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Maxwell's and Wave Equations in Media Containing Electromagnetic Field Sources

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SUMMARY

This paper presents a new system of Maxwell's equations and corresponding wave equations for conducting media containing electromagnetic field sources. The theoretical approach begins with a consistent generalisation of Maxwell's differential equations in a slowly moving conducting medium. From this generalised form, both differential and integral forms of Maxwell's equations are derived for slowly moving and stationary conducting media. The well-known Maxwell's equations for a perfect dielectric containing sources appear as a special case of the equations for a conducting medium containing sources. Notably, Maxwell's third differential equation is derived from the first, resulting in a new form of the third equation specific to conducting media. This revised equation differs from the traditional form valid in perfect dielectrics – namely, Gauss's law for the electric field – which holds in a conducting medium only when the medium is linear, isotropic, homogeneous and source-free. A key contribution of this work is the introduction of the volume density of the source leakage electric current in conducting media as a novel physical quantity, analogous to the volume charge density in perfect dielectrics. The volume density of the leakage electric current injected by the source into the surrounding medium is reduced to the volume density of the displacement electric current in the case of a dielectric, where the conducting component is absent. Finally, wave equations for electromagnetic fields and electromagnetic potentials are derived from the generalised Maxwell's differential equations.

KEY WORDS: *Maxwell's equations; wave equations; electromagnetic field sources; conducting media; perfect dielectrics; volume density of the source leakage electric current.*

1. INTRODUCTION

Maxwell's equations are the fundamental equations of classical electromagnetism, or classical electrodynamics, and can be expressed in both differential and integral forms. The conversion from the differential to the integral form can be performed using the Ostrogradsky-Green-Gauss theorem and Stokes' theorem. Numerous versions of Maxwell's differential equations can be found in the literature [1–15]. To the best of my knowledge, correct formulations of Maxwell's equations in conducting media with electromagnetic field sources are generally not available in the literature, including geophysical electromagnetism texts [16–22], with the exception of report [23], which addresses the time-harmonic electromagnetic field in a conducting, linear, isotropic and homogeneous medium.

In conducting media, the sources of the static component of the electric field are electric current sources rather than electric charges. In other words, Gauss’s law for the electric field is not valid in conducting media containing electromagnetic field sources. Maxwell’s third differential equation in conducting media must be derived from Maxwell’s first differential equation; in other words, Maxwell’s differential equations must not be mutually contradictory. Moreover, Maxwell’s equations in perfect dielectrics should be treated as a special case of the more general equations in conducting media. It is also important to note that this paper is an expanded version of my conference paper [24].

James Clerk Maxwell identified an inconsistency in the original formulation of Ampère’s law when applied to time-varying electric fields, particularly in situations such as the charging of a capacitor. In such cases, although no actual current flows between the capacitor plates, the changing electric field suggests the presence of a magnetic field, which the original Ampère’s law could not account for. To resolve this inconsistency and ensure compatibility with the continuity equation (which expresses conservation of electric charge), Maxwell introduced the concept of displacement electric current. By adding this term to Ampère’s law, he extended it to include the contribution of time-varying electric fields, thereby restoring consistency with the continuity equation [14], [15]. This modification allowed Maxwell to mathematically predict the existence of electromagnetic waves propagating through space at the speed of light, a key step in unifying electricity and magnetism into a single theory of electromagnetism.

Maxwell's inclusion of the displacement electric current in Ampère's law stemmed from his understanding that his equations were contradictory in a perfect dielectric without this correction. In this paper, it is precisely from the postulation that Maxwell’s equations must not be contradictory even in a conducting medium that the proposed new system of Maxwell’s equations emerged.

In this paper, for a conducting medium, a system of Maxwell's equations has been derived that is analogous to the system of Maxwell's equations in a perfect dielectric, instead of deriving the subordinate equations based on physical analogy.

2. MAXWELL’S DIFFERENTIAL EQUATIONS

The framework of Maxwell’s electrodynamics for slowly moving media can be formulated based on the following consistent generalised equations:

$$\nabla \times \vec{H} = \vec{J}_{tot} \tag{1}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \tag{2}$$

$$\nabla \cdot \vec{J}_{tot} = 0 \tag{3}$$

$$\nabla \cdot \vec{B} = 0 \tag{4}$$

Where $\vec{H}$ is the magnetic field intensity vector, $\vec{E}$ is the electric field intensity vector, $\vec{B}$ is the magnetic flux density vector, $\vec{J}_{tot}$ is the vector of the surface density of the total electric current, and ∇ is the Hamiltonian (nabla) operator.

Equation (1) represents the well-known Ampère's law in differential form referred to as Maxwell's first differential equation. Equation (2) represents the well-known Faraday’s law of electromagnetic induction in differential form – let it be called Maxwell's second differential equation. Equation (3) represents Gauss's law for the total electric current, from which Maxwell's third differential equation can be derived, valid in both conducting media and perfect dielectrics. Equation (4) represents Gauss's law for the magnetic field, referred to as Maxwell's fourth differential equation. Equations (3) and (4) are continuity equations, which assert that the vector fields $\vec{J}_{tot}$ and $\vec{B}$ contain neither sources nor sinks.

Since the divergence of the curl of any vector field is zero, Eqs. (3) and (4) can be derived from Eqs. (1) and (2):

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \vec{J}_{tot} = 0 \tag{5}$$

$$\nabla \cdot \left(\nabla \times \vec{E}\right) = -\frac{d\,(\nabla \cdot \vec{B})}{d\,t} = 0 \tag{6}$$

It follows from Eq. (6) that the divergence of the vector is time-independent, and Eq. (4) is a special case of the equation derived from Eq. (6). It is important to emphasise that Eqs. (4) and (6) are not in contradiction – this is the fundamental physical premise of this paper.

It is important to emphasise that Eq. (3) represents Maxwell's original continuity equation, expressed in Heaviside's modern vector notation. In this paper, the very Maxwell continuity equation is applied within a conducting medium. Accordingly, this equation must be satisfied in both conducting media and perfect dielectrics.

In the slowly moving media, Maxwell's second differential equation can be written as [4], [11]:

$$\nabla \times \vec{E} = -\frac{d\,\vec{B}}{d\,t} = \nabla \times \left(\vec{v} \times \vec{B}\right) - \frac{\partial \vec{B}}{\partial\,t} \tag{7}$$

where $\vec{v}$ denotes a constant velocity vector with respect to the observer.

In stationary media, Eq. (7) takes the following form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial\,t} \tag{8}$$

If the convection currents that occur in gases and liquids are excluded from consideration, then the vector of the surface density of the total electric current can be expressed as:

$$\vec{J}_{tot} = \vec{J}_s + \vec{J}_c + \vec{J}_{disp} \tag{9}$$

where $\vec{J}_s$ is the vector of the surface density of the source (impressed) electric current, $\vec{J}_c$ is the vector of the surface density of the conduction electric current, and $\vec{J}_{disp}$ is the vector of the surface density of the displacement electric current.

In slowly moving media, the following equation holds [4], [11]:

$$\vec{J}_{disp} = \frac{d\,\vec{D}}{d\,t} = -\nabla \times \left(\vec{v} \times \vec{D}\right) + \left(\nabla \cdot \vec{D}\right)\vec{v} + \frac{\partial \vec{D}}{\partial\,t} \tag{10}$$

which, in stationary media, takes on a new form:

$$\vec{J}_{disp} = \frac{\partial \vec{D}}{\partial\,t} \tag{11}$$

where $\vec{D}$ is the electric flux density vector.

