \to +\infty.$$

Asymptotic series expansion of function $\log\Gamma(z + a)$ (see e.g. [22, Section 2.11, relation (4)]) applied to $\log\Gamma\left(\lambda_j(\sigma + it) + \mu_j\right)$, with $z = i\lambda_j t$ and $a = \lambda_j\sigma + \mu_j$, and to $\log\Gamma\left(\lambda_j(\sigma_1 - \sigma - it) + \overline{\mu_j}\right)$, with $z = -i\lambda_j t$ and $a = \lambda_j(\sigma_1 - \sigma) + \overline{\mu_j}$, respectively, for all $j = 1, \ldots, r$, yields (3.1). Note that, in both cases a depends on σ, so a constant appearing in asymptotic series expansion of function $\log\Gamma(z + a)$ (see e.g. [22, Section 2.11, relation (4)]) depends on σ. If σ lies in a closed and bounded subset of $\mathbb{R}$, the constant in asymptotic series expansion is uniform in σ, so the limit is uniform. The proof is complete. ◻

LEMMA 3.2. *Let F be a function from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$, then for an arbitrary $\varepsilon > 0$,*

$$F(\sigma+it) = \begin{cases} O_\varepsilon(1) & \text{if} \quad \sigma \geq \sigma_0+\varepsilon, \\ O_\varepsilon\left(|\,t\,|^{\frac{d_F}{2}(\sigma_0+\varepsilon-\sigma)}\right) & \text{if} \quad \sigma_1-\sigma_0-\varepsilon < \sigma < \sigma_0+\varepsilon, \\ O_{\varepsilon,\sigma}\left(|\,t\,|^{\frac{d_F}{2}(\sigma_1-2\sigma)}\right) & \text{if} \quad \sigma \leq \sigma_1-\sigma_0-\varepsilon, \end{cases}$$

as $|\,t\,| \to +\infty$.

PROOF. For $\mathrm{Re}s = \sigma \geq \sigma_0+\varepsilon > \sigma_0$ the function $F(s)$ is given by absolutely convergent Dirichlet series (2.1), so

$$F(\sigma+it) = O_\varepsilon(1),$$

as $|\,t\,| \to +\infty$. For $\mathrm{Re}s = \sigma \leq \sigma_1-\sigma_0-\varepsilon < \sigma_1-\sigma_0$, the functional equation for function $F(s)$ given by (2.2 and relation (3.1) imply

$$F(\sigma+it) = O_{\varepsilon,\sigma}\left(|t|^{\frac{d_F}{2}(\sigma_1-2\sigma)}\right),$$

as $|\,t\,| \to +\infty$, where d_F denotes the degree of function F, introduced in axiom (iii'). Note that, when σ lies in a closed and bounded subset of $\mathbb{R}$, a constant in O notation is uniform in σ and depends on ε.

For σ such that $\sigma_1-\sigma_0-\varepsilon < \sigma < \sigma_0+\varepsilon$, Phragmén-Lindelöf theorem for strips can be used to derive desired result. From the meromorphic continuation axiom, as proved in [12], follows that ξ_F is an entire function of order one. When combined with the well known asymptotic properties of reciprocal of gamma function (see e.g. [34, Theorem 1.6, p. 165]) implies that

$$|F(\sigma+it)| = O\left(\exp(\exp(\delta|t|))\right),$$

holds true for sufficiently large $|t|$ and any $\delta > 0$. The result [28, Proposition 8.15] can be applied to the function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ in the strip $\sigma_1-\sigma_0-\varepsilon \leq \sigma \leq \sigma_0+\varepsilon$ and it implies

$$F(\sigma+it) = O_\varepsilon\left(|\,t\,|^{\frac{d_F}{2}(\sigma_0+\varepsilon-\sigma)}\right),$$

as $|\,t\,| \to +\infty$. The proof is complete. ◻

4. GENERALIZED EULER-STIELTJES CONSTANTS ASSOCIATED TO FUNCTION $F \in \mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ AND ITS INTEGRAL REPRESENTATION

Let $\rho = \alpha + i\beta$ be a pole of function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ such that $\alpha = \max\limits_{k=1,\dots,N} \mathrm{Re}s_k$, let l be a corresponding index of the pole ρ, i.e. $\rho = s_l$ and let $m = m_l$ be order of the pole ρ.

Denote by $\gamma_n(F)$ coefficients in Laurent series expansion of function $F(s)$ from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ at pole $s=\rho$, i.e.

$$F(s)=\sum_{n=-m}^{\infty}\gamma_n(F)(s-\rho)^n. \tag{4.1}$$

Here, we will derive an integral representation for these coefficients. Classical method based on contour integrals (see e.g. [35, Sect. 4.14]) is used for the proof. Cauchy integral formula implies

$$\gamma_n(F)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{F(s)}{(s-\rho)^{n+1}}ds, \tag{4.2}$$

where contour $\mathcal{C}$ is positively oriented circle with center $s=\rho$ and radius r such that it contains $s=\rho$ as the only singularity of the integrand.

Remark 4.1. If a function $F\in\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ does not have a pole, then we define the Euler-Stieltjes constants $\gamma_F(n)$ of function F as the coefficients in Taylor series expansion at $s=\sigma_0$

$$F(s)=\sum_{n=0}^{\infty}\gamma_F(n)(s-\sigma_0)^n. \tag{4.3}$$

Note, by definition, σ_0 determines the region of absolute convergence of Dirichlet series representation for an L function under consideration.

