ISSN 1330-3651 (Print), ISSN 1848-6339 (Online)
https://doi.org/10.17559/TV-20240829001952
Original scientific paper

Enhanced Multi-Attribute Group Decision-Making under Uncertainty: A Novel Interval-Valued Fuzzy Linguistic Approach

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Abstract: Handling uncertainty and hesitancy is a fundamental challenge in multi-attribute group decision-making (MAGDM). To address this, we introduce Interval-Valued Probabilistic Uncertain Linguistic Pythagorean Fuzzy Sets (IVPULPFSs), a new decision model that integrates interval-valued probabilistic representations with uncertain linguistic evaluations. This study proposes two novel MAGDM approaches: (1) an aggregation-based ranking method, and (2) a TODIM-based decision model incorporating psychological behaviours of decision-makers. To validate the effectiveness of these methods, we apply them to real-world decision-making problems, including human resource selection and doctoral thesis evaluation. The results demonstrate that IVPULPFS-based methods outperform existing fuzzy decision models by providing greater accuracy, flexibility, and robustness in handling uncertainty. This study offers a scalable decision-support framework for applications in finance, risk assessment, supply chain management, and intelligent transportation systems.

Keywords: interval-valued probabilistic uncertain linguistic Pythagorean fuzzy sets; multi-attribute group decision-making; probabilistic dual hesitant linguistic Pythagorean fuzzy set;

1 INTRODUCTION

Multi-attribute group decision-making (MAGDM) refers to a kind of decision-making problems in which a group of decision makers (DMs) are invited to evaluate the performance of a collection of alternatives under a set of attributes, which is an interesting and significantly important research topic in the fields of decision sciences, operations management, information sciences, etc., that has received much attention [1, 2]. In MAGDM, one of the most important issues is "How can DMs' evaluations be appropriately expressed?" It is acknowledged that fuzziness and uncertainty widely exist in real and when depicting DMs' evaluation information in MAGDM problems, such kind of fuzziness and vagueness should not be ignored. In other word, crisp numbers are powerless to be employed to describe DMs' evaluation opinions in decision-making problems. Hence, new information representation tools that can effectively handle inherent fuzziness are of high necessity.

Fuzzy sets (FS) originated by Zadeh [3] and intuitionistic fuzzy sets (IFSs) proposed by Atanassov [4] are two important and useful theories, which has been proved to be effective to handle fuzzy evaluation information in MAGDM procedure. However, both FSs and IFSs have some drawbacks in some special decision-making situations. For example, an expert is invited to evaluate the performance of a tourism destination under the attribute “attractions of the tourism”, He/she is hesitant between 0.4, 0.5 and 0.7 when providing the membership degree (MD). Obviously, none of FS and IFS can handle this situation. To effectively deal with such situations, Torra [5] proposed the concept of hesitant fuzzy sets (HFSs), which permit multiple MDs and the above tourism destination evaluation information provided by the DM can be denoted by {0.4, 0.5, 0.7}. HFSs are consistent with realistic MAGDM scenarios, therefore, they have been widely applied in solving decision-making problems. Recent years have witnessed the outbreak of HFSs based MAGDM methods, and we can still find novel achievements in recent publications [6, 7].

Above-mentioned publications revealed the good performance and high efficiency of HFSs in depicting fuzzy decision-making information, however, their drawbacks should not be ignored. In HFSs, multiple MDs are permitted, however, their importance is neglected. In other words, HFSs allow more than one MD, but multiple appearance of some MDs is overlooked. To overcome such limitation of HFS, Zhang et al. [8] proposed the concept of probabilistic HFSs (PHFSs), which consider not only the MDs, but also their probabilistic information. Obviously, compared with HFSs, PHFSs can depict DMs' evaluations more comprehensively and accurately. Due to this characteristic, PHFSs have been extensively applied in solving real MAGDM problems [9, 10]. Zhang et al.'s works [8] enlightened related scholars that not only MDs but also their probabilistic values should be taken into consideration. Hence, more and more analogous achievements have been published. As an extension of dual hesitant linguistic Pythagorean fuzzy set (DHLPFSs) [11], the recently proposed probabilistic dual hesitant linguistic Pythagorean fuzzy set (PDHLPFS) [11] is a powerful tool for describing DMs' evaluation values. The characteristic of DHLPFS is that it allows multiple linguistic MDs and NMDs. Furthermore, by considering probabilistic value of each MDs and NMDs, the PDHLPFS was originated. In contemporary MAGDM scenarios, the inherent complexity of human cognition often necessitates more nuanced uncertainty representation tools. While PDHLPFSs [12] have advanced decision modelling by integrating probabilistic and linguistic hesitancy, two critical limitations hinder their real-world applicability:

(1) Rigid Linguistic Constraints: PDHLPFSs employ linguistic term to denote the linguistic MDs and NMDs, mandate precise linguistic terms (e.g., "good") for MDs/NMDs degrees, failing to accommodate prevalent uncertain linguistic evaluations like "between good and very good". In some complicated decision-making problems, rather than linguistic term, DMs prefer to use uncertain linguistic terms or uncertain linguistic variables (ULVs) to express MDs and NMDs. Many scholars have realized this phenomenon and conducted some pioneering works. For instance, Liu and Qin [13] utilized ULVs to

denote the MDs and NMDs in linguistic intuitionistic fuzzy set [14] and proposed interval-valued linguistic fuzzy set as well as its corresponding MAGDM method. Hence, it is of high necessity to consider ULVs in PDHLPFSs to more accurately capture DMs' evaluation information.

(2) Over-Simplified Probabilistic Modelling: The requirement for crisp probability values (e.g., 0.3) in PDHLPFSs contradicts practitioners' tendency to provide interval probabilities (e.g., [0.25, 0.35]) under cognitive uncertainty. In PDHLPFS, probabilistic information of MDs and NMDs is noted by crisp numbers. However, in some real decision-making problems, it is difficult for DMs to use crisp numbers to denote probabilistic values. As a matter of fact, DMs prefer to use interval values to denote probabilistic values. Song et al. [15] extended PDHFSs to interval-valued PDHFSs, where the probabilistic information of hesitant MDs is expressed by crisp numbers. Similarly, Liu and Chen [16] introduced interval-valued probabilistic dual hesitant fuzzy sets by considering interval-valued probabilistic information in probabilistic dual hesitant fuzzy sets. Hence, it is necessary to consider interval-valued probabilistic information in PDHLPFSs.

To bridge these gaps, this study proposes extensions of PDHLPFSs and investigates applications of these extensions in MAGDM. Considering the drawbacks of PDHLPFSs in depicting DMs' evaluation opinions in real MAGDM problems, this paper proposes an extension of PDHLPFSs, called IVPULPFSs, where MDs and NMDs are constructed by two series of ULVs and their corresponding probabilistic information is denoted by interval values. Afterwards, we investigate a MAGDM approaches under IVPULPFSs. When considering the approach, the performance of the traditional TODIM method (an acronym in Portuguese for Iterative Multi-criteria Decision Making) [17] has impressed deeply. TODIM is based on prospect theory [18] which takes DMs' psychological behaviours into account. Hence, decision-making results produced by the TODIM method is more reliable and reasonable. Due to the good performance in MAGDM, the traditional TODIM method has been generalized into different fuzzy decision-making environments. Scholars studied TODIM under IFSs, HFSs, interval-valued HFSs, probabilistic HFSs, hesitant fuzzy linguistic term sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, etc. Therefore, this study further extends TODIM into IVPULPFSs and proposed TODIM based MAGDM approach. Finally, the proposed MAGDM methods are employed to solve real decision-making problems to illustrate their effectiveness.