Note: The first complication in the physically accurate naming of linear, surface and volume densities of electric current lies in the use of the term *surface current density* to denote the linear density of surface electric current, measured perpendicular to the direction of the current – where *surface electric current* refers to current flowing along a surface. Surface current density is a vector quantity whose direction coincides with that of the surface current. The second complication arises from the inconsistent use of the two terms, *current density* and *volume* *current density,* to describe the surface density of electric current. To avoid physical ambiguity, this paper adopts longer and unambiguous terms for all three types of electric current density. This clarification is especially important given that this paper introduces a new physical quantity: the volume density of the source leakage electric current.

2.1 MAXWELL'S THIRD DIFFERENTIAL EQUATION

Based on (3) and (9), Maxwell's third differential equation for conducting media containing electromagnetic field sources takes the following form:

$$\nabla \cdot \left(\vec{J}_c + \vec{J}_{disp}\right) = -\nabla \cdot \vec{J}_s = g_{st} \quad \textbf{(12)}$$

where g_{st} is the volume density of the source leakage electric current (A/m^3), whose physical nature becomes evident from the definition of divergence:

$$g_{st} = \nabla \cdot \left(\vec{J}_c + \vec{J}_{disp}\right) = \lim_{V \to 0} \frac{\oint_{\partial V} (\vec{J}_c + \vec{J}_{disp}) \cdot d\vec{S}}{V} \quad \textbf{(13)}$$

where V is the volume enclosed by the surface ∂V (Figure 1). It is always true that the divergence of the surface density of a physical quantity yields the volume density of that quantity. The volume density of the source leakage electric current is used in reference [23], but in that report, this physical quantity is not referred to by its proper name. The source leakage electric current is the portion of the electric current lost by the source, flowing out of the intended electric circuit through which the source (impressed) electric current is conducted.

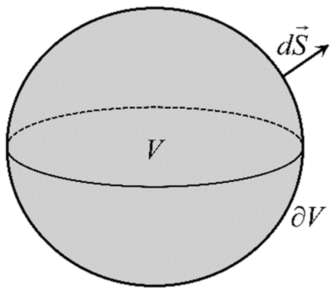

Fig. 1 *Volume V enclosed by the surface ∂V*

According to (10) and (12), and taking into account that the divergence of the curl of any vector is zero, Maxwell's third differential equation in slowly moving conducting media containing sources can be written as:

$$\nabla \cdot \left(\vec{J}_c + \frac{d\,\vec{D}}{d\,t}\right) = \nabla \cdot \left(\vec{J}_c + \left(\nabla \cdot \vec{D}\right)\vec{v} + \frac{\partial\,\vec{D}}{\partial\,t}\right) = g_{st} \quad \textbf{(14)}$$

which, in stationary conducting media containing sources, takes on a new form:

$$\nabla \cdot \left(\vec{J}_c + \frac{\partial\,\vec{D}}{\partial\,t}\right) = g_{st} \quad \textbf{(15)}$$

Based on (14), Maxwell's third differential equation for source-free, slowly moving conducting media takes the following form:

$$\nabla \cdot \left(\vec{J}_c + \frac{d\,\vec{D}}{d\,t}\right) = \nabla \cdot \left(\vec{J}_c + \left(\nabla \cdot \vec{D}\right)\vec{v} + \frac{\partial\,\vec{D}}{\partial\,t}\right) = 0 \quad \textbf{(16)}$$

which, in the source-free stationary conducting media takes on a new form:

$$\nabla \cdot \left(\vec{J}_c + \frac{\partial\,\vec{D}}{\partial\,t}\right) = 0 \quad \textbf{(17)}$$

Based on (14), Maxwell’s third differential equation for a slowly moving perfect dielectric containing sources, under the condition $\vec{J}_c = 0$, can be expressed as:

$$\nabla \cdot \left(\frac{d\,\vec{D}}{d\,t}\right) = \frac{d\,(\nabla \cdot \vec{D})}{d\,t} = g_{st} = -\nabla \cdot \vec{J}_s \tag{18}$$

In the slowly moving perfect dielectric containing sources, the well-known law of conservation of electric charge is valid:

$$\nabla \cdot \vec{J}_s = -\,g_{st} = -\,\frac{d\,\rho_s}{d\,t} \tag{19}$$

where ρ_s denotes the volume density of the electric charge. Thus, in a perfect dielectric, the volume density of the leakage electric current g_{st} injected by the source into the surrounding medium is reduced to the volume density of the displacement electric current in dielectric where the conducting component is absent.

From (18) and (19), Maxwell’s third differential equation – Gauss’s law for the electric field in differential form – can be obtained as:

$$\nabla \cdot \vec{D} = \rho_s \tag{20}$$

This equation is valid in both stationary and slowly moving perfect dielectrics containing sources, but it is not valid in conducting media. In the literature, it is often incorrectly stated that Eq. (20) also holds in stationary and slowly moving conducting media containing sources.

In linear, isotropic and homogeneous (LIH) media, the following constitutive relations hold in the time domain:

$$\vec{D} = \varepsilon\,\vec{E} \quad ; \quad \vec{J}_c = \sigma\,\vec{E} \quad ; \quad \vec{B} = \mu\,\vec{H} \tag{21}$$

where ε is the permittivity, σ is the conductivity, and μ is the magnetic permeability of the medium. In an LIH medium, the material properties are scalar constants.

According to (14) and (21), in the slowly moving conducting LIH media containing sources, Maxwell's third differential equation can be expressed as:

$$\left(\sigma + \varepsilon\,\frac{d}{d\,t}\right)\left(\nabla \cdot \vec{E}\right) = \left(\sigma + \varepsilon\,\vec{v} + \varepsilon\,\frac{\partial}{\partial\,t}\right)\left(\nabla \cdot \vec{E}\right) = g_{st} \tag{22}$$

which, in stationary conducting LIH media containing sources, takes on a new form:

$$\left(\sigma + \varepsilon\,\frac{\partial}{\partial\,t}\right)\left(\nabla \cdot \vec{E}\right) = g_{st} \tag{23}$$

According to (20) - (22) and (23), Maxwell’s third differential equation takes the following form in both source-free conducting LIH media and source-free perfect LIH dielectrics:

$$\nabla \cdot \vec{E} = 0 \tag{24}$$

from which follows that, in source-free LIH media, the following holds:

$$\nabla \cdot \vec{J}_c = 0 \quad ; \quad \nabla \cdot \vec{D} = 0 \tag{25}$$

2.2 MAXWELL’S DIFFERENTIAL EQUATIONS IN CONDUCTING MEDIA

Based on the previous considerations, it is straightforward to conclude that Maxwell’s differential equations in the time domain, in slowly moving conducting media containing sources, are given by the following expressions:

$$\nabla \times \vec{H} = \vec{J}_s + \vec{J}_c - \nabla \times \left(\vec{v} \times \vec{D}\right) + \left(\nabla \cdot \vec{D}\right)\vec{v} + \frac{\partial \vec{D}}{\partial\,t} \tag{26}$$

$$\nabla \times \vec{E} = \nabla \times \left(\vec{v} \times \vec{B}\right) - \frac{\partial\,\vec{B}}{\partial\,t} \tag{27}$$

$$\nabla \cdot \left(\vec{J}_c + \left(\nabla \cdot \vec{D} \right) \vec{v} + \frac{\partial \vec{D}}{\partial t} \right) = g_{st} \tag{28}$$