Theorem 4.2. *Let $F\in\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$, for some fixed $\sigma_0\geq\sigma_1>0$ and let $\rho=\alpha+i\beta$ be a pole of function $F(s)$ such that $\alpha=\max_{k=1,\ldots,N}\mathrm{Re}s_k$, let l be a corresponding index of the pole ρ, i.e. $\rho=s_l$. Let n be a positive integer and a be a real number such that $\sigma_0<\sigma_0+\varepsilon<a<\frac{n+1}{d_F}+\frac{\sigma_1}{2}$ and $\lambda_j(\sigma_1-a)+\mathrm{Re}\mu_j\notin\mathbb{Z}$ for all $j=1,\ldots,r$. Then,*

$$\gamma_n(F)=\frac{(-1)^n}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\overline{F(\overline{s})}G_F(s)}{(s-\sigma_1+\rho)^{n+1}}ds-\sum_{\substack{k=1\\k\neq l}}^{N}\operatorname*{Res}_{s=s_k}\frac{F(s)}{(s-\rho)^{n+1}}, \tag{4.4}$$

where

$$\begin{aligned}G_F(s)&=\omega\frac{Q_F^{2s-\sigma_1}}{\pi^r}\prod_{k=1}^{2M+\delta(\sigma_1)}\left(\frac{s-\overline{s_k}}{\sigma_1-s-s_k}\right)^{m_k}\\&\times\prod_{j=1}^{r}\left(\Gamma\left(\lambda_js+\overline{\mu_j}\right)\Gamma\left(1-\lambda_j(\sigma_1-s)-\mu_j\right)\sin\pi\left(\lambda_j(\sigma_1-s)+\mu_j\right)\right).\end{aligned} \tag{4.5}$$

Proof. The proof is based on the integral representation (4.2). The contour $\mathcal{C}$ is deformed to a suitable rectangular $\mathcal{R}_{a,A,T}$ and the integral is decomposed into integrals over its sides.

Let A and T be sufficiently large positive numbers such that $A \geq \sigma_0 + 1$ and $T > \max_{k=1,\dots,N} |\mathrm{Im} s_k|$. Let $\mathcal{R}_{a,A,T}$ be positively oriented rectangle determined by vertices $-a+\sigma_1-iT$, $A-iT$, $A+iT$ and $-a+\sigma_1+iT$. Additional contributions, compared to the integral (4.2) over $\mathcal{C}$, are from poles $s=s_k$ for $k \in \{1,2,\dots,N\} \setminus \{l\}$ of the function $F(s)$. By Cauchy's formula, we can write

$$\frac{1}{2\pi i} \int_{\mathcal{R}_{a,A,T}} \frac{F(s)}{(s-\rho)^{n+1}} ds = \gamma_n(F) + \sum_{\substack{k=1 \\ k \neq l}}^{N} \operatorname*{Res}_{s=s_k} \frac{F(s)}{(s-\rho)^{n+1}}.$$

Therefore,

$$\gamma_n(F) = \frac{1}{2\pi i} \int_{\mathcal{R}_{a,A,T}} \frac{F(s)}{(s-\rho)^{n+1}} ds - \sum_{\substack{k=1 \\ k \neq l}}^{N} \operatorname*{Res}_{s=s_k} \frac{F(s)}{(s-\rho)^{n+1}}. \tag{4.6}$$

Now, integral over $\mathcal{R}_{a,A,T}$ can be written as a sum of integrals over line segments $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ joining $-a+\sigma_1+iT$, $-a+\sigma_1-iT$, $A-iT$, $A+iT$ and $-a+\sigma_1+iT$, respectively.

For integral over $\mathcal{L}_2$, we will use decomposition of real parts based on Lemma 3.2

$$\begin{aligned}\int_{\mathcal{L}_2} \frac{F(s)}{(s-\rho)^{n+1}} ds &= \int_{-a+\sigma_1-iT}^{A-iT} \frac{F(s)}{(s-\rho)^{n+1}} ds \\ &= \left(\int_{-a+\sigma_1-iT}^{\sigma_1-\sigma_0-\varepsilon-iT} + \int_{\sigma_1-\sigma_0-\varepsilon-iT}^{\sigma_0+\varepsilon-iT} + \int_{\sigma_0+\varepsilon-iT}^{A-iT} \right) \frac{F(s)}{(s-\rho)^{n+1}} ds.\end{aligned}$$

The asymptotic bounds proved in Lemma 3.2 imply

$$\left| \int_{-a+\sigma_1-iT}^{\sigma_1-\sigma_0-\varepsilon-iT} \frac{F(s)}{(s-\rho)^{n+1}} ds \right| = O_\varepsilon \left(\left| \frac{T}{T+\beta} \right|^{n+1} |\, T\,|^{d_F\left(a-\frac{\sigma_1}{2}\right)-n-1} \right),$$

$$\left| \int_{\sigma_1-\sigma_0-\varepsilon-iT}^{\sigma_0+\varepsilon-iT} \frac{F(s)}{(s-\rho)^{n+1}} ds \right| = O_\varepsilon \left(\left| \frac{T}{T+\beta} \right|^{n+1} |\, T\,|^{d_F\left(\sigma_0+\varepsilon-\frac{\sigma_1}{2}\right)-n-1} \right),$$

and

$$\left| \int_{\sigma_0+\varepsilon-iT}^{A-iT} \frac{F(s)}{(s-\rho)^{n+1}} ds \right| = O_\varepsilon \left(\frac{1}{|\, T+\beta\,|^{n+1}} (A-\sigma_0-\varepsilon) \right).$$

as $|T| \to \infty$. Hence, for a under consideration and for $n \geq \frac{d_F}{2}(2a-\sigma_1)$ we have

$$\int\limits_{\mathcal{L}_2} \frac{F(s)}{(s-\rho)^{n+1}} ds \to 0, \quad \text{as} \quad |T| \to \infty.$$