The main contribution of this paper are three-folds. (1) A novel information representation tool, namely IVPULPFSs are developed. IVPULPFSs are characterized by uncertain linguistic MDs and NMDs and interval probabilistic values. (2) A novel MAGDM method is developed to help DMs select the best alternatives in MAGDM problems under IVPULPFSs. The main steps and algorithms of the two MAGDM methods are clearly illustrated. (3) The new MAGDM methods is applied in a real problem to demonstrate their effectiveness.

The rest of this paper is organized as follows. Section 2 proposes the concept of IVPULPFSs and studies their properties. AOs for IVPULPFSs are also studied in this section. Two novel MAGDM methods under IVPULPFSs are provided in Section 3. An application of our proposed method in a real MAGDM problem is presented in Section 4. Findings and future researches are summarized in Section 5.

2 INTERVAL-VALUED PROBABILISTIC UNCERTAIN LINGUISTIC PYTHAGOREAN FUZZY SET

In this section, we propose the concept of IVPULPFS. Let X be a given fixed set and $\tilde{S}=\{s_\alpha | 0 \le \alpha \le l\}$ be a continuous linguistic term set with odd cardinality. An interval-valued probabilistic uncertain linguistic Pythagorean fuzzy set (IVPULPFS) $\tilde{D}$ definition on X is expressed as:

$$\tilde{D}=\left\{x, h_{\tilde{D}}(x) \middle| \tilde{p}_{\tilde{D}}(x), g_{\tilde{D}}(x) \middle| \tilde{t}_{\tilde{D}}(x) \middle| x \in X\right\} \tag{1}$$

where $h_{\tilde{D}}(x)$ and $g_{\tilde{D}}(x)$ are two sets of ULVs defined on $\tilde{S}$, denoting the possible MDs and NMDs of the element $x \in X$ to the set $\tilde{D}$. Moreover, $\tilde{p}_{\tilde{D}}(x)=\left[\tilde{p}_{\tilde{D}}^{U}(x), \tilde{p}_{\tilde{D}}^{L}(x)\right]$ $\left(\tilde{p}_{\tilde{D}}^{U}(x)=\inf \tilde{p}_{\tilde{D}}(x)\right.$ and $\left.\tilde{p}_{\tilde{D}}^{L}(x)=\sup \tilde{p}_{\tilde{D}}(x)\right)$ and $\tilde{t}_{\tilde{D}}(x)=\left[\tilde{t}_{\tilde{D}}^{U}(x), \tilde{t}_{\tilde{D}}^{L}(x)\right]$ $\left(\tilde{t}_{\tilde{D}}^{U}(x)=\inf \tilde{t}_{\tilde{D}}(x)\right.$ and $\left.\tilde{t}_{\tilde{D}}^{L}(x)=\sup \tilde{t}_{\tilde{D}}(x)\right)$ are two collections of interval values, denoting the interval-valued probabilistic information of $h_{\tilde{D}}(x)$ and $g_{\tilde{D}}(x)$, respectively, such that

$$0 \le \gamma^L, \gamma^U, \eta^L, \eta^U \le l,\ \left(\left(\gamma^U\right)^+\right)^2+\left(\left(\eta^U\right)^+\right)^2 \le l^2,$$
$$\tilde{p}_{\tilde{D}}(x),\ \tilde{t}_{\tilde{D}}(x) \subseteq [0,1],\ \sum_{\left[\tilde{p}_{\tilde{D}}^{U}(x), \tilde{p}_{\tilde{D}}^{L}(x)\right] \in \tilde{p}_{\tilde{D}}(x)} \tilde{p}_{\tilde{D}}^{L}(x)=1 \tag{2}$$
$$\sum_{\left[\tilde{t}_{\tilde{D}}^{U}(x), \tilde{t}_{\tilde{D}}^{L}(x)\right] \in \tilde{t}_{\tilde{D}}(x)}, \tilde{t}_{\tilde{D}}^{L}(x)=1$$

where $\left[\gamma^L, \gamma^U\right] \in h_{\tilde{D}}(x)$, $\left[\eta^L, \eta^U\right] \in g_{\tilde{D}}(x)$, $\left(\gamma^U\right)^+=\cup_{\left[\gamma^L, \gamma^U\right] \in h_{\tilde{D}}(x)} \max\left\{\gamma^U\right\}$, and $\left(\eta^U\right)^+=\cup_{\left[\eta^L, \eta^U\right] \in g_{\tilde{D}}(x)} \max\left\{\eta^U\right\}$. For easy description, we call the ordered pair $\tilde{d}(x)=\left(\tilde{h}(x) \middle| \tilde{p}_{\tilde{h}(x)}, \tilde{g}(x) \middle| \tilde{t}_{\tilde{g}(x)}\right)$ an interval-valued probabilistic uncertain linguistic Pythagorean fuzzy element (IVPULPFE), which can be denoted by $\tilde{d}=\left(\tilde{h} \middle| \tilde{p}_{\tilde{h}}, \tilde{g} \middle| \tilde{t}_{\tilde{g}}\right)$ for simplicity. Moreover, we call $\tilde{h} \middle| \tilde{p}_{\tilde{h}}$ and $\tilde{g} \middle| \tilde{t}_{\tilde{g}}$ the interval-valued probabilistic uncertain linguistic Pythagorean fuzzy membership elements (IVPULPFMEs) and interval-valued probabilistic uncertain linguistic Pythagorean fuzzy non-membership elements (IVPULPFNMEs) respectively. In addition, the basic operations for IVPULPFEs are defined as follows.

Let $\tilde{d}=\left(\tilde{h}\middle|\tilde{p}_{\tilde{h}},\tilde{g}\middle|\tilde{t}_{\tilde{g}}\right)$, $\tilde{d}_1=\left(\tilde{h}_1\middle|p_{\tilde{h}_1},\tilde{g}_1\middle|\tilde{t}_{\tilde{g}_1}\right)$ and $\tilde{d}_2=\left(\tilde{h}_2\middle|p_{\tilde{h}_2},\tilde{g}_2\middle|\tilde{t}_{\tilde{g}_2}\right)$ be any three IVPULPFEs defined on a continuous linguistic term set $\tilde{S}=\{s_\alpha|0\le\alpha\le l\}$, and λ be a positive real number, then we have

(1) $$\tilde{d}_1\oplus\tilde{d}_2=\cup_{[\gamma_1^L,\gamma_1^U]\in\tilde{h}_1,[\gamma_2^L,\gamma_2^U]\in\tilde{h}_2,[\eta_1^L,\eta_1^U]\in\tilde{g}_1,[\eta_2^L,\eta_2^U]\in\tilde{g}_2}$$
$$\left\{\begin{array}{l}\left\{\left[s_{\left((\gamma_1^L)^2+(\gamma_2^L)^2-(\gamma_1^L\gamma_2^L)^2/l^2\right)^{1/2}},s_{\left((\gamma_1^U)^2+(\gamma_2^U)^2-(\gamma_1^U\gamma_2^U)^2/l^2\right)^{1/2}}\right] \middle| \left[\tilde{p}^L_{[\gamma_1^L,\gamma_1^U]}\tilde{p}^L_{[\gamma_2^L,\gamma_2^U]},\tilde{p}^U_{[\gamma_1^L,\gamma_1^U]}\tilde{p}^U_{[\gamma_2^L,\gamma_2^U]}\right]\right\},\\ \left\{\left[s_{(\eta_1^L\eta_2^L/l)},s_{(\eta_1^U\eta_2^U/l)}\right]\middle|\left[\tilde{t}^L_{[\eta_1^L,\eta_1^U]}\tilde{t}^L_{[\eta_2^L,\eta_2^U]},\tilde{t}^U_{[\eta_1^L,\eta_1^U]}\tilde{t}^U_{[\eta_2^L,\eta_2^U]}\right]\right\}\end{array}\right\}$$