$$\nabla \cdot \vec{B} = 0 \tag{29}$$

which, in stationary conducting media containing sources, take on a new form:

$$\nabla \times \vec{H} = \vec{J}_s + \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \tag{30}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{31}$$

$$\nabla \cdot \left(\vec{J}_c + \frac{\partial \vec{D}}{\partial t} \right) = g_{st} \tag{32}$$

$$\nabla \cdot \vec{B} = 0 \tag{33}$$

whereas in stationary good conductors containing sources, where displacement electric currents can be neglected, the magnetodynamic approximation of the dynamic electromagnetic field [8] is described by the following Maxwell's differential equations:

$$\nabla \times \vec{H} = \vec{J}_s + \vec{J}_c \tag{34}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{35}$$

$$\nabla \cdot \vec{J}_c = g_{st} \tag{36}$$

$$\nabla \cdot \vec{B} = 0 \tag{37}$$

According to (21) and (30) - (33), Maxwell's differential equations in the time domain, in stationary conducting LIH media containing sources, are given by the following expressions:

$$\nabla \times \vec{H} = \vec{J}_s + \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{38}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \tag{39}$$

$$\left(\sigma + \varepsilon \frac{\partial}{\partial t} \right) \left(\nabla \cdot \vec{E} \right) = g_{st} \tag{40}$$

$$\nabla \cdot \vec{H} = 0 \tag{41}$$

whereas in stationary good LIH conductors containing sources, the magnetodynamic approximation of the dynamic electromagnetic field is described by the following Maxwell's differential equations:

$$\nabla \times \vec{H} = \vec{J}_s + \sigma \vec{E} \tag{42}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{43}$$

$$\nabla \cdot \vec{E} = \frac{g_{st}}{\sigma} \tag{44}$$

$$\nabla \cdot \vec{H} = 0 \tag{45}$$

Maxwell's differential equations in the phasor domain, for slowly moving conducting media containing sources, are described by the following expressions:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + \underline{\vec{J}_c} - \nabla \times \left(\vec{v} \times \underline{\vec{D}} \right) + \left(\nabla \cdot \underline{\vec{D}} \right) \vec{v} + j\,\omega\, \underline{\vec{D}} \tag{46}$$

$$\nabla \times \underline{\vec{E}} = \nabla \times \left(\vec{v} \times \underline{\vec{B}} \right) - j\,\omega\, \underline{\vec{B}} \tag{47}$$

$$\nabla \cdot \left(\underline{\vec{J}_c} + \left(\nabla \cdot \underline{\vec{D}} \right) \vec{v} + j\,\omega\, \underline{\vec{D}} \right) = \underline{g}_{st} \tag{48}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{49}$$

which, in stationary conducting media containing sources, take on the following form:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + \underline{\vec{J}_c} + j\,\omega\,\underline{\vec{D}} \quad \textbf{(50)}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\underline{\vec{B}} \quad \textbf{(51)}$$

$$\nabla \cdot \left(\underline{\vec{J}_c} + j\,\omega\,\underline{\vec{D}}\right) = \underline{g}_{st} \quad \textbf{(52)}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \quad \textbf{(53)}$$

whereas in stationary good conductors containing sources, the magnetodynamic approximation of the dynamic electromagnetic field in the phasor domain is described by the following Maxwell's differential equations:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + \underline{\vec{J}_c} \quad \textbf{(54)}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\underline{\vec{B}} \quad \textbf{(55)}$$

$$\nabla \cdot \underline{\vec{J}_c} = \underline{g}_{st} \quad \textbf{(56)}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \quad \textbf{(57)}$$

where $\underline{\vec{E}}$, $\underline{\vec{H}}$, $\underline{\vec{D}}$, $\underline{\vec{B}}$, $\underline{\vec{J}_c}$ and $\underline{\vec{J}_s}$ are the phasors of the corresponding vectors; $\underline{g}_{st}$ is the phasor of the volume density of the source leakage electric current, j is the imaginary unit, and ω is the angular frequency.

According to (21), the following constitutive relations apply in the phasor domain for linear, isotropic and homogeneous (LIH) media:

$$\underline{\vec{D}} = \varepsilon\,\underline{\vec{E}} \quad ; \quad \underline{\vec{J}_c} = \sigma\,\underline{\vec{E}} \quad ; \quad \underline{\vec{B}} = \mu\,\underline{\vec{H}} \quad \textbf{(58)}$$

According to (50) - (53) and (58), Maxwell's differential equations in the phasor domain for stationary conducting LIH media containing sources are given by the following expressions:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + (\sigma + j\,\omega\,\varepsilon)\,\underline{\vec{E}} \quad \textbf{(59)}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\mu\,\underline{\vec{H}} \quad \textbf{(60)}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\underline{g}_{st}}{\sigma + j\,\omega\,\varepsilon} \quad \textbf{(61)}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \quad \textbf{(62)}$$

whereas in stationary good LIH conductors containing sources, the magnetodynamic approximation of the dynamic electromagnetic field in the phasor domain is described by the following Maxwell's differential equations:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + \sigma\,\underline{\vec{E}} \quad \textbf{(63)}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\mu\,\underline{\vec{H}} \quad \textbf{(64)}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\underline{g}_{st}}{\sigma} \quad \textbf{(65)}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \quad \textbf{(66)}$$

2.3 MAXWELL'S DIFFERENTIAL EQUATIONS IN PERFECT DIELECTRICS

According to (20) and (26) - (29), for $\vec{J}_c = 0$, Maxwell's differential equations in the time domain for slowly moving perfect dielectrics containing sources are given by the following expressions:

$$\nabla \times \vec{H} = \vec{J}_s - \nabla \times \left(\vec{v} \times \vec{D}\right) + \rho_s\,\vec{v} + \frac{\partial \vec{D}}{\partial\, t} \quad \textbf{(67)}$$

$$\nabla \times \vec{E} = \nabla \times \left(\vec{v} \times \vec{B}\right) - \frac{\partial\, \vec{B}}{\partial\, t} \quad \textbf{(68)}$$

$$\nabla \cdot \left(\frac{d\,\vec{D}}{d\,t}\right) = g_{st} \quad \Rightarrow \quad \nabla \cdot \vec{D} = \rho_s \tag{69}$$

$$\nabla \cdot \vec{B} = 0 \tag{70}$$

which, in stationary perfect dielectrics containing sources, take on the following form:

$$\nabla \times \vec{H} = \vec{J}_s + \frac{\partial \vec{D}}{\partial\, t} \tag{71}$$

$$\nabla \times \vec{E} = -\frac{\partial\,\vec{B}}{\partial\, t} \tag{72}$$

$$\nabla \cdot \vec{D} = \rho_s \tag{73}$$

$$\nabla \cdot \vec{B} = 0 \tag{74}$$

whereas in stationary perfect LIH dielectrics containing sources, Maxwell's differential equations in the time domain are described by the following expressions:

$$\nabla \times \vec{H} = \vec{J}_s + \varepsilon\,\frac{\partial \vec{E}}{\partial\, t} \tag{75}$$

$$\nabla \times \vec{E} = -\mu\,\frac{\partial\,\vec{H}}{\partial\, t} \tag{76}$$

$$\nabla \cdot \vec{E} = \frac{\rho_s}{\varepsilon} \tag{77}$$

$$\nabla \cdot \vec{H} = 0 \tag{78}$$

If displacement electric currents are neglected in stationary perfect LIH dielectrics containing sources, then Maxwell's differential equations in the time domain are given by the following expressions:

$$\nabla \times \vec{H} = \vec{J}_s \tag{79}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial\,\vec{H}}{\partial\, t} \tag{80}$$

$$\nabla \cdot \vec{E} = \frac{\rho_s}{\varepsilon} \tag{81}$$

$$\nabla \cdot \vec{H} = 0 \tag{82}$$

and these equations represent the magnetoquasistatic approximation of the dynamic electromagnetic field.