Analogous procedure yields analogous asymptotic bound for the integral over $\mathcal{L}_4$, i.e.

$$\int\limits_{\mathcal{L}_4} \frac{F(s)}{(s-\rho)^{n+1}} ds \to 0, \quad \text{as} \quad |T| \to \infty.$$

For the integral over $\mathcal{L}_3$, notice that $s = A + it$, and by the choice of A, we are in the region of absolute convergence of function $F(s)$. Lemma 3.2 and substitution $u = t - \beta$ imply

$$\left| \int\limits_{\mathcal{L}_3} \frac{F(s)}{(s-\rho)^{n+1}} ds \right| \leq 2K \int\limits_0^{+\infty} \frac{du}{\left((A-\alpha)^2 + u^2\right)^{\frac{n+1}{2}}},$$

where K is a positive constant such that $|F(A+it)| \leq K$. Since, for the integrand

$$f_A(t) = \frac{1}{\left((A-\alpha)^2 + t^2\right)^{\frac{n+1}{2}}},$$

and function

$$g(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t \leq 1, \\ \frac{1}{t^{n+1}} & \text{if} \quad t > 1, \end{cases}$$

the following properties are satisfied $f_A(t) \leq g(t)$ on $[0, +\infty)$, for a positive integer n, $g(t)$ is integrable, and $\lim\limits_{A \to +\infty} f_A(t) = 0$, Lebesgue's convergence theorem my be applied. It implies that the contribution of the integral over $\mathcal{L}_3$ tends to zero, as $|T| \to \infty$, for all positive integers n.

It follows that the only contribution to the integral given in (4.6), when $|T| \to \infty$, for all n under consideration is from the integral over $\mathcal{L}_1$.

Simple substitution $s = \sigma_1 - u$ in the integral over $\mathcal{L}_1$, when $|T| \to \infty$, yields

$$\int\limits_{-a+\sigma_1+i\infty}^{-a+\sigma_1-i\infty} \frac{F(s)}{(s-\rho)^{n+1}} ds = (-1)^n \int\limits_{a-i\infty}^{a+i\infty} \frac{F(\sigma_1 - u)}{(u - \sigma_1 + \rho)^{n+1}} du.$$

Integrand in the last integral can be transformed using functional equation (2.4) and functional equation factor given by (2.5). Follows that

$$F(\sigma_1 - s) = \overline{F(\overline{s})}\frac{Q_F^{2s-\sigma_1}}{\overline{\omega}}\prod_{j=1}^{r}\frac{\Gamma(\lambda_j s + \overline{\mu_j})}{\Gamma(\lambda_j(\sigma_1 - s) + \mu_j)} \prod_{k=1}^{2M+\delta(\sigma_1)}\left(\frac{s-\overline{s_k}}{\sigma_1 - s - s_k}\right)^{m_k}. \tag{4.7}$$

The application of Euler's reflection formula for gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

valid for all $z \notin \mathbb{Z}$, implies that (4.7) can be written as

$$F(\sigma_1 - s) = \overline{F(\overline{s})}G_F(s),$$

where $G_F(s)$ is defined by (4.5), whenever $\lambda_j(\sigma_1 - s) + \mu_j \notin \mathbb{Z}$ for all integers $j = 1, \ldots, r$. Now, the representation (4.4) follows from (4.6), passing to the limit as $|\,T\,| \to \infty$, for all positive integers n such that $n \geq \frac{d_F}{2}(2a - \sigma_1)$. This completes the proof. ∎

Remark 4.3. When function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ does not have a pole, integral representation formula is reduced, i.e. $\gamma_F(n)$ defined by (4.3) are given by

$$\gamma_n(F) = \frac{(-1)^n}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\overline{F(\overline{s})}G_F(s)}{(s-\sigma_1+\sigma_0)^{n+1}}ds,$$

where

$$G_F(s) = \omega\frac{Q_F^{2s-\sigma_1}}{\pi^r}\prod_{j=1}^{r}\left[\Gamma\left(\lambda_j s + \overline{\mu_j}\right)\Gamma\left(1 - \lambda_j(\sigma_1 - s) - \mu_j\right)\right. \\ \left. \times \sin\pi\left(\lambda_j(\sigma_1 - s) + \mu_j\right)\right]$$

for a positive integer n and a real number a such that $\sigma_0 < \sigma_0 + \varepsilon < a < \frac{n+1}{d_F} + \frac{\sigma_1}{2}$ and $\lambda_j(\sigma_1 - a) + \mathrm{Re}\mu_j \notin \mathbb{Z}$ for all $j = 1, \ldots, r$.

5. Bounds for the generalized Euler-Stieltjes constants associated to function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$

In this section, we prove the main result of the paper, the theorem that gives an upper bound for the Euler-Stieltjes coefficients $\gamma_n(F)$ defined by (4.1). The proof is based on the integral representation (4.4) and an appropriate bound for the function $G_F(s)$, proved in the following lemma.