(2) $$\tilde{d}_1\otimes\tilde{d}_2=\cup_{[\gamma_1^L,\gamma_1^U]\in\tilde{h}_1,[\gamma_2^L,\gamma_2^U]\in\tilde{h}_2,[\eta_1^L,\eta_1^U]\in\tilde{g}_1,[\eta_2^L,\eta_2^U]\in\tilde{g}_2}$$
$$\left\{\begin{array}{l}\left\{\left[s_{(\gamma_1^L\gamma_2^L/l)},s_{(\gamma_1^U\gamma_2^U/l)}\right]\middle|\left[\tilde{p}^L_{[\gamma_1^L,\gamma_1^U]}\tilde{p}^L_{[\gamma_2^L,\gamma_2^U]},\tilde{p}^U_{[\gamma_1^L,\gamma_1^U]}\tilde{p}^U_{[\gamma_2^L,\gamma_2^U]}\right]\right\},\\ \left\{\left[s_{\left((\eta_1^L)^2+(\eta_2^L)^2-(\eta_1^L\eta_2^L)^2/l^2\right)^{1/2}},s_{\left((\eta_1^U)^2+(\eta_2^U)^2-(\eta_1^U\eta_2^U)^2/l^2\right)^{1/2}}\right]\middle|\left[\tilde{t}^L_{[\eta_1^L,\eta_1^U]}\tilde{t}^L_{[\eta_2^L,\eta_2^U]},\tilde{t}^U_{[\eta_1^L,\eta_1^U]}\tilde{t}^U_{[\eta_2^L,\eta_2^U]}\right]\right\}\end{array}\right\};$$

(3) $$\lambda\tilde{d}=\cup_{[\gamma^L,\gamma^U]\in h,[\eta^L,\eta^U]\in g}$$
$$\left\{\begin{array}{l}\left\{\left[s_{l\left(1-\left(1-(\gamma^L)^2/l^2\right)^\lambda\right)^{1/2}},s_{l\left(1-\left(1-(\gamma^U)^2/l^2\right)^\lambda\right)^{1/2}}\right]\middle|\left[\tilde{p}^L_{[\gamma^L,\gamma^U]},\tilde{p}^U_{[\gamma^L,\gamma^U]}\right]\right\},\\ \left\{\left[s_{l(\eta^L/l)^\lambda},s_{l(\eta^U/l)^\lambda}\right]\middle|\left[\tilde{t}^L_{[\eta^L,\eta^U]},\tilde{t}^U_{[\eta^L,\eta^U]}\right]\right\}\end{array}\right\};$$

(4) $$\tilde{d}^\lambda=\cup_{[\gamma^L,\gamma^U]\in h,[\eta^L,\eta^U]\in g}$$
$$\left\{\begin{array}{l}\left\{\left[s_{l(\gamma^L/l)^\lambda},s_{l(\gamma^U/l)^\lambda}\right]\middle|\left[\tilde{p}^L_{[\gamma^L,\gamma^U]},\tilde{p}^U_{[\gamma^L,\gamma^U]}\right]\right\},\\ \left\{\left[s_{l\left(1-\left(1-(\eta^L)^2/l^2\right)^\lambda\right)^{1/2}},s_{l\left(1-\left(1-(\eta^U)^2/l^2\right)^\lambda\right)^{1/2}}\right]\middle|\left[\tilde{t}^L_{[\eta^L,\eta^U]},\tilde{t}^U_{[\eta^L,\eta^U]}\right]\right\}\end{array}\right\}$$

Example 1:

Let $\tilde{d}_1=\left\{\{[s_2,s_3]|[0.6,1]\},\{[s_1,s_4]|[0.8,0.9]\}\right\}$, $\tilde{d}_2=\left\{\{[s_4,s_5]|[1,1]\},\{([s_2,s_3]|[0.6,0.7],[s_1,s_2]|[0,0.1])\}\right\}$ be two IVPULPFEs on a continuous linguistic term set $\tilde{S}=\{s_\alpha|0\le\alpha\le 5\}$. According to the basic operations for IVPULPFEs, we can obtain

(1) $$\tilde{d}_1\oplus\tilde{d}_2=\left\{\begin{array}{c}\{[s_{4.1761},s_5]|[0.6,0.1]\},\\ \{([s_{0.4},s_{2.4}]|[0.48,0.63],[s_{0.2},s_{1.6}]|[0,0.09])\}\end{array}\right\}$$

(2) $$\tilde{d}_1\otimes\tilde{d}_2=\left\{\begin{array}{c}\{[s_{1.6},s_3]|[0.6,0.1]\},\\ \{([s_{2.2},s_{4.3863}]|[0.48,0.63],[s_{1.4},s_{4.1761}]|[0,0.09])\}\end{array}\right\}$$

(3) $$2\tilde{d}_1=\left\{\begin{array}{c}\{[s_{2.7129},s_{3.8419}]|[0.6,0.1]\},\\ \{[s_{0.2},s_{3.2}]|[0.8,0.9]\}\end{array}\right\}$$

(4) $$\tilde{d}_1^2=\left\{\begin{array}{c}\{[s_{0.8},s_{1.8}]|[0.6,0.1]\},\\ \{[s_{1.4},s_{4.6648}]|[0.8,0.9]\}\end{array}\right\}$$

In the following, we propose a method to rank any two IVPULPFEs. In order to do this, we first introduce the concepts of score function and accuracy function of an IVPULPFE. Afterward, the method for comparing two IVPULPFEs is presented. Let $\tilde{d}=\left(\tilde{h}\middle|\tilde{p}_{\tilde{h}},\tilde{g}\middle|\tilde{t}_{\tilde{g}}\right)$ be an IVPULPFE defined on a continuous linguistic term set $\tilde{S}=\{s_\alpha|0\le\alpha\le l\}$, then the score function of $\tilde{d}$ is defined as:

$$S\left(\tilde{d}\right)=s_{\sqrt{\left(2l^2+\sum_{i=1}^{\#\tilde{h}}\left(\left(\gamma_i^L\right)^2\tilde{p}^L_{\left[\gamma_i^L,\gamma_i^U\right]}+\left(\gamma_i^U\right)^2\tilde{p}^U_{\left[\gamma_i^L,\gamma_i^U\right]}\right)-\sum_{j=1}^{\#\tilde{g}}\left(\left(\eta_j^L\right)^2\tilde{t}^L_{\left[\eta_j^L,\eta_j^U\right]}+\left(\eta_j^U\right)^2\tilde{t}^U_{\left[\eta_j^L,\eta_j^U\right]}\right)\right)/4}} \tag{3}$$

where $\#\tilde{h}$ and $\#\tilde{g}$ denote the number of values in $\tilde{h}$ and $\tilde{g}$, respectively. In addition, let $\tilde{d}=\left(\tilde{h}\left|\tilde{p}_{\tilde{h}},\tilde{g}\right|\tilde{t}_{\tilde{g}}\right)$ be an IVPULPFE defined on a continuous linguistic term set $\tilde{S}=\left\{s_\alpha|0\le\alpha\le l\right\}$, then the accuracy function of $\tilde{d}$ is defined as

$$H\left(\tilde{d}\right)=s_{\sqrt{\left(\sum_{i=1}^{\#\tilde{h}}\left(\left(\gamma_i^L\right)^2\tilde{p}^L_{\left[\gamma_i^L,\gamma_i^U\right]}+\left(\gamma_i^U\right)^2\tilde{p}^U_{\left[\gamma_i^L,\gamma_i^U\right]}\right)+\sum_{j=1}^{\#\tilde{g}}\left(\left(\eta_j^L\right)^2\tilde{t}^L_{\left[\eta_j^L,\eta_j^U\right]}+\left(\eta_j^U\right)^2\tilde{t}^U_{\left[\eta_j^L,\eta_j^U\right]}\right)\right)/4}} \tag{4}$$

where $\#\tilde{h}$ and $\#\tilde{g}$ denote the number of values in $\#\tilde{h}$ and $\#\tilde{g}$, respectively.