According to (58) and (67) - (70), Maxwell's differential equations in the phasor domain for slowly moving perfect dielectrics containing sources are described by the following expressions:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} - \nabla \times \left(\vec{v} \times \underline{\vec{D}}\right) + \bar{\rho}_s\,\vec{v} + j\,\omega\,\underline{\vec{D}} \tag{83}$$

$$\nabla \times \underline{\vec{E}} = \nabla \times \left(\vec{v} \times \underline{\vec{B}}\right) - j\,\omega\,\underline{\vec{B}} \tag{84}$$

$$\nabla \cdot \underline{\vec{D}} = \bar{\rho}_s \tag{85}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{86}$$

which in stationary perfect dielectrics take on a new form:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + j\,\omega\,\underline{\vec{D}} \tag{87}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\underline{\vec{B}} \tag{88}$$

$$\nabla \cdot \underline{\vec{D}} = \bar{\rho}_s \tag{89}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{90}$$

whereas in stationary perfect LIH dielectrics containing sources, Maxwell's differential equations in the phasor domain are described by the following expressions:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} + j\,\omega\,\varepsilon\,\underline{\vec{E}} \tag{91}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\mu\,\underline{\vec{H}} \tag{92}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\bar{\rho}_s}{\varepsilon} \tag{93}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{94}$$

where $\bar{\rho}_s$ is the phasor of the volume density of the electric charge.

According to (58) and (79) - (82), Maxwell's differential equations for the magnetoquasistatic approximation of the dynamic electromagnetic field in the phasor domain for stationary perfect LIH dielectrics containing sources are given by the following expressions:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}_s} \tag{95}$$

$$\nabla \times \underline{\vec{E}} = -j\,\omega\,\mu\,\underline{\vec{H}} \tag{96}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\bar{\rho}_s}{\varepsilon} \tag{97}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{98}$$

3. MAXWELL'S INEGRAL EQUATIONS

By applying the Ostrogradsky-Green-Gauss integral theorem (also known as the divergence theorem) and Stokes' integral theorem, Maxwell's differential equations can be transformed into their integral form. The divergence theorem can be expressed as follows (see Figure 1):

$$\oint_{\partial V} \vec{a} \cdot d\,\vec{S} = \int_V (\nabla \cdot \vec{a})\, dV = \int_V \operatorname{div} \vec{a}\, dV \tag{99}$$

whereas Stokes' integral theorem can be written as follows (see Figure 2):

$$\oint_C \vec{a} \cdot d\vec{\ell} = \int_S (\nabla \times \vec{a}) \cdot d\vec{S} = \int_S \operatorname{curl} \vec{a} \cdot d\vec{S} \tag{100}$$

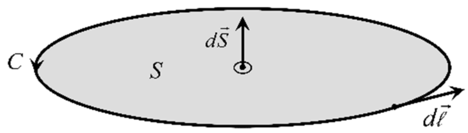

Fig. 2 *A surface S bounded by a closed curve C*

3.1 MAXWELL'S INTEGRAL EQUATIONS IN CONDUCTING MEDIA

According to (26) – (37), (99) and (100), Maxwell's integral equations in the time domain for slowly moving conducting media containing sources are described by the following expressions:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \vec{J}_c + \vec{v}\left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \vec{D}\right) \cdot d\vec{\ell} \tag{101}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \vec{B}\right) \cdot d\vec{\ell} \tag{102}$$

$$\oint_{\partial V} \left(\vec{J}_c + \left(\nabla \cdot \vec{D}\right) \vec{v} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S} = \int_V g_{st}\, dV \tag{103}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{104}$$

which, in stationary conducting media containing sources, take the following form:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \vec{J}_c + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S} = i_s + i_c + i_{disp} \tag{105}$$

$$e \;=\; \oint_C \vec{E}\cdot d\vec{\ell} \;=\; -\int_S \frac{\partial \vec{B}}{\partial\, t}\cdot d\vec{S} \qquad \textbf{(106)}$$

$$\oint_{\partial V}\left(\vec{J}_c \;+\; \frac{\partial\,\vec{D}}{\partial\, t}\right)\cdot d\vec{S} \;=\; \int_V g_{st}\, dV \;=\; i_{st} \qquad \textbf{(107)}$$

$$\oint_{\partial V} \vec{B}\cdot d\vec{S} \;=\; 0 \qquad \textbf{(108)}$$

whereas, in stationary good conductors containing sources, the magnetodynamic approximation of the dynamic electromagnetic field is described by the following Maxwell's integral equations:

$$\oint_C \vec{H}\cdot d\vec{\ell} \;=\; \int_S (\vec{J}_s \;+\; \vec{J}_c\,)\cdot d\vec{S} = i_s \;+\; i_c \qquad \textbf{(109)}$$

$$e \;=\; \oint_C \vec{E}\cdot d\vec{\ell} \;=\; -\int_S \frac{\partial \vec{B}}{\partial\, t}\cdot d\vec{S} \qquad \textbf{(110)}$$

$$\oint_{\partial V} \vec{J}_c\cdot d\vec{S} \;=\; \int_V g_{st}\, dV \;=\; i_{st} \qquad \textbf{(111)}$$

$$\oint_{\partial V} \vec{B}\cdot d\vec{S} \;=\; 0 \qquad \textbf{(112)}$$

where e is the induced electromotive force, i_{st} is the source leakage electric current flowing into the surrounding conducting medium within the considered volume V (Figure 3), i_s is the source electric current flowing through the surface S (Figure 4), i_c is the conduction electric current flowing through the surface S (Figure 4), and i_{disp} is the displacement electric current flowing through the surface S (Figure 4).

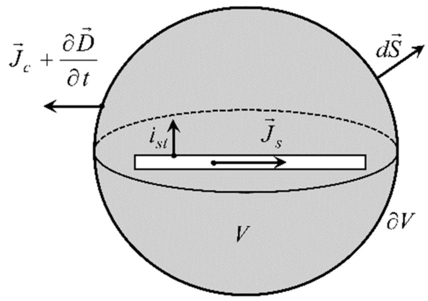

Fig. 3 *Graphical illustration of the expression (107)*

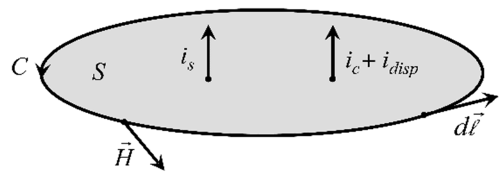

Fig. 4 *Graphical illustration of the expression (105)*

According to (105) and (107), the following expressions hold:

$$i_s \;=\; \int_S \vec{J}_s\cdot d\vec{S} \qquad \textbf{(113)}$$

$$i_c \;=\; \int_S \vec{J}_c\cdot d\vec{S} \qquad \textbf{(114)}$$

$$i_{disp} \;=\; \int_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{S} \qquad \textbf{(115)}$$