Before we state the lemma, some notation needs to be introduced. For $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$, let us denote $\lambda_M = \max\limits_{j=1,\ldots,r}\lambda_j$, $\lambda_m = \min\limits_{j=1,\ldots,r}\lambda_j$, $\mu_M =$

$\max_{j=1,\ldots,r} \mathrm{Re}\mu_j$, $\mu_m = \min_{j=1,\ldots,r} \mathrm{Re}\mu_j$ and $\alpha_M = \max_{j=1,\ldots,r} |\,\mathrm{Im}\mu_j\,|$, where λ_j and μ_j, for $j = 1, \cdots, r$, are parameters defined in the functional equation axiom (iii') of the class $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. Also, axiom (iii') implies that poles of $F(s)$ are arranged such that $s_{2j-1} + \overline{s}_{2j} = \sigma_1$, for $j = 1, \ldots, M$, i.e. s_{2j-1} and s_{2j} are symmetric with respect to the line $\mathrm{Re}s = \sigma_1/2$. So it implies that one of them is with real part less or equal to $\sigma_1/2$. Let assume that they correspond to indices $2j-1$ for $j = 1, \ldots, M$ and that those with real parts strictly less than $\sigma_1/2$ are with indices $2j-1$ for $j = 1, \ldots, M' \le M$.

Lemma 5.1. *Let $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ and let λ_m, λ_M, μ_m, μ_M and α_M be as above. Let $\nu = \max\{|\,\mu_m\,|, |\,\mu_M\,| - 1\}$. The following bound for function G_F, defined by (4.5), holds true for $a > \max\left\{\sigma_0 + 1, -\frac{\mu_m}{\lambda_m}, \frac{\mu_M - 1}{\lambda_m} + \sigma_1\right\}$,*

$$(5.1)\qquad \begin{aligned} |\,G_F(a+it)\,| &\le Q_F^{2a-\sigma_1} C_F(a) \\ &\quad \times \left((1 + \lambda_M a + \nu)^2 + (\lambda_M |t| + \alpha_M)^2 \right)^{\frac{d_F}{4}(2a-\sigma_1)}, \end{aligned}$$

where Q_F is a positive constant, an analytic conductor of the function F, appearing in axiom (iii') and $C_F(a)$ is constant given by

$$\begin{aligned} C_F(a) &= 2^r \exp\left(\frac{1}{12} \sum_{j=1}^{r} \frac{1 + \lambda_j(2a - \sigma_1)}{(\lambda_j a + \mathrm{Re}\mu_j)(1 + \lambda_j(a - \sigma_1) - \mathrm{Re}\mu_j)} \right) \\ &\quad \times (1 + 2\sigma_0 - \sigma_1)^{\sum_{j=1}^{M'} m_{2j-1}}. \end{aligned}$$

Proof. From axiom (iii') we know that $|\,\omega\,| = 1$ and $Q_F > 0$, so definition (4.5) of function $G_F(s)$ for $s = a + it$, $(a, t \in \mathbb{R})$ implies

$$(5.2)\qquad \begin{aligned} |\,G_F(a+it)\,| &= \frac{Q_F^{2a-\sigma_1}}{\pi^r} \prod_{k=1}^{2M+\delta(\sigma_1)} \left| \frac{a + it - \overline{s_k}}{\sigma_1 - a - it - s_k} \right|^{m_k} \\ &\quad \prod_{j=1}^{r} \left[|\,\Gamma\left(\lambda_j(a+it) + \overline{\mu_j}\right) \Gamma\left(1 - \lambda_j(\sigma_1 - a - it) - \mu_j\right)| \right. \\ &\quad \left. \times |\sin\pi\left(\lambda_j(\sigma_1 - a - it) + \mu_j\right)| \right]. \end{aligned}$$

Let us firstly consider product over poles s_k of function F. Using the assumption for the parameter a and the fact that $\sigma_0 > \sigma_1$ simple calculations yield

$$\left| \frac{a + it - \overline{s_k}}{\sigma_1 - a - it - s_k} \right|^{m_k} \le 1,$$

for all poles s_k such that $\mathrm{Re}s_k \ge \sigma_1/2$. Thus, the main contribution in this product comes from the poles such that $\mathrm{Re}s_k < \sigma_1/2$. Using notation introduced just before the formulation of the theorem, these are poles with the indices $2j-1$ for $j=1,\dots,M'\le M$. It follows

$$\begin{aligned}\prod_{k=1}^{2M+\delta(\sigma_1)}\left|\frac{a+it-\overline{s_k}}{\sigma_1-a-it-s_k}\right|^{m_k} &\le \prod_{j=1}^{M'}\left|\frac{a+it-\overline{s_{2j-1}}}{\sigma_1-a-it-s_{2j-1}}\right|^{m_{2j-1}}\\ &\le \prod_{j=1}^{M'}\left|\frac{a-\mathrm{Re}s_{2j-1}}{\sigma_1-a-\mathrm{Re}s_{2j-1}}\right|^{m_{2j-1}}\\ &\le (1+2\sigma_0-\sigma_1)^{\sum_{j=1}^{M'}m_{2j-1}},\end{aligned}\tag{5.3}$$

since $\sigma_1-\sigma_0\le \mathrm{Re}s_{2j-1}<\sigma_1/2$, for all $j=1,\dots,M'$ and $a>\sigma_0+1$.