Based on the concepts of score and accuracy functions, a method for comparing any two IVPULPFEs is presented as follows. Let $\tilde{d}_1=\left(\tilde{h}_1\left|p_{\tilde{h}_1},\tilde{g}_1\right|\tilde{t}_{\tilde{g}1}\right)$ and $\tilde{d}_2=\left(\tilde{h}_2\left|p_{\tilde{h}_2},\tilde{g}_2\right|\tilde{t}_{\tilde{g}2}\right)$ be any two IVPULPFEs defined on a continuous linguistic term set $\tilde{S}=\left\{s_\alpha|0\le\alpha\le l\right\}$, then

(1) If $S\left(\tilde{d}_1\right)>S\left(\tilde{d}_2\right)$, then $\tilde{d}_1>\tilde{d}_2$,

(2) If $S\left(\tilde{d}_1\right)=S\left(\tilde{d}_2\right)$, then

If $H\left(\tilde{d}_1\right)>H\left(\tilde{d}_2\right)$, then $\tilde{d}_1>\tilde{d}_2$;

If $H\left(\tilde{d}_1\right)=H\left(\tilde{d}_2\right)$, then $\tilde{d}_1=\tilde{d}_2$.

The score function is designed to quantify the overall tendency of a preference set by aggregating its interval-valued probabilistic uncertain linguistic terms. This provides a numerical representation of the "central trend" of the preference, which is crucial for ranking alternatives in decision-making. The accuracy function, on the other hand, captures the degree of uncertainty or hesitancy within the preference set. By evaluating the range of interval values and their associated probabilities, the accuracy function ensures that alternatives with higher certainty are prioritized, which is especially important in scenarios where reducing uncertainty is critical for decision-making.

In the following, we discuss aggregations operators for IVPULPFEs. Let $\tilde{d}_i=\left(\tilde{h}_i\left|p_{\tilde{h}_i},\tilde{g}_i\right|\tilde{t}_{\tilde{g}_i}\right)(i=1,2,\ldots,n)$ be a collection of IVPULPFEs defined on a continuous linguistic term set $\tilde{S}=\left\{s_\alpha|0\le\alpha\le l\right\}$ that to be aggregated, and $\varpi=\left(\varpi_1,\varpi_2,\ldots,\varpi_n\right)^T$ be the weight vector, such that $0\le\varpi_i\le1$ and $\sum_{i=1}^n\varpi_i=1$. The interval-valued probabilistic uncertain linguistic Pythagorean fuzzy weighted average (IVPULPFWA) operator is defined as

$$IVPULPFWA\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\oplus_{i=1}^n\varpi_i\tilde{d}_i \tag{5}$$

In addition, if $\varpi=\left(1/n,1/n,\ldots,1/n\right)^T$, then IVPULPFWA reduces to interval-valued probabilistic uncertain linguistic Pythagorean fuzzy average (IVPULPFA) operator, i.e.,

$$IVPULPFA\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\frac{1}{n}\oplus_{i=1}^n\tilde{d}_i \tag{6}$$

Based on the operations of IVPULPFEs, we can obtain the following aggregation results. Let $\tilde{d}_i=\left(\tilde{h}_i\left|p_{\tilde{h}_i},\tilde{g}_i\right|\tilde{t}_{\tilde{g}_i}\right)(i=1,2,\ldots,n)$ be a collection of IVPULPFEs, then the aggregated value by the IVPULPFWA is also an IVPULPFE, and

$$IVPULPFWA\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\cup_{\left[\gamma_i^L,\gamma_i^U\right]\in\tilde{h}_i,\left[\eta_j^L,\eta_j^U\right]\in\tilde{g}_j}\left\{\begin{array}{l}\left\{\left[s_{l\left(1-\prod_{i=1}^n\left(1-\left(\gamma_i^L\right)^2/l^2\right)^{\varpi_i}\right)^{1/2}},s_{l\left(1-\prod_{i=1}^n\left(1-\left(\gamma_i^U\right)^2/l^2\right)^{\varpi_i}\right)^{1/2}}\right]\left|\left[\prod_{i=1}^n\tilde{p}^L_{\left[\gamma_i^L,\gamma_i^U\right]},\prod_{i=1}^n\tilde{p}^U_{\left[\gamma_i^L,\gamma_i^U\right]}\right]\right.\right\}\\ \left\{\left[s_{l\prod_{j=1}^n\left(\eta_j^L/l\right)^{\varpi_j}},s_{l\prod_{j=1}^n\left(\eta_j^U/l\right)^{\varpi_j}}\right]\left|\left[\prod_{j=1}^n\tilde{t}^L_{\left[\eta_j^L,\eta_j^U\right]},\prod_{j=1}^n\tilde{t}^U_{\left[\eta_j^L,\eta_j^U\right]}\right]\right.\right\}\end{array}\right\} \tag{7}$$

Similarly, we can obtain the definition of the interval-valued probabilistic uncertain linguistic Pythagorean fuzzy weighted geometric operator and its corresponding result. Let $\tilde{d}_i=\left(\tilde{h}_i\left|p_{\tilde{h}_i},\tilde{g}_i\right|\tilde{t}_{\tilde{g}_i}\right)(i=1,2,\ldots,n)$ be a collection of IVPULPFEs defined on a continuous linguistic term set $\tilde{S}=\left\{s_\alpha|0\le\alpha\le l\right\}$ that to be aggregated, and $\varpi=\left(\varpi_1,\varpi_2,\ldots,\varpi_n\right)^T$ be the weight vector, such that $0\le\varpi_i\le1$ and $\sum_{i=1}^n\varpi_i=1$. The interval-valued probabilistic uncertain linguistic Pythagorean fuzzy weighted geometric (IVPULPFWG) operator is defined as

$$IVPULPFWG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\otimes_{i=1}^{n}\tilde{d}_i^{\varpi_i} \tag{8}$$

In addition, if $\varpi=\left(1/n,1/n,\ldots,1/n\right)^T$, then IVPULPFWG reduces to interval-valued probabilistic uncertain linguistic Pythagorean fuzzy geometric (IVPULPFG) operator, i.e.,

$$IVPULPFG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\otimes_{i=1}^{n}\tilde{d}_i^{1/n} \tag{9}$$

Furthermore, its aggregation results its presented as follows.