$$i_{st} \;=\; \int_V g_{st}\, dV \qquad \textbf{(116)}$$

According to (101) - (104), in the phasor domain for slowly moving conducting media containing sources, Maxwell's differential equations are given by the following expressions:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_s + \underline{\vec{J}}_c + \vec{v}\,(\nabla \cdot \underline{\vec{D}}) + j\,\omega\,\underline{\vec{D}}\right) \cdot d\vec{S} - \oint_C (\vec{v} \times \underline{\vec{D}}) \cdot d\vec{\ell} \quad \textbf{(117)}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -\,j\,\omega \int_S \underline{\vec{B}} \cdot d\vec{S} + \oint_C (\vec{v} \times \underline{\vec{B}}) \cdot d\vec{\ell} \quad \textbf{(118)}$$

$$\oint_{\partial V} \left(\underline{\vec{J}}_c + (\nabla \cdot \underline{\vec{D}})\,\vec{v} + j\,\omega\,\underline{\vec{D}}\right) \cdot d\vec{S} = \int_V \bar{g}_{st}\, dV \quad \textbf{(119)}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \quad \textbf{(120)}$$

which, in stationary conducting media containing sources, take the following form:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_s + \underline{\vec{J}}_c + j\,\omega\,\underline{\vec{D}}\right) \cdot d\vec{S} = \bar{I}_s + \bar{I}_c + \bar{I}_{disp} \quad \textbf{(121)}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -\,j\,\omega \int_S \underline{\vec{B}} \cdot d\vec{S} \quad \textbf{(122)}$$

$$\oint_{\partial V} \left(\underline{\vec{J}}_c + j\,\omega\,\underline{\vec{D}}\right) \cdot d\vec{S} = \int_V \bar{g}_{st}\, dV = \bar{I}_{st} \quad \textbf{(123)}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \quad \textbf{(124)}$$

whereas in stationary good conductors containing sources, the magnetodynamic approximation of the dynamic electromagnetic field is described by the following Maxwell's integral equations:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_s + \underline{\vec{J}}_c\right) \cdot d\vec{S} = \bar{I}_s + \bar{I}_c \quad \textbf{(125)}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -\,j\,\omega \int_S \underline{\vec{B}} \cdot d\vec{S} \quad \textbf{(126)}$$

$$\oint_{\partial V} \underline{\vec{J}}_c \cdot d\vec{S} = \int_V \bar{g}_{st}\, dV = \bar{I}_{st} \quad \textbf{(127)}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \quad \textbf{(128)}$$

where $\bar{I}_{st}$ is the phasor of the source leakage electric current i_{st} flowing into the surrounding conducting medium within the considered volume V (Figure 3), $\bar{I}_s$ is the phasor of the source electric current i_s flowing through the surface S (Figure 4), $\bar{I}_c$ is the phasor of the conduction electric current i_c flowing through the surface S (Figure 4), whereas $\bar{I}_{disp}$ is the phasor of the displacement electric current i_{disp} flowing through the surface S (Figure 4).

3.2 MAXWELL'S INTEGRAL EQUATIONS IN PERFECT DIELECTRICS

According to (67) - (74), (99) and (100), for $\vec{J}_c = 0$, Maxwell's integral equations in the time domain for slowly moving perfect dielectrics containing sources are given by the following expressions:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \rho_s\,\vec{v} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S} - \oint_C (\vec{v} \times \vec{D}) \cdot d\vec{\ell} \quad \textbf{(129)}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{\ell} \quad \textbf{(130)}$$

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \int_V \rho_s\, dV = q_s \quad \textbf{(131)}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \quad \textbf{(132)}$$

which, in stationary perfect dielectrics containing sources, take the following form:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{S} = i_s + i_{disp} \quad \textbf{(133)}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \quad \textbf{(134)}$$

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \int_V \rho_s\, dV = q_s \quad \textbf{(135)}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \quad \textbf{(136)}$$

where q_s is the electric charge inside volume V.

According to (129) - (132), Maxwell's integral equations in the phasor domain for slowly moving perfect dielectrics containing sources are described by the following expressions:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} \;=\; \int_S \left(\underline{\vec{J}_s} \;+\; \bar{\rho}_s\,\vec{v} \;+\; j\,\omega\,\underline{\vec{D}} \right) \cdot d\vec{S} \;-\; \oint_C \left(\vec{v} \times \underline{\vec{D}} \right) \cdot d\vec{\ell} \tag{137}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} \;=\; -\,j\,\omega\,\int_S \underline{\vec{B}} \cdot d\vec{S} \;+\; \oint_C \left(\vec{v} \times \underline{\vec{B}} \right) \cdot d\vec{\ell} \tag{138}$$

$$\oint_{\partial V} \underline{\vec{D}} \cdot d\vec{S} \;=\; \int_V \bar{\rho}_s\, dV = \bar{Q}_s \tag{139}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} \;=\; 0 \tag{140}$$

which, in stationary perfect dielectrics containing sources, take the following form:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} \;=\; \int_S \left(\underline{\vec{J}_s} \;+\; j\,\omega\,\underline{\vec{D}} \right) \cdot d\vec{S} \;=\bar{I}_s \;+\; \bar{I}_{disp} \tag{141}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} \;=\; -\,j\,\omega\,\int_S \underline{\vec{B}} \cdot d\vec{S} \tag{142}$$

$$\oint_{\partial V} \underline{\vec{D}} \cdot d\vec{S} \;=\; \int_V \bar{\rho}_s\, dV = \bar{Q}_s \tag{143}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} \;=\; 0 \tag{144}$$

where $\overline{Q}_s$ is the phasor of the electric charge q_s inside volume V.

4. WAVE EQUATIONS IN LIH MEDIA

4.1 WAVE EQUATIONS FOR ELECTROMAGNETIC FIELDS IN SOURCE-FREE LIH MEDIA

In source-free, stationary, conducting linear, isotropic and homogeneous (LIH) media, the well-known homogeneous damped wave equations for the electric and magnetic fields in the time domain can be written as [3], [25]:

$$\Delta\,\vec{E} \;-\; \mu\,\varepsilon\,\frac{\partial^2 \vec{E}}{\partial\, t^2} \;-\; \mu\,\sigma\,\frac{\partial \vec{E}}{\partial\, t} \;=\; 0 \tag{145}$$

$$\Delta\,\vec{H} \;-\; \mu\,\varepsilon\,\frac{\partial^2 \vec{H}}{\partial\, t^2} \;-\; \mu\,\sigma\,\frac{\partial \vec{H}}{\partial\, t} \;=\; 0 \tag{146}$$

which, in source-free, stationary, perfect LIH dielectrics, become undamped homogeneous wave equations:

$$\Delta\,\vec{E} \;-\; \mu\,\varepsilon\,\frac{\partial^2 \vec{E}}{\partial\, t^2} \;=\; 0 \tag{147}$$

$$\Delta\,\vec{H} \;-\; \mu\,\varepsilon\,\frac{\partial^2 \vec{H}}{\partial\, t^2} \;=\; 0 \tag{148}$$

whereas in source-free, stationary, good LIH conductors, homogeneous wave equations become homogeneous diffusion equations:

$$\Delta\,\vec{E} \;-\; \mu\,\sigma\,\frac{\partial \vec{E}}{\partial\, t} \;=\; 0 \tag{149}$$

$$\Delta\,\vec{H} \;-\; \mu\,\sigma\,\frac{\partial \vec{H}}{\partial\, t} \;=\; 0 \tag{150}$$

where Δ is the Laplace operator.