Factors in the last product in (5.2) will be bounded separately. Using simple representation of sinus function in terms of exponential functions follows that $|\sin z|\le e^{|\,\mathrm{Im}z\,|}$ for $z\in\mathbb{C}$, so

$$|\sin\pi\left(\lambda_j(\sigma_1-a-it)+\mu_j\right)|\le\exp\left(\pi\,|\,\lambda_j t-\mathrm{Im}\mu_j\,|\right),\tag{5.4}$$

for all $j=1,\dots,r$. Bounds for the factors containing gamma functions will be derived using Binet formula [36, p. 258] combined with the inequality

$$\mathrm{Im}z\cdot\arctan\left(\frac{\mathrm{Im}z}{\mathrm{Re}z}\right)+\mathrm{Re}z\ge\frac{\pi}{2}\,|\,\mathrm{Im}z\,|,$$

which holds true for $\mathrm{Re}z>0$. We obtain the following inequality

$$\begin{aligned}\log|\,\Gamma(z)\,|\le&\left(\mathrm{Re}z-\frac12\right)\log|\,z\,|-\frac{\pi}{2}\,|\,\mathrm{Im}z\,|+\frac12\log(2\pi)\\ &+\mathrm{Re}\left[\int_0^{+\infty}\left(\frac12-\frac1t+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}dt\right],\end{aligned}\tag{5.5}$$

valid for $\mathrm{Re}z>0$, which can be simplified using some properties of the function $g(t)=\left(\frac12-\frac1t+\frac{1}{e^t-1}\right)\frac1t$. Specially, the function $g(t)$ attains its maximum $1/12$, at $t=0$. So

$$\mathrm{Re}\left[\int_0^{+\infty}\left(\frac12-\frac1t+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}dt\right]\le\frac{1}{12\mathrm{Re}z},$$

and (5.5) implies

$$\log|\,\Gamma(z)\,|\le\left(\mathrm{Re}z-\frac12\right)\log|\,z\,|-|\,\mathrm{Im}z\,|\frac{\pi}{2}+\frac12\log(2\pi)+\frac{1}{12\mathrm{Re}z},\tag{5.6}$$

for $\mathrm{Re}z>0$.

For $z=\lambda_j(a+it)+\overline{\mu_j}$ and $z=1-\lambda_j(\sigma_1-a-it)-\mu_j$ by the assumption for parameter a follows that

$$\mathrm{Re}\,(\lambda_j(a+it)+\overline{\mu_j})>0 \quad \text{and} \quad \mathrm{Re}\,(1-\lambda_j(\sigma_1-a-it)-\mu_j)>0,$$

for all $j=1,\dots,r$, thus inequality (5.6) may be applied for the gamma factors in (5.2).

In addition, definition of numbers ν, λ_M and α_M implies the following inequalities

$$\begin{aligned}
&(\lambda_j t-\mathrm{Im}\mu_j)^2 \le (\lambda_M\,|\,t\,|+\alpha_M)^2\,,\\
&(\lambda_j a+\mathrm{Re}\mu_j)^2 \le (\lambda_M a+|\,\mu_M\,|)^2 \le (1+\lambda_M a+\nu)^2,\\
&(1-\lambda_j(\sigma_1-a)-\mathrm{Re}\mu_j)^2 \le (1+\lambda_M a+|\,\mu_m\,|)^2 \le (1+\lambda_M a+\nu)^2,
\end{aligned}$$

$j=1,\dots,r$ and from (5.6) we obtain

$$\begin{aligned}
&\log|\,\Gamma\,(\lambda_j(a+it)+\overline{\mu_j})\,|+\log|\,\Gamma\,(1-\lambda_j(\sigma_1-a-it)-\mu_j)\,|\\
&\quad\le \frac{\lambda_j(2a-\sigma_1)}{2}\log\Big((1+\lambda_M a+\nu)^2+(\lambda_M\,|\,t\,|+\alpha_M)^2\Big)\\
&\qquad+\frac{1}{12}\frac{1+\lambda_j(2a-\sigma_1)}{(\lambda_j a+\mathrm{Re}\mu_j)(1+\lambda_j(a-\sigma_1)-\mathrm{Re}\mu_j)}\\
&\qquad+\log 2\pi-\pi\,|\,\lambda_j t-\mathrm{Im}\mu_j\,|\,,
\end{aligned}$$

for all $j=1,\dots,r$. This bound combined with (5.4) implies

$$\begin{aligned}
&|\Gamma\,(\lambda_j(a+it)+\overline{\mu_j})|\,|\,\Gamma\,(1-\lambda_j(\sigma_1-a-it)-\mu_j)\,|\\
&\quad\times|\sin\pi\,(\lambda_j(\sigma_1-a-it)+\mu_j)\,|\\
&\quad\le \exp\Bigg[\frac{\lambda_j(2a-\sigma_1)}{2}\log\Big((1+\lambda_M a+\nu)^2+(\lambda_M\,|\,t\,|+\alpha_M)^2\Big)\\
&\qquad+\frac{1}{12}\frac{1+\lambda_j(2a-\sigma_1)}{(\lambda_j a+\mathrm{Re}\mu_j)(1+\lambda_j(a-\sigma_1)-\mathrm{Re}\mu_j)}+\log 2\pi\Bigg].
\end{aligned}$$

Substituting the last relation and (5.3) into (5.2), we obtain (5.1), and the proof is complete. □

The following theorem is the main result of the paper, it gives a bound for generalized Euler-Stieltjes constants attached to functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$.