$$IVPULPFWG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\cup_{\left[\gamma_i^L,\gamma_i^U\right]\in\tilde{h}_i,\left[\eta_j^L,\eta_j^U\right]\in\tilde{g}_j}$$

$$\left\{\begin{array}{c}\left\{\begin{array}{c}\left[s_{l\prod_{i=1}^{n}\left(\gamma_i^L/l\right)^{\varpi_i}},s_{l\prod_{i=1}^{n}\left(\eta_i^U/l\right)^{\varpi_i}}\right]\\ \mid\left[\prod_{i=1}^{n}\tilde{p}^L_{\left[\gamma_i^L,\gamma_i^U\right]},\prod_{i=1}^{n}\tilde{p}^U_{\left[\gamma_i^L,\gamma_i^U\right]}\right]\end{array}\right\}\\ \left\{\begin{array}{c}\left[s_{l\left(1-\prod_{j=1}^{n}\left(1-\left(\eta_j^L\right)^2/l^2\right)^{\varpi_j}\right)^{1/2}},s_{l\left(1-\prod_{j=1}^{n}\left(1-\left(\eta_j^U\right)^2/l^2\right)^{\varpi_j}\right)^{1/2}}\right]\\ \mid\left[\prod_{j=1}^{n}\tilde{t}^L_{\left[\eta_j^L,\eta_j^U\right]},\prod_{j=1}^{n}\tilde{t}^U_{\left[\eta_j^L,\eta_j^U\right]}\right]\end{array}\right\}\end{array}\right\} \tag{10}$$

3 NOVEL MAGDM METHODS BASED ON IVPULPFSS

This section proposes two novel MAGDM under IVPULPFSs decision-making environment. Let's consider a MAGDM problem where decision makes' evaluation values are expressed by IVPULPFSs. Assume that there are m alternatives, i.e., $\left\{A_1,A_2,\ldots,A_m\right\}$ and a group of decision makers are required to evaluate all the alternatives under n attributes which can be denoted as $\left\{G_1,G_2,\ldots,G_n\right\}$. Weight vector of attributes is $w=\left(w_1,w_2,\ldots,w_n\right)^T$, such that $\sum_{j=1}^{n}w_j=1$. Let $\tilde{S}=\left\{s_\alpha|0\le\alpha\le l\right\}$ be a pre-defined continuous linguistic term set and DMs use an IVPULPFE $\tilde{d}_{ij}=\left(\tilde{h}_{ij}\left|p_{\tilde{h}_{ij}},\tilde{g}_{ij}\right|\tilde{t}_{\tilde{g}_{ij}}\right)$ to denote their evaluation value of the attribute $G_j\left(j=1,2,\ldots,n\right)$ of alternative $A_i\left(i=1,2,\ldots,n\right)$. Hence, DMs' evaluation opinions can be denoted by an interval-valued probabilistic uncertain linguistic Pythagorean fuzzy decision matrix is, which can be denoted as $\tilde{D}=\left(\tilde{d}_{ij}\right)_{m\times n}$. In the following, we present two novel approaches to solve this problem.

3.1 Approach I: An Aggregation Operator Based MAGDM Method

Step 1. Standardize the original decision matric. In real MAGDM problems, there are usually two types of attributes. Different types of attributes have effect on alternatives. Therefore, the original decision matrix should be standardized according to:

$$\tilde{d}_{ij}=\begin{cases}\left(\tilde{h}_{ij}\left|p_{\tilde{h}_{ij}},\tilde{g}_{ij}\right|\tilde{t}_{\tilde{g}_{ij}}\right)\\ \left(\tilde{g}_{ij}\left|\tilde{t}_{\tilde{g}_{ij}},\tilde{h}_{ij}\right|p_{\tilde{h}_{ij}}\right)\end{cases} \tag{11}$$

Step 2. For alternative $A_i\left(i=1,2\ldots,m\right)$, use the IVPULPFWA operator

$$\tilde{d}_i=IVPULPFWA\left(\tilde{d}_{i1},\tilde{d}_{i2},\ldots,\tilde{d}_{in}\right) \tag{12}$$

or the IVPULPFWG operator

$$\tilde{d}_i=IVPULPFWG\left(\tilde{d}_{i1},\tilde{d}_{i2},\ldots,\tilde{d}_{in}\right) \tag{13}$$

to calculate the comprehensive evaluation value.

Step 3. Compute the score values of alternatives.

Step 4. Rank alternatives according to their score values.

3.2 Approach II: A TODIM Based MAGDM Method

Step 1. Standardize the original decision matric, which is the same as the step 1 in Approach I.

Step 2. Compute the relative weight of attribute G_j to the reference attribute G_r according to:

$$w_{jr}=w_j/w_r \tag{14}$$

where w_j denotes the weight of G_j and $w_r=\max\left\{w_j\mid j=1,2,\ldots,n\right\}$.

Step 3. Compute the dominance of each alternative A_i over each alternative A_k according to

$$\vartheta\left(A_i,A_k\right)=\sum_{j=1}^{n}\phi_j\left(A_i,A_k\right),\ i,s=1,2,\ldots,m \tag{15}$$

wherein

$$\phi_j\left(A_i,A_s\right)=\begin{cases}\sqrt{w_{jk}/\sum_{j=1}^{n}w_{jk}\,d\left(\tilde{d}_{ij},\tilde{d}_{kj}\right)} & if\tilde{d}_{ij}>\tilde{d}_{kj}\\ 0 & if\ \tilde{d}_{ij}=\tilde{d}_{kj}\\ -\frac{1}{\theta}\sqrt{\left(\sum_{j=1}^{n}w_{jk}\right)/w_{jk}\,d\left(\tilde{d}_{ij},\tilde{d}_{kj}\right)} & if\ \tilde{d}_{ij}<\tilde{d}_{kj}\end{cases} \tag{16}$$

wherein $d\left(\tilde{d}_{ij},\tilde{d}_{kj}\right)$ denotes the distance between $\tilde{d}_{ij}$ and $\tilde{d}_{kj}$, $\phi_j\left(A_i,A_k\right)$ denotes the contributions of the attribute G_j to the function $\vartheta\left(A_i,A_k\right)$, and the parameter ϑ represents the attenuation factor of the losses.

Step 4. Compute the overall prospect value of alternative A_i based on the following formula

$$\phi\left(A_i\right)=\frac{\sum_{k=1}^{m}\vartheta\left(A_i,A_k\right)-\min_i\left\{\sum_{k=1}^{m}\vartheta\left(A_i,A_k\right)\right\}}{\max_i\left\{\sum_{k=1}^{m}\vartheta\left(A_i,A_k\right)\right\}-\min_i\left\{\sum_{k=1}^{m}\vartheta\left(A_i,A_k\right)\right\}} \tag{17}$$

Step 5. Rank alternatives according to their overall prospect values.

4 APPLICATIONS OF THE NEW MAGDM METHODS

Example 2:

A human resources department of a company is now recruiting a sales consultant. After primary evaluation and selection, there are four candidates (A_1, A_2, A_3 and A_4) on the final list. A group of human resources specialists are invited to evaluate the four candidates under four attributes, i.e., experiences (G_1), competencies (G_2), abilities in foreign language (G_3), and human relationship management (G_4). Weight vector of attributes is $w=\left(0.15,0.25,0.25,0.35\right)^T$. The group of specialists use Based on a continuous linguistic term set S = {s_0 = *"Extremely poor"*, s_1 = *"Very poor"*, s_2 = *"Poor"*, s_3 = *"Slightly poor"*, s_4 = *"Fair"*, s_5 = *"Slightly good"*, s_6 = *"Good"*, s_7 = *"Very good"*, s_8 = *"Extremely good"*}, DMs use an IVPULPFE to denote the evaluation value of attribute $G_j\left(j=1,2,3\right)$ of alternative $A_i\left(i=1,2,3,4\right)$. The original decision matrix is listed in Tab. 1.