4.2 WAVE EQUATIONS FOR THE LORENZ-GAUGE POTENTIALS IN LIH MEDIA

The mathematical treatment of the electromagnetic field in LIH media can be simplified by using electromagnetic potential functions instead of the fields themselves. In the time domain, for stationary LIH media, these potentials are described by:

$$\vec{B} = \nabla \times \vec{A} \quad ; \quad \vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \tag{151}$$

which, in the phasor domain, can be written as:

$$\underline{\vec{B}} = \nabla \times \underline{\vec{A}} \quad ; \quad \underline{\vec{E}} = -\nabla \bar{\varphi} - j\,\omega\,\underline{\vec{A}} \tag{152}$$

where φ is the electric scalar potential, $\bar{\varphi}$ is the phasor of the electric scalar potential, $\vec{A}$ is the magnetic vector potential and $\underline{\vec{A}}$ is the phasor of the magnetic vector potential. The most used gauge in wave propagation problems is the Lorenz gauge, which, in stationary conducting LIH media, can be written in the time domain as:

$$\nabla \cdot \vec{A} + \mu\,\varepsilon\,\frac{\partial \varphi}{\partial t} + \mu\,\sigma\,\varphi = 0 \tag{153}$$

and in the phasor domain as:

$$\nabla \cdot \underline{\vec{A}} + \mu\,\bar{\kappa}\,\bar{\varphi} = 0 \tag{154}$$

which, in stationary perfect LIH dielectrics, take on a new form:

$$\nabla \cdot \vec{A} + \mu\,\varepsilon\,\frac{\partial \varphi}{\partial t} = 0 \tag{155}$$

$$\nabla \cdot \underline{\vec{A}} + j\,\omega\,\mu\,\varepsilon\,\bar{\varphi} = 0 \tag{156}$$

where:

$$\bar{\kappa} = \sigma + j\,\omega\,\varepsilon \tag{157}$$

is the complex conductivity of the conducting LIH medium.

From (38) - (41) and (151), the following damped wave equations for the Lorenz-gauge potentials in the stationary conducting LIH media containing sources in the time domain can be obtained:

$$\Delta \vec{A} - \mu\,\varepsilon\,\frac{\partial^2 \vec{A}}{\partial t^2} - \mu\,\sigma\,\frac{\partial \vec{A}}{\partial t} = -\mu\,\vec{J}_s \tag{158}$$

$$\left(\sigma + \varepsilon\,\frac{\partial}{\partial t}\right)\left(\Delta \varphi - \mu\,\varepsilon\,\frac{\partial^2 \varphi}{\partial t^2} - \mu\,\sigma\,\frac{\partial \varphi}{\partial t}\right) = -g_{st} \tag{159}$$

which in stationary good LIH conductors containing sources, where displacement electric currents can be neglected, take on a new form:

$$\Delta \vec{A} - \mu\,\sigma\,\frac{\partial \vec{A}}{\partial t} = -\mu\,\vec{J}_s \tag{160}$$

$$\Delta \varphi - \mu\,\sigma\,\frac{\partial \varphi}{\partial t} = -\frac{g_{st}}{\sigma} \tag{161}$$

and such equations are known as inhomogeneous diffusion equations.

From (75) - (78) and (151), the following wave equations for the Lorenz-gauge potentials in stationary perfect LIH dielectrics in the time domain can be obtained:

$$\Delta \vec{A} - \mu\,\varepsilon\,\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\,\vec{J}_s \tag{162}$$

$$\Delta \varphi - \mu\,\varepsilon\,\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho_s}{\varepsilon} \tag{163}$$

whose particular solutions are known as the retarded potentials [3], [6].

In many references, incorrect expressions appear instead of Eqs. (159) and (161), as these references are based on incorrect Maxwell's differential equations in conducting LIH media.

4.3 HELMHOLTZ DIFFERENTIAL EQUATIONS

The Helmholtz differential equation represents the spatial behaviour of wave phenomena in the phasor domain and is fundamentally equivalent to the wave equation under time-harmonic conditions.

From wave Eqs. (158) and (159), the following inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials in stationary conducting LIH media containing sources in the phasor domain can be obtained:

$$\Delta \underline{\vec{A}} - \bar{\gamma}^2 \underline{\vec{A}} = -\mu \underline{\vec{J}_s} \tag{164}$$

$$\Delta \bar{\varphi} - \bar{\gamma}^2 \bar{\varphi} = -\frac{\bar{g}_{st}}{\bar{\kappa}} \tag{165}$$

where:

$$\bar{\gamma} = \sqrt{j\,\omega\,\mu\,\bar{\kappa}} \tag{166}$$

is the propagation constant of the LIH medium, which is generally a complex quantity.

Particular solutions of the Helmholtz differential Eqs. (164) and (165) are given by:

$$\underline{\vec{A}} = \frac{\mu}{4\pi} \int_V \frac{\underline{\vec{J}_s}\, e^{-\bar{\gamma}\, r}\, dV}{r} \tag{167}$$

$$\bar{\varphi} = \frac{1}{4\pi\bar{\kappa}} \int_V \frac{\bar{g}_{st}\, e^{-\bar{\gamma}\, r}\, dV}{r} \tag{168}$$

which, for a current-carrying thin-wire conductor in a stationary conducting LIH medium, can be written as (Figure 5):

$$\underline{\vec{J}_s}\, dV \rightarrow \bar{I}_\ell\, d\vec{\ell} \quad ; \quad \underline{\vec{A}} = \frac{\mu}{4\pi} \int_C \frac{\bar{I}_\ell\, e^{-\bar{\gamma}\, r}\, d\vec{\ell}}{r} \tag{169}$$

$$\bar{g}_{st}\, dV \rightarrow \bar{\tau}_s\, d\ell \quad ; \quad \bar{\varphi} = \frac{1}{4\pi\bar{\kappa}} \int_C \frac{\bar{\tau}_s\, e^{-\bar{\gamma}\, r}\, d\ell}{r} \tag{170}$$

where r is the distance between the field point and the source point, $\bar{I}_\ell$ is the phasor of the longitudinal electric current flowing along the axis of the thin-wire conductor, whereas:

$$\bar{\tau}_s = -\frac{\partial \bar{I}_\ell}{\partial l} \tag{171}$$

is the phasor representing the linear density of the leakage (transverse) electric current (A/m) that escapes from the axis of the thin-wire conductor into the surrounding conducting medium.