Theorem 5.2. *Let $F\in\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ and $\rho=\alpha+i\beta$ be a pole of function $F(s)$ such that $\alpha=\max\limits_{k=1,\dots,N}\mathrm{Re}s_k$ and let l be a corresponding index of the pole ρ, i.e. $\rho=s_l$. Let numbers λ_M, λ_m, μ_M, μ_m, α_M and ν be as in Lemma 5.1 and $\vartheta_M=\max\{\nu+\lambda_M(\sigma_1-\alpha),\alpha_M+\lambda_M\,|\,\beta\,|\}$.*

For a real number a such that $a > \max\left\{\sigma_0 + 1, -\frac{\mu_m}{\lambda_m}, \frac{\mu_M - 1}{\lambda_m} + \sigma_1\right\}$ and $\lambda_j(\sigma_1 - a) + \mathrm{Re}\mu_j \notin \mathbb{Z}$ for all $j = 1, \ldots, r$ we have

$$
\begin{aligned}
|\gamma_n(F)| &\leq \frac{D_F(a)}{(a-\sigma_1+\alpha)^n}\left(1 + \frac{1}{n - \frac{d_F}{2}(2a-\sigma_1)}\right) \\
&\quad + \sum_{\substack{k=1\\k\neq l}}^{N} \operatorname*{Res}_{s=s_k}\left|\frac{F(s)}{(s-\rho)^{n+1}}\right|,
\end{aligned} \tag{5.7}
$$

where constant $D_F(a)$ is given by

$$
\begin{aligned}
D_F(a) = \exp&\left(\frac{1}{12}\sum_{j=1}^{r}\frac{1+\lambda_j(2a-\sigma_1)}{(\lambda_j a + \mathrm{Re}\mu_j)(1+\lambda_j(a-\sigma_1)-\mathrm{Re}\mu_j)}\right) \\
&\times \frac{2^{\frac{d_F}{4}(2a-\sigma_1)+r} Q_F^{2a-\sigma_1}}{\pi}\,\frac{1+\lambda_M+\vartheta_M}{\lambda_M}\,(1+2\sigma_0-\sigma_1)^{\sum_{j=1}^{M'} m_{2j-1}} \\
&\times \left((1+\lambda_M+\vartheta_M)(a-\sigma_1+\alpha)\right)^{\frac{d_F}{2}(2a-\sigma_1)}\left(\sum_{k=1}^{+\infty}\frac{|a_F(k)|}{k^a}\right).
\end{aligned}
$$

for all positive integere n such that $n > \frac{d_F}{2}(2a-\sigma_1)$.

Proof. The proof is based on the integral representation of generalized Euler-Stieltjes coefficients given in Theorem 4.2 and the bound derived in Lemma 5.1. Simple substitution in integral in (4.4) and (5.1) yield

$$
\begin{aligned}
|\gamma_n(F)| \leq C_F(a)\frac{Q_F^{2a-\sigma_1}}{2\pi}\int_{-\infty}^{+\infty}&\left((1+\lambda_M a+\nu)^2 + (\lambda_M|t|+\alpha_M)^2\right)^{\frac{d_F}{4}(2a-\sigma_1)} \\
&\times \frac{\left|\overline{F(a-it)}\right|}{\left((a-\sigma_1+\alpha)^2+(t+\beta)^2\right)^{\frac{n+1}{2}}}dt \\
&+ \sum_{\substack{k=1\\k\neq l}}^{N}\left|\operatorname*{Res}_{s=s_k}\frac{F(s)}{(s-\rho)^{n+1}}\right|,
\end{aligned}
$$

where $C_F(a)$ is defined in Lemma 5.1. For $a > \sigma_0 + 1 > \sigma_0$, it hold true

$$
\left|\overline{F(a-it)}\right| \leq \sum_{k=1}^{\infty}\frac{|a_F(k)|}{k^a} < +\infty,
$$

by Dirichlet series representation axiom (i'). Hence,

$$|\gamma_n(F)| \le C_F(a)\frac{Q_F^{2a-\sigma_1}}{2\pi}\sum_{k=1}^{\infty}\frac{|a_F(k)|}{k^a}I + \sum_{\substack{k=1\\k\neq l}}^{N}\left|\operatorname*{Res}_{s=s_k}\frac{F(s)}{(s-\rho)^{n+1}}\right|, \tag{5.8}$$

where

$$I = \int_{-\infty}^{+\infty}\frac{\left((1+\lambda_M a+\nu)^2+(\lambda_M|t|+\alpha_M)^2\right)^{\frac{d_F}{4}(2a-\sigma_1)}}{((a-\sigma_1+\alpha)^2+(t+\beta)^2)^{\frac{n+1}{2}}}dt.$$

Thus, it is left to derive a bound for the integral I. Depending on the sign of β, we examine two cases.

Let $\beta \ge 0$. Then

$$I = \int_0^{+\infty}\left(\frac{1}{((a-\sigma_1+\alpha)^2+(t-\beta)^2)^{\frac{n+1}{2}}}+\frac{1}{((a-\sigma_1+\alpha)^2+(t+\beta)^2)^{\frac{n+1}{2}}}\right) \times\left((1+\lambda_M a+\nu)^2+(\lambda_M t+\alpha_M)^2\right)^{\frac{d_F}{4}(2a-\sigma_1)}dt. \tag{5.9}$$

The interval of integration we split into two parts $(0,B)$ and $(B,+\infty)$, where $B=\frac{1}{\lambda_M}+a-\sigma_1+\alpha+\beta$. Denote by I_1 and I_2 corresponding integrals, respectively. For I_1 we have

$$I_1 \le B\frac{2^{\frac{d_F}{4}(2a-\sigma_1)+1}}{a-\sigma_1+\alpha}\frac{(1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M)^{\frac{d_F}{2}(2a-\sigma_1)}}{(a-\sigma_1+\alpha)^n}, \tag{5.10}$$

where $\vartheta_M = \max\{\nu+\lambda_M(\sigma_1-\alpha), \alpha_M+\lambda_M\beta\} > 0$.