Table 1 The original decision matrix

	A_1
G_1	$\left\{\left\{\left[s_2,s_3\right]\mid\left[0.6,1\right]\right\},\left\{\left(\left[s_4,s_5\right]\mid\left[0.1,0.3\right]\right),\left(\left[s_5,s_6\right]\mid\left[0.6,0.7\right]\right)\right\}\right\}$
G_2	$\left\{\left\{\left(\left[s_1,s_4\right]\mid\left[0.5,0.8\right]\right),\left(\left[s_3,s_5\right]\mid\left[0.1,0.2\right]\right)\right\},\left\{\left(\left[s_5,s_6\right]\mid\left[0.7,1\right]\right)\right\}\right\}$
G_3	$\left\{\left\{\left(\left[s_3,s_4\right]\mid\left[0.3,1\right]\right)\right\},\left\{\left(\left[s_4,s_6\right]\mid\left[0.6,1\right]\right)\right\}\right\}$
G_4	$\left\{\left\{\left(\left[s_1,s_2\right]\mid\left[0.2,0.4\right]\right),\left(\left[s_2,s_3\right]\mid\left[0.2,0.3\right]\right),\left(\left[s_3,s_4\right]\mid\left[0.1,0.3\right]\right)\right\},\left\{\left(\left[s_3,s_4\right]\mid\left[0.8,1\right]\right)\right\}\right\}$
	A_2
G_1	$\left\{\left\{\left(\left[s_4,s_5\right]\mid\left[0.5,1\right]\right)\right\},\left\{\left(\left[s_2,s_4\right]\mid\left[0.1,0.3\right]\right),\left(\left[s_3,s_6\right]\mid\left[0.5,0.7\right]\right)\right\}\right\}$
G_2	$\left\{\left\{\left(\left[s_1,s_3\right]\mid\left[0.5,1\right]\right)\right\},\left\{\left(\left[s_2,s_6\right]\mid\left[0.1,0.2\right]\right),\left(\left[s_4,s_5\right]\mid\left[0.7,0.8\right]\right)\right\}\right\}$
G_3	$\left\{\left\{\left(\left[s_2,s_3\right]\mid\left[0.4,0.5\right]\right),\left(\left[s_4,s_5\right]\mid\left[0.3,0.5\right]\right)\right\},\left\{\left(\left[s_1,s_3\right]\mid\left[0.6,1\right]\right)\right\}\right\}$
G_4	$\left\{\left\{\left(\left[s_4,s_6\right]\mid\left[0.6,1\right]\right)\right\},\left\{\left(\left[s_2,s_5\right]\mid\left[0.8,1\right]\right)\right\}\right\}$
	A_3
G_1	$\left\{\left\{\left(\left[s_3,s_5\right]\mid\left[0.7,1\right]\right)\right\},\left\{\left(\left[s_3,s_6\right]\mid\left[0.9,1\right]\right)\right\}\right\}$
G_2	$\left\{\left\{\left(\left[s_1,s_3\right]\mid\left[0.8,1\right]\right)\right\},\left\{\left(\left[s_3,s_4\right]\mid\left[0.6,0.7\right]\right),\left(\left[s_5,s_6\right]\mid\left[0.2,0.3\right]\right)\right\}\right\}$
G_3	$\left\{\left\{\left(\left[s_1,s_3\right]\mid\left[0.6,0.8\right]\right),\left(\left[s_5,s_6\right]\mid\left[0.1,0.2\right]\right)\right\},\left\{\left(\left[s_3,s_4\right]\mid\left[0.9,1\right]\right)\right\}\right\}$
G_4	$\left\{\left\{\left(\left[s_2,s_4\right]\mid\left[0.5,1\right]\right)\right\},\left\{\left(\left[s_4,s_6\right]\mid\left[0.1,0.4\right]\right),\left(\left[s_3,s_7\right]\mid\left[0.4,0.6\right]\right)\right\}\right\}$
	A_4
G_1	$\left\{\left\{\left(\left[s_1,s_4\right]\mid\left[0.7,0.8\right]\right),\left(\left[s_5,s_6\right]\mid\left[0.1,0.2\right]\right)\right\},\left\{\left(\left[s_2,s_4\right]\mid\left[0.8,1\right]\right)\right\}\right\}$
G_2	$\left\{\left\{\left(\left[s_3,s_4\right]\mid\left[0.7,1\right]\right)\right\},\left\{\left(\left[s_5,s_6\right]\mid\left[0.4,1\right]\right)\right\}\right\}$
G_3	$\left\{\left\{\left(\left[s_2,s_5\right]\mid\left[0.5,1\right]\right)\right\},\left\{\left(\left[s_5,s_6\right]\mid\left[0.4,0.6\right]\right),\left(\left[s_1,s_5\right]\mid\left[0.2,0.4\right]\right)\right\}\right\}$
G_4	$\left\{\left\{\left(\left[s_3,s_4\right]\mid\left[0.7,1\right]\right)\right\},\left\{\left(\left[s_1,s_4\right]\mid\left[0.1,0.2\right]\right),\left(\left[s_5,s_6\right]\mid\left[0.6,0.8\right]\right)\right\}\right\}$

4.1 The Decision-Making Process

In the following, we use the above proposed MAGDM method to solve the example and the detailed calculation process is presented as follows.

4.1.1 The Decision-Making Process by Approach I

Step 1. It is obvious that all attributes are benefit type and the decision matrix does not need to be standardized.

Step 2. For each alternative, use the IVPULPFWA operator to aggregate the attribute values and a series of collective evaluation values are obtained. The comprehensive evaluation values are shown as follows

$\tilde{d}_1$ = {{[$s_{1.8834}$, $s_{3.3212}$] | [0.0180,0.3200], [$s_{2.1379}$, $s_{3.5523}$] | [0.0180,0.2400], [$s_{2.5205}$, $s_{3.8742}$] | [0.0090, 0.2400], [$s_{2.3590}$, $s_{3.7027}$] | [0.0036, 0.0800], [$s_{2.5602}$, $s_{3.9011}$] | [0.0036, 0.0600], [$s_{2.8771}$, $s_{4.1819}$] | [0.0018, 0.0600]}, {[$s_{3.8244}$, $s_{5.0657}$] | [0.0336, 0.3000], [$s_{3.9545}$, $s_{5.2062}$] | [0.2016, 0.7000]}};

$\tilde{d}_2$ = {{[$s_{3.1106}$, $s_{4.7705}$] | [0.0600, 0.5000], [$s_{3.5529}$, $s_{5.1141}$] | [0.0450, 0.5000]}, {[$s_{1.6818}$, $s_{4.4542}$] | [0.0048, 0.0600], [$s_{2.0000}$, $s_{4.2557}$] | [0.0336, 0.2400], [$s_{1.7873}$, $S_{4.7335}$] | [0.0240, 0.1400], [$s_{2.1254}$, $s_{4.5226}$] | [0.1680, 0.5600]}};

$\tilde{d}_3$ = {[$s_{1.8209}$, $s_{3.7676}$] | [0.1680, 0.8000], [$s_{3.1893}$, $s_{4.6829}$] | [0.0280, 0.2000]}, {[$s_{3.3178}$, $s_{4.8990}$] | [0.0486, 0.2800], [$s_{3.0000}$, $s_{5.1706}$] | [0.1944, 0.4200], [$s_{3.7697}$, $s_{5.4216}$] | [0.0162, 0.1200], [$s_{3.4087}$, $s_{5.7222}$] | [0.0648, 0.1800]}};

$\tilde{d}_4$ = {{[$s_{2.5756}$, $s_{4.2928}$] | [0.1715, 0.8000], [$s_{3.2581}$, $s_{4.6870}$] | [0.0245, 0.2000]}, {[$s_{2.4811}$, $s_{4.8990}$] | [0.0128, 0.1200], [$s_{4.3579}$, $s_{5.6460}$] | [0.0768, 0.4800], [$s_{1.6592}$, $s_{4.6807}$] | [0.0064, 0.0800], [$s_{2.9143}$, $s_{5.3944}$] | [0.0384, 0.3200]}}.