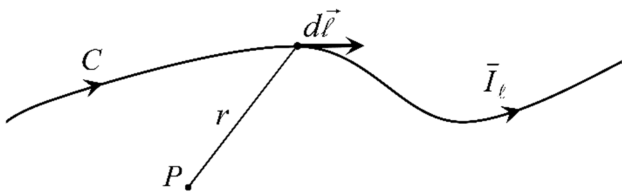

Fig. 5 *Current-carrying thin-wire conductor*

From the diffusion Eqs. (160) and (161), the following inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials in stationary good LIH conductors containing sources in the phasor domain can be obtained:

$$\Delta \underline{\vec{A}} - \bar{\gamma}^2 \underline{\vec{A}} = -\mu \underline{\vec{J}}_s \tag{172}$$

$$\Delta \bar{\varphi} - \bar{\gamma}^2 \bar{\varphi} = -\frac{\bar{g}_{st}}{\sigma} \tag{173}$$

where:

$$\bar{\gamma} = \sqrt{j\,\omega\,\mu\,\sigma} \tag{174}$$

Particular solutions of the Helmholtz differential Eqs. (172) and (173) are given by:

$$\underline{\vec{A}} = \frac{\mu}{4\,\pi} \int_V \frac{\underline{\vec{J}}_s\, e^{-\bar{\gamma}\,r}\, dV}{r} \tag{175}$$

$$\bar{\varphi} = \frac{1}{4\,\pi\,\sigma} \int_V \frac{\bar{g}_{st}\, e^{-\bar{\gamma}\,r}\, dV}{r} \tag{176}$$

which, for a current-carrying thin-wire conductor in stationary good LIH conductors, can be written as (Figure 5):

$$\underline{\vec{J}}_s\, dV \to \bar{I}_\ell\, d\vec{\ell} \quad ; \quad \underline{\vec{A}} = \frac{\mu}{4\,\pi} \int_C \frac{\bar{I}_\ell\, e^{-\bar{\gamma}\,r}\, d\vec{\ell}}{r} \tag{177}$$

$$\bar{g}_{st}\, dV \to \bar{\tau}_s\, d\ell \quad ; \quad \bar{\varphi} = \frac{1}{4\,\pi\,\sigma} \int_C \frac{\bar{\tau}_s\, e^{-\bar{\gamma}\,r}\, d\ell}{r} \tag{178}$$

From wave Eqs. (162) and (163), the following inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials in stationary perfect LIH dielectrics containing sources can be obtained in the phasor domain:

$$\Delta \underline{\vec{A}} + k^2 \underline{\vec{A}} = -\mu \underline{\vec{J}}_s \tag{179}$$

$$\Delta \bar{\varphi} + k^2 \bar{\varphi} = -\frac{\bar{\rho}_s}{\varepsilon} \tag{180}$$

where:

$$k = \omega \sqrt{\mu\,\varepsilon} \tag{181}$$

denotes the real wave number of a perfect LIH dielectric.

Particular solutions of the Helmholtz differential Eqs. (179) and (180) are given by:

$$\underline{\vec{A}} = \frac{\mu}{4\,\pi} \int_V \frac{\underline{\vec{J}}_s\, e^{-j\,k\,r}\, dV}{r} \tag{182}$$

$$\bar{\varphi} = \frac{1}{4\,\pi\,\varepsilon} \int_V \frac{\bar{\rho}_s\, e^{-j\cdot k\cdot r}\, dV}{r} \tag{183}$$

which, for a current-carrying thin-wire conductor in a stationary perfect LIH dielectric, can be written as (Figure 5):

$$\underline{\vec{J}}_s\, dV \to \bar{I}_\ell\, d\vec{\ell} \quad ; \quad \underline{\vec{A}} = \frac{\mu}{4\,\pi} \int_C \frac{\bar{I}_\ell\, e^{-j\,k\,r}\, d\vec{\ell}}{r} \tag{184}$$

$$\bar{\rho}_s\, dV \to \bar{\lambda}_s\, d\ell \quad ; \quad \bar{\varphi} = \frac{1}{4\,\pi\,\varepsilon} \int_C \frac{\bar{\lambda}_s\, e^{-\bar{\gamma}\,r}\, d\ell}{r} \tag{185}$$

where $\bar{\lambda}_s$ is the phasor of the linear charge density (C/m).

4.4 SPECIAL CASE 1: STATIONARY CURRENT FIELD

According to (43) and (44), Maxwell's differential equations for a stationary current field in a conducting LIH medium are given by:

$$\nabla \times \vec{E} = 0 \quad ; \quad \nabla \cdot \vec{E} = \frac{g_{st}}{\sigma} \tag{186}$$

According to (161), the electric scalar potential of a stationary current field in conducting LIH media is described by Poisson's differential equation:

$$\Delta\,\varphi \;=\; -\,\frac{g_{st}}{\sigma} \tag{187}$$

whose particular solution is:

$$\varphi \;=\; \frac{1}{4\,\pi\,\sigma}\int_V \frac{g_{st}\,dV}{r} \tag{188}$$

If the source leakage electric current flows from the surface S, then Eq. (188) takes on a new form [26], [27]:

$$g_{st}\,dV \rightarrow J_n\,dS \quad ; \quad \varphi \;=\; \frac{1}{4\,\pi\,\sigma}\int_V \frac{J_n\,dS}{r} \tag{189}$$

where J_n is the normal component of the surface density of the source leakage electric current flowing from the surface S.

If the source leakage electric current flows from the axis C of the thin-wire conductor, then Eq. (188) takes on a new form [27], [28]:

$$g_{st}\,dV \rightarrow \tau_s\,d\ell \quad ; \quad \varphi \;=\; \frac{1}{4\,\pi\,\sigma}\int_V \frac{\tau_s\,d\ell}{r} \tag{190}$$

where τ_s is the linear density of the source leakage electric current flowing from the axis C of the thin-wire conductor.

If the source of the leakage electric current is a point current source, then Eq. (188) takes on a new form [27], [29]:

$$g_{st} \;\rightarrow\; I\,\delta(r) \quad ; \quad \varphi \;=\; \frac{I}{4\,\pi\,\sigma\,r} \tag{191}$$

where I is the source leakage electric current, and $\delta\,(r)$ is the Dirac delta function.

4.5 SPECIAL CASE 2: ELECTROSTATIC FIELD

According to (76) and (77), Maxwell's differential equations for the electrostatic field in stationary perfect LIH dielectrics are given by:

$$\nabla \times \vec{E} \;=\; 0 \quad ; \quad \nabla \cdot \vec{E} \;=\; \frac{\rho_s}{\varepsilon} \tag{192}$$

According to (163), the electric scalar potential of the electrostatic field in stationary perfect LIH dielectrics is described by Poisson's differential equation:

$$\Delta\,\varphi \;=\; -\,\frac{\rho_s}{\varepsilon} \tag{193}$$

whose particular solution is:

$$\varphi \;=\; \frac{1}{4\,\pi\,\varepsilon}\int_V \frac{\rho_s\,dV}{r} \tag{194}$$

If electric charge is present on the surface S, then (194) takes on a new form:

$$\rho_s\,dV \rightarrow \sigma_s\,dS \quad ; \quad \varphi \;=\; \frac{1}{4\,\pi\,\varepsilon}\int_V \frac{\sigma_s\,dS}{r} \tag{195}$$

where σ_s represents the surface density of the electric charge.

If electric charge is present on the axis C of the thin-wire conductor, then (194) takes on a new form:

$$\rho_s\,dV \rightarrow \lambda_s\,d\ell \quad ; \quad \varphi \;=\; \frac{1}{4\,\pi\,\varepsilon}\int_V \frac{\lambda_s\,d\ell}{r} \tag{196}$$

where λ_s is the linear density of the electric charge present on the axis C of the thin-wire conductor.

If electric charge is present at a point, then (194) takes on a new form:

$$\rho_s \rightarrow Q\,\delta(r) \quad ; \quad \varphi = \frac{Q}{4\,\pi\,\varepsilon\,r} \tag{197}$$

where Q is the point electric charge.

5. SOURCE LEAKAGE ELECTRIC CURRENT IN THE EQUATIONS OF A TWO-WIRE TRANSMISSION LINE

Let R, L, C and G be the resistance, inductance, capacitance and conductance per unit length of the two-wire transmission line. These are also known as the primary parameters of the transmission line. Both the electric voltage u between the conductors and the electric current i in the conductors depend on the position along the line, i.e., on the coordinate z (Figure 6).