By the assumptions of the theorem, it holds true $\frac{B}{a-\sigma_1+\alpha} \le 1+q+\frac{1}{\lambda_M}$ and $1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M \le (\lambda_M+1+\vartheta_M)(a-\sigma_1+\alpha)$, so from relation (5.10) we obtain the following bound for the integral I_1

$$I_1 \le \frac{2^{\frac{d_F}{4}(2a-\sigma_1)+1}}{\lambda_M}(\lambda_M+1+\vartheta_M)^{\frac{d_F}{2}(2a-\sigma_1)+1}(a-\sigma_1+\alpha)^{\frac{d_F}{2}(2a-\sigma_1)-n}. \tag{5.11}$$

Therefore, $I_1 \to 0$, as $n \to +\infty$.

In order to bound I_2, note that for $t \ge B$ and the following inequalities hold true $1+\lambda_M a+\nu \le 1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M \le \lambda_M(t-\beta)+\vartheta_M$ and $\lambda_M t+\alpha_M \le \lambda_M(t-\beta)+\vartheta_M$, so

$$(1+\lambda_M a+\nu)^2+(\lambda_M t+\alpha_M)^2 \le 2(\lambda_M(t-\beta)+\vartheta_M)^2.$$

Hence,

$$I_2 \leq \int\limits_B^{+\infty} \frac{2}{(t-\beta)^{n+1}} \left(2\left(\lambda_M(t-\beta)+\vartheta_M\right)^2\right)^{\frac{d_F}{4}(2a-\sigma_1)} dt$$
$$= 2\left(2\lambda_M^2\right)^{\frac{d_F}{4}(2a-\sigma_1)} \int\limits_{B-\beta}^{+\infty} \left(1+\frac{\vartheta_M}{\lambda_M u}\right)^{n+1} \left(u+\frac{\vartheta_M}{\lambda_M}\right)^{\frac{d_F}{2}(2a-\sigma_1)-n-1} du.$$

Furthermore, since function $g(t) = 1 + \frac{\vartheta_M}{\lambda_M t}$ is monotonically decreasing for $t \geq B-\beta = a-\sigma_1+\alpha+\frac{1}{\lambda_M} \geq 1$, $g(t) \geq 1$ and $\lim\limits_{t\to+\infty} g(t) = 1$ follow that maximal value of $g(t)$ is at point $t = B-\beta$. Thus,

$$1 \leq g(t) \leq g(B-\beta) = \frac{1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M}{1+\lambda_M(a-\sigma_1+\alpha)},$$

hence,

$$I_2 \leq 2\left(2\lambda_M^2\right)^{\frac{d_F}{4}(2a-\sigma_1)} \left(\frac{1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M}{1+\lambda_M(a-\sigma_1+\alpha)}\right)^{n+1}$$
$$\times \int\limits_{B-\beta}^{+\infty} \left(u+\frac{\vartheta_M}{\lambda_M}\right)^{\frac{d_F}{2}(2a-\sigma_1)-n-1} du.$$

For all n under consideration, i.e. $n > \frac{d_F}{2}(2a-\sigma_1)$ the above integral converges and yields

$$I_2 \leq \frac{2^{\frac{d_F}{4}(2a-\sigma_1)+1}}{n-\frac{d_F}{2}(2a-\sigma_1)} \frac{\left(1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M\right)^{\frac{d_F}{2}(a-\sigma_1)+1}}{\lambda_M\left(a-\sigma_1+\alpha\right)^{n+1}}.$$

Additionally, since $\frac{1+\lambda_M(a-\sigma_1+\alpha)+\vartheta_M}{a-\sigma_1+\alpha} \leq 1+\lambda_M+\vartheta_M$, we obtain

$$I_2 \leq \frac{2^{\frac{d_F}{4}(2a-\sigma_1)+1}(1+\lambda_M+\vartheta_M)^{\frac{d_F}{2}(2a-\sigma_1)+1}}{\lambda_M\left(n-\frac{d_F}{2}(2a-\sigma_1)\right)} (a-\sigma_1+\alpha)^{\frac{d_F}{2}(2a-\sigma_1)-n}. \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.9), combined with (5.8) implies (5.7) and the proof is completed in this case.