Step 3. Compute the score values of alternatives and we can obtain the following results

$S\left(\tilde{d}_1\right) = s_{5.2718}$, $S\left(\tilde{d}_2\right) = s_{5.7553}$, $S\left(\tilde{d}_3\right) = s_{5.3375}$, $S\left(\tilde{d}_4\right) = s_{5.4177}$

Step 4. Rank alternatives according to their score values and we can obtain $A_2 \succ A_4 \succ A_3 \succ A_1$. Hence A_2 is the best alternative.

In step2, if the IVPULPFWG operator is used to aggregate the attribute values of alternatives, then we can obtain the following results

$\tilde{d}_1$ = {{[$s_{1.4603}$, $s_{3.0058}$] | [0.0180, 0.3200], [$s_{1.8612}$, $s_{3.4641}$] | [0.0180, 0.2400], [$s_{2.1450}$, $s_{3.8311}$] | [0.0090, 0.2400], [$s_{1.9218}$, $s_{3.1782}$] | [0.0036, 0.0800], [$s_{2.4495}$, $s_{3.6628}$] | [0.0036, 0.0600], [$s_{2.8230}$, $s_{4.0509}$] | [0.0018, 0.0600]}, {[$s_{4.0254}$, $s_{5.3351}$] | [0.0336, 0.3000], [$s_{4.2036}$, $s_{5.4942}$] | [0.2016, 0.7000]}};

$\tilde{d}_2$ = {{[$s_{1.5029}$, $s_{3.5820}$] | [0.1680, 0.8000], [$s_{2.2474}$, $s_{4.2598}$] | [0.0280, 0.2000]}, {[$s_{3.3999}$, $s_{5.2287}$] | [0.0486, 0.2800], [$s_{3.0000}$, $s_{5.8765}$] | [0.1944, 0.4200], [$s_{3.9847}$, $s_{5.6534}$] | [0.0162, 0.1200], [$s_{3.6782}$, $s_{6.1845}$] | [0.0648, 0.1800]}};

$\tilde{d}_3$ = {{[$s_{1.5029}$, $s_{3.5820}$] | [0.1680, 0.8000], [$s_{2.2474}$, $s_{4.2598}$] | [0.0280, 0.2000]}, {[$s_{3.3999}$, $s_{5.2287}$] | [0.0486, 0.2800], [$s_{3.0000}$, $s_{5.8765}$] | [0.1944, 0.4200], [$s_{3.9847}$, $s_{5.6534}$] | [0.0162, 0.1200], [$s_{3.6782}$, $s_{6.1845}$] | [0.0648, 0.1800]}};

$\tilde{d}_4$ = {{[$s_{2.2990}$, $s_{4.2295}$] | [0.1715, 0.8000], [$s_{2.9267}$, $s_{4.4947}$] | [0.0245, 0.2000]}, {[$s_{3.8462}$, $s_{5.2287}$] | [0.0128, 0.1200], [$s_{4.7325}$, $s_{5.8002}$] | [0.0768, 0.4800], [$s_{2.9199}$, $s_{4.9166}$] | [0.0064, 0.0800], [$s_{4.1349}$, $s_{5.5696}$] | [0.0384, 0.3200]}};

The score values of the four alternatives are

$S\left(\tilde{d}_1\right) = s_{5.1552}$, $S\left(\tilde{d}_2\right) = s_{5.5557}$, $S\left(\tilde{d}_3\right) = s_{5.1500}$, $S\left(\tilde{d}_4\right) = s_{5.3282}$

Therefore, the ranking order of the four candidates is $A_2 \succ A_4 \succ A_1 \succ A_3$, and A_2 is the best alternative.

4.1.2 The Decision-Making Process by Approach II

Step 1. It is obvious that all attributes are benefit type and the decision matrix does not need to be standardized.

Step 2. Compute the relative weight of attribute G_j to the reference attribute G_r according to Eq. (14)

$$w_{jr} = \left(0.4286, 0.7143, 0.7143, 1\right)^T$$

Step 3. Compute the dominance of each alternative A_i over each alternative A_k according to Eq. (15) (Without loss of generality, we assume $\theta = 1$)

$$\vartheta\left(A_i, A_k\right) = \begin{bmatrix} 0 & -0.8856 & -0.6452 & -0.5689 \\ 0.2057 & 0 & -0.0225 & -0.4548 \\ 0.0509 & -0.5967 & 0 & -0.4091 \\ -0.0681 & -0.3606 & -0.2094 & 0 \end{bmatrix},$$
$$i, k = 1, 2, 3, 4$$

Step 4. Compute the overall prospect value of alternative A_i according to Eq. (16)

$\phi\left(A_1\right) = -1.3709$, $\phi\left(A_2\right) = 0.2774$
$\phi\left(A_3\right) = -0.5530$, $\phi\left(A_4\right) = -0.7696$

Step 5. Rank alternatives according to their prospect values and we can obtain $A_2 \succ A_3 \succ A_4 \succ A_1$, which indicates that A_2 is the best alternative.

4.2 Comparison Analysis

To better demonstrate the advantages and superiorities of our proposed methods, we provide comparison analysis. We use our method and that developed by Wang et al. [12] to solve the same example and compare their decision results.

Example 3:

(Revised from Wang et al. [12]). A college plans to evaluate the quality of doctoral students' theses. Three professors are invited to evaluate the quality of five theses. The weight vector of the three professors is $\lambda = \left(0.243, 0.514, 0.243\right)^T$. Five theses (A_1, A_2, A_3, A_4, and A_5) are assess under four attributes, i.e., significant degree of the study (G_1), the degree of innovation (G_2), the degree of compliance with academic norms (G_3), and the significant degree methodology (G_4). The weight vector of attributes is $w=(0.3, 0.1, 0.2, 0.4)^T$. Let S = {s_0 = *"Extremely poor"*, s_1 = *"Very poor"*, s_2 = *"Poor"*, s_3 = *"Slightly poor"*, s_4 = *"Fair"*, s_5 = *"Slightly good"*, s_6 = *"Good"*, s_7 = *"Very good"*, s_8 = *"Extremely good"*} be a linguistic term set and the three professors use PDHLPFEs defined on S to express their evaluation values. The original decision matrices are presented in Tabs. 2, 3, and 4.