Fig. 6 *Two-wire transmission line*

The well-known first-order partial differential equations of the two-wire transmission line, which do not account for mutual electromagnetic coupling between infinitesimal segments of the line, are given by the following expressions:

$$-\frac{\partial u}{\partial z} = R \cdot i + L \cdot \frac{\partial i}{\partial t} \tag{198}$$

$$-\frac{\partial i}{\partial z} = G \cdot u + C \cdot \frac{\partial u}{\partial t} \tag{199}$$

where:

$$-\frac{\partial i}{\partial z} = \tau_s \tag{200}$$

representing the linear density of the source leakage (transverse) electric current that flows between the conductors of a two-wire transmission line. It is a special case – the one-dimensional (1D) case – of the volume density of the source leakage electric current, which corresponds to the general three-dimensional (3D) case.

6. CONCLUSION

In this paper, generalised Maxwell's equations, in their physically consistent form, are formulated for a slowly moving conducting medium containing sources. The derivation of these equations is based on the postulation that the system of Maxwell's equations must be self-consistent. This same criterion was used by James Clerk Maxwell in deriving his equations for a perfect dielectric.

The generalised Maxwell's equations provide a unified mathematical framework that describes both time-varying and static electric and magnetic fields, encompassing electrostatics, magnetostatics, stationary current fields and dynamic wave phenomena. Through generalisation, Maxwell's equations incorporate material properties such as conductivity, permittivity and permeability, enabling accurate modelling of electromagnetic fields in conducting media. The generalised form also supports the treatment of complex source distributions – volume, surface, or line leakage electric currents and electric charges – making the equations applicable to a broad range of physical and engineering problems.

The volume density of the source leakage electric current in conducting media is introduced here as a new physical quantity, analogous to the volume charge density in perfect dielectrics. Maxwell's and wave equations in a source-free medium emerge as a special case of those in media containing sources. The substitution of leakage electric current sources for electric charge as the origin of the static component of the electric field in conducting media is an innovative concept.

This work opens new avenues for both theoretical investigation and practical applications in electromagnetics involving conducting materials. The most significant application of the proposed system of Maxwell's and wave equations is the numerical analysis of grounding grid systems, as well as in geophysical electromagnetics.

7. REFERENCES

[1] Stratton, J.A., *Electromagnetic Theory*, New York: McGraw-Hill Book Company, 1941.

[2] Jackson, J.D., *Classical Electrodynamics*, New York: John Wiley & Sons, 1962.

[3] Haznadar, Z., Štih, Ž., *Electromagnetic Fields*, Waves and Numerical Methods, Amsterdam: IOS Press, 2000.

[4] Poljak, D., *Advanced Modeling in Computational Electromagnetic Compatibility*, New York: John Wiley & Sons, 2007. https://doi.org/10.1002/0470116889

[5] Popović, Z., Popović, B.D., *Introductory Electromagnetics*, New Jersey: Prentice Hall, 1999.

[6] Barkeshli, K., *Advanced Electromagnetics and Scattering Theory*, New York: Springer, 2015. https://doi.org/10.1007/978-3-319-11547-4

[7] Balanis, C.A., *Advanced Engineering Electromagnetics*, 2nd ed., New Jersey: John Wiley & Sons, 2012.

[8] Ida, N., Bastos, J.P.A., *Electromagnetics and Calculation of Fields*, New York: Springer-Verlag, 1992. https://doi.org/10.1007/978-1-4684-0526-2

[9] Sunde, E.D., *Earth Conduction Effects in Transmission Systems*, New York: D. Van Nostrand Company, 1949.

[10] Lorrain, P., Corson, D.L., *Electromagnetic Fields and Waves – Including Electric Circuits*, 3rd ed., New York: W. H. Freeman and Company, 1988.

[11] Panofsky, W.K.H., Phillips, M., *Classical Electricity and Magnetism*, 2nd ed., London: Addison-Wesley Publishing Company, 1962.

[12] Tretyakov, O.A., Erden, F., A Novel Simple Format of Maxwell's Equations in SI Unit, IEEE Access, Vol. 9, pp. 88272–88278, June 2021.

https://doi.org/10.1109/ACCESS.2021.3089673

[13] Huray, P.G., *Maxwell's Equations*, New Jersey: John Wiley & Sons, 2010.

https://doi.org/10.1002/9780470549919

[14] Siegel, D.M., *Innovation in Maxwell's Electromagnetic Theory: Molecular Vortices, Displacement Current, and Light*, Cambridge: Cambridge University Press, 1991.

https://doi.org/10.1017/CBO9780511529290

[15] Griffiths, D.J., *Introduction to Electrodynamics*, 4th ed., Boston: Pearson, 2013,

[16] Zhdanov, M.S., *Geophysical Electromagnetic Theory and Methods*, Amsterdam: Elsevier, 2009.

[17] Zhdanov, M.S., *Inverse Theory and Applications in Geophysics*, 2nd ed., Amsterdam: Elsevier, 2015.

[18] Zhdanov, M.S., *Foundations of Geophysical Electromagnetic Theory and Methods*, 2nd ed., Amsterdam: Elsevier, 2018. https://doi.org/10.1016/B978-0-44-463890-8.00011-6

[19] Patra H.P., Mallick, K., *Geosounding Principles, 2 – Time-Varying Geoelectric Soundings*, Amsterdam: Elsevier, 1980.

[20] Kaufman, A.A., *Geophysical Field Theory and Method, Part B - Electromagnetic Fields I*, San Diego: Academic Press, 1994.

[21] Kaufman, A.A., *Geophysical Field Theory and Method, Part C - Electromagnetic Fields II*, New York: Academic Press, 1994.

[22] Wait, J.R., Geo-Electromagnetism, New York: Academic Press, 1982.

[23] Baños, A., Jr., Wesley, J.P., *The Horizontal Electric Dipole in a Conducting Half-Space*, University of California, Marine Physical Laboratory of the Scripps Institution of Oceanography, 1953.

[24] Vujević, S., A New Look to the Maxwell's Equations and Wave Equations in Conducting Media, 2024 International Conference on Software, Telecommunications and Computer Networks (SoftCOM), Split, Croatia, pp. 1-7, 2024.

https://doi.org/10.23919/SoftCOM62040.2024.10721727

[25] Erden F., Çoşan, A.A., A novel format of the wave equation in SI units, *Journal of Naval Sciences and Engineering*, Vol. 18, No. 2, pp. 231–250, November 2022.

https://doi.org/10.56850/jnse.1181938

[26] Berberovic, S., Haznadar, Z., Stih,Z., Method of moments in analysis of grounding systems, *Engineering Analysis with Boundary Elements*, Vol. 27, No. 4, pp. 351–360, April 2003. https://doi.org/10.1016/S0955-7997(02)00123-6

[27] Ollendorf, F., *Erdströme – Zweite, neubearbeitete Auflage*, Basel: Verlag, 1969.

[28] Vujević, S., Kurtović, M., Numerical analysis of earthing grids buried in horizontal stratified multilayer earth, *International Journal for Numerical Methods in Engineering*, Vol. 41, No. 7, pp. 1297–1319, December 1998.

https://doi.org/10.1002/(SICI)1097-0207(19980415)41:7<1297::AID-NME338>3.0.CO;2-Z

[29] Vujević, S., Numerical analysis of potential distribution for a point current source in horizontally stratified multilayer earth, *International Journal for Engineering Modelling*, Vol. 8, No. 1–2, pp. 21–29, 1995. https://hrcak.srce.hr/clanak/461835