When $\beta < 0$ procedure is completely analogous as in the previous case, after simple substitution $\beta_1 = -\beta > 0$. This completes the proof of theorem. ∎

Remark 5.3. Bound for coefficients $\gamma_F(n)$ in the case when function $F \in \mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ does not have a pole can be derived completely analogously as in previous theorem. Thus, for $\gamma_F(n)$ defined by (4.3) follows that

$$|\,\gamma_n(F)\,| \leq \frac{D_F(a)}{(a-\sigma_1+\sigma_0)^n}\left(1+\frac{1}{n-\frac{d_F}{2}(2a-\sigma_1)}\right),$$

where

$$\begin{aligned} D_F(a) = {}& \exp\left(\frac{1}{12}\sum_{j=1}^{r}\frac{1+\lambda_j(2a-\sigma_1)}{(\lambda_j a+\mathrm{Re}\mu_j)(1+\lambda_j(a-\sigma_1)-\mathrm{Re}\mu_j)}\right) \\ & \times \frac{2^{\frac{d_F}{4}(2a-\sigma_1)+r}Q_F^{2a-\sigma_1}}{\pi}\,\frac{1+\lambda_M+\vartheta_M}{\lambda_M} \\ & \times \left((1+\lambda_M+\vartheta_M)(a-\sigma_1+\sigma_0)\right)^{\frac{d_F}{2}(2a-\sigma_1)}\left(\sum_{k=1}^{+\infty}\frac{|\,a_F(k)\,|}{k^a}\right), \end{aligned}$$

for all positive integere n such that $n > \frac{d_F}{2}(2a-\sigma_1)$ and a is a real number such that $a > \max\left\{\sigma_0+1, -\frac{\mu_m}{\lambda_m}, \frac{\mu_M-1}{\lambda_m}+\sigma_1\right\}$ and $\lambda_j(\sigma_1-a)+\mathrm{Re}\mu_j \notin \mathbb{Z}$ for all $j=1,\ldots,r$. Numbers λ_M, λ_m, μ_M, μ_m, α_M and ν are same as in Lemma 5.1 and $\vartheta_M = \max\{\nu+\lambda_M(\sigma_1-\sigma_0), \alpha_M\}$.

6. Numerical examples

Examples of functions from the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ (not in the Selberg class) are products of suitably shifted Riemann zeta functions. In [12] it is proved that $H(s)=\zeta(s-h)\zeta(s+h)$ for some real $h>0$ is an element of the class $\mathcal{S}^{\sharp\flat}(h+1,1)$. In this section we will demonstrate an application of derived result to the function $H(s)$. We will derive a bound for the Euler-Stieltjes constants attached to $H(s)$ implied by Theorem 5.2.

Using properties of the Riemann zeta function we can easily deduce that axioms of the class $\mathcal{S}^{\sharp\flat}(\sigma_0,\sigma_1)$ are satisfied with the following parameters: $N=2$, $s_1=1-h$, $s_2=1+h$, $m_1=m_2=1$, $Q_H=\pi^{-1}$, $\omega=1$, $r=2$, $\lambda_1=\lambda_2=1/2$, $\mu_1=h/2$, $\mu_2=-h/2$ and the degree of function $H(s)$ is $d_H=2$. For $h\neq 1/2$ easily follows that $M=0$ and $\delta(\sigma_1)=\delta(1)=0$, while for $h=1/2$ it follows that $M=0$ and $\delta(\sigma_1)=\delta(1)=1$.

In both cases for generalized Euler-Stieltjes constants defined by (4.1), with $\rho=1+h$, i.e. $\alpha=1+h$, $\beta=0$ the following bound holds true

$$(6.1)\quad |\,\gamma_n(H)\,| \leq \frac{D_H(a)}{(a+h)^n}\left(1+\frac{1}{n-2a+1}\right)+\operatorname*{Res}_{s=1-h}\left|\frac{H(s)}{(s-h-1)^{n+1}}\right|,$$

where constant $D_H(a)$ is given by

$$D_H(a) = \exp\left(\frac{1+2a}{6}\left(\frac{1}{(a-h)(a+h+1)} + \frac{1}{(a-h+1)(a+h)}\right)\right)$$
$$\times 2^{-a+\frac{5}{2}}\left(\frac{3}{\pi}\right)^{2a}(a+h)^{2a-1}\left(\sum_{k=1}^{+\infty}\frac{|\,a_F(k)\,|}{k^a}\right),$$

since $\lambda_M = \lambda_m = 1/2$, $\mu_M = h/2$, $\mu_m = -h/2$, $\alpha_M = 0$, $\nu = h/2$ and $\vartheta_M = 0$. Here, real number a is such that $a > h+2$ and $\frac{1}{2}(1-a) \pm \frac{h}{2} \notin \mathbb{Z}$ and the bound (6.1) is valid for all n such that $n > 2h+3$. Additionally, by the fact that the residuum of Riemann zeta function at pole $s=1$ is 1 implies that contribution from the pole $1-h$ in (6.1) is equal to $\frac{|\,\zeta(1-2h)\,|}{(2h)^{n+1}}$.

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O EULER–STIELTJESOVIM KONSTANTAMA ZA FUNKCIJE IZ GENERALIZIRANE SELBERGOVE KLASE

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Sažetak. Klasa $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$ je vrlo široka klasa L-funkcija koja sadrži Selbergovu klasu, klasu svih automorfnih L-funkcija i Rankin–Selbergove L-funkcije, kao i umnoške prikladnih pomaka tih funkcija. U ovom radu razmatramo generalizirane Euler–Stieltjesove konstante $\gamma_n(F)$ pridružene funkcijama $F(s)$ iz klase $\mathcal{S}^{\sharp\flat}(\sigma_0, \sigma_1)$. To su koeficijenti u Laurentovom razvoju funkcije $F(s)$ u njezinom polu. Izvodimo integralnu reprezentaciju i gornju ogradu za te konstante. Prikazana je i primjena dobivenih rezultata u slučaju umnoška prikladnih pomaka Riemannove zeta funkcije.