Table 2 The original decision matrix provided by the first professor in Example 3

	G_1	G_2
A_1	{{s_7 (0.3), s_6 (0.3), s_5 (0.4)}, {s_2 (1)}}	{{s_7 (1)}, {s_2 (1)}}
A_2	{{s_1 (1)}, {s_4 (1)}}	{{s_3 (1)}, {s_7 (1)}}
A_3	{{s_6 (1)}, {s_3 (1)}}	{{s_5 (1)}, {s_2 (1)}}
A_4	{{s_5 (0.7), s_2 (0.3)}, {s_5 (1)}}	{{s_3 (0.5), s_2 (0.5)}, {s_6 (1)}}
	G_3	G_4
A_1	{{s_2 (1)}, {s_2 (1)}}	{{s_7 (0.5), s_6 (0.5)}, {s_3 (1)}}
A_2	{{s_7 (1)}, {s_3 (0.5), s_2 (0.5)}}	{{s_3 (1)}, {s_3 (1)}}
A_3	{{s_1 (1)}, {s_7 (1)}}	{{s_4 (0.4), s_2 (0.6)}}, {s_4 (1)}}
A_4	{{s_8 (1)}, {s_1 (1)}}	{{s_2 (1)}, {s_6 (1)}}

Table 3 The original decision matrix provided by the second professor in Example 3

	G_1	G_2
A_1	{{s_7 (1) }, {s_2 (0.1), s_5 (0.9)}}	{{s_3 (1) }, {s_2 (0.4), s_6 (0.6)}}
A_2	{{s_8 (1)}, {s_7 (0.3), s_8 (0.7)}}	{{s_6 (1)}, {s_2 (0.5), s_6 (0.5)}}
A_3	{{s_7 (0.6), s_1 (0.4)}, {s_3 (1)}}	{{s_2 (0.5), s_6 (0.3), s_5 (0.2)}, {s_1 (0.2), s_6 (0.8)}}
A_4	{{s_5 (0.6), s_3 (0.4)}, {s_6 (0.4), s_2 (0.6)}}	{{s_3 (1)}, {s_1 (0.3), s_8 (0.7)}}
	G_3	G_4
A_1	{{s_3 (1) }, {s_5 (1)}}	{{s_8 (1) }, {s_5 (0.3), s_7 (0.7)}}
A_2	{{s_6 (0.7), s_4 (0.3)}, {s_4 (1)}}	{{s_4 (0.4), s_6 (0.6)}, {s_8 (0.1), s_6 (0.9)}}
A_3	{{s_2 (1)}, {s_5 (1)}}	{{s_7 (1)}, {s_5 (0.4), s_6 (0.6)}}
A_4	{{s_7 (1)}, {s_3 (1)}}	{{s_5 (1)}, {s_5 (1)}}

Table 4 The original decision matrix provided by the third professor in Example 3

	G_1	G_2
A_1	{{s_4 (1) }, {s_3 (1)}}	{{s_3 (1) }, {s_4 (1)}}
A_2	{{s_3 (0.4), s_6 (0.6)}, {s_5 (1)}}	{{s_2 (1)}, {s_4 (0.4), s_7 (0.6)}}
A_3	{{s_7 (1)}, {s_2 (1)}}	{{s_1 (1)}, {s_5 (1)}}
A_4	{{s_6 (1)}, {s_5 (1)}}	{{s_2 (0.3), s_1 (0.7)}, {s_3 (0.3), s_7 (0.7)}}
	G_3	G_4
A_1	{{s_5 (0.3), s_6 (0.1), s_3 (0.6)}, {s_4 (1)}}	{{s_3 (1)}, {s_4 (1)}}
A_2	{{s_5 (1)}, {s_6 (1)}}	{{s_3 (0.2), s_6 (0.8)}, {s_6 (1)}}
A_3	{{s_1 (1)}, {s_7 (1)}}	{{s_2 (0.6), s_4 (0.4)}, {s_4 (1)}}
A_4	{{s_5 (1)}, {s_3 (0.3), s_5 (0.7)}}	{{s_4 (1)}, {s_1 (1)}}

We use Wang et al.'s [12] method based on the probabilistic dual hesitant linguistic Pythagorean fuzzy weighted average (PDHLPFWA) operator and Approach I in the present paper to solve Example 3 and the final decision results are shown in Tab. 5.

It should be pointed out that Example 3 is based on PDHLPFSs. As pointed out in the above section, IVPULPFSs is more powerful than PDHLPFSs and our method can still be applied in solving MAGDM problems. As a matter of fact, a PDHLPFE can be transferred in to an IVPULPFE. For example, a PDHLPFE {{s_4 (0.4), s_6 (0.6)}, {s_8 (0.1), s_6 (0.9)}} can be transferred into {{[s_4, s_4] | [0.4, 0.4], [s_6, s_6] | [0.6, 0.6]}, {[s_8, s_8] | [0.1, 0.1], [s_6, s_6] | [0.9, 0.9]}}, which is an IVPULPFE. As it is seen from Tab. 5, the two MAGDM methods produces different score values of alternatives as well as different ranking results. Moreover, our proposed method is still more powerful than that developed by Wang et al. [12]. This is because our method is based on IVPULPFSs, while Wang et al.'s [12] method is based on PDHLPFSs. As mentioned above, IVPULPFS is more powerful than PDHLPFSs. In other word, we proposed method can be applied in AMGDM problems under PDHLPFSs, while Wang et al.'s [12] method fails to handle MAGDM problems where attributes' values are denoted by IVPULPFSs. For instance, in Example 3, when the first professor is evaluating the significant degree of the study of A_1 and he/she thinks the importance of the linguistic term s_7 should be [0.3, 0.4]. Then, it is obviously that Wang et al.'s [12] method fails to handle such a new case.

In summary, we compare the advantages and disadvantages of the proposed method in this paper with those of the method in [12] and traditional fuzzy decision-making methods, as shown in Tab. 6.

Table 5 The decision results of Example 3 by using different methods

	Score values $S(\tilde{d}_i)(i=1,2,3,4)$	Ranking order of alternatives
Wang et al.'s [12] method	$S(\tilde{d}_1)=s_{5.6266}$, $S(\tilde{d}_2)=s_{5.6583}$, $S(\tilde{d}_3)=s_{5.5646}$, $S(\tilde{d}_4)=s_{5.6453}$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Approach I	$S(\tilde{d}_1)=s_{7.4947}$, $S(\tilde{d}_2)=s_{7.1656}$, $S(\tilde{d}_3)=s_{6.2568}$, $S(\tilde{d}_4)=s_{7.5958}$	$A_4 \succ A_1 \succ A_2 \succ A_3$

Table 6 Comparative analysis of decision models

Method	Model Basis	Uncertainty Handling	Computational Efficiency	Decision Accuracy	Key Limitation	Overall Superiority
IVPULPFSs (Proposed)	Interval-Valued Probabilistic Uncertain Linguistic Pythagorean Fuzzy Sets	Captures both probabilistic and interval-valued uncertainty	Moderate computational complexity due to interval-valued processing	High accuracy due to better handling of hesitancy and uncertainty	Slightly higher computational cost due to interval-valued processing	High flexibility, accuracy, and applicability in handling complex and uncertain decision problems
LPFS [19-21]	Linguistic Pythagorean Fuzzy Sets	Captures linguistic uncertainty and hesitancy effectively	Moderate efficiency due to linguistic term processing	High accuracy in handling linguistic and hesitant information	Cannot handle probabilistic or interval-valued linguistic uncertainties	Lacks flexibility for probabilistic and interval-valued scenarios
PDHLPFSs [12]	Probabilistic Dual Hesitant Linguistic Pythagorean Fuzzy Sets	Captures probabilistic uncertainty only	Higher efficiency compared to IVPULPFSs due to simpler representation	Moderate accuracy; struggles with scenarios involving interval-valued uncertainty	Cannot handle interval-valued probabilistic linguistic evaluations	Moderate flexibility but fails in interval-valued cases
Traditional Fuzzy Models	Single-valued or basic fuzzy sets	Limited ability to handle uncertainty	High efficiency but limited applicability in complex scenarios	Lower accuracy; limited ability to model complex decision environments	Inability to model hesitancy or probabilistic linguistic uncertainty	Limited flexibility and applicability

5 CONCLUSION AND FUTURE RESEARCH

5.1 Conclusion

This study introduces IVPULPFSs, a novel fuzzy set model that improves MAGDM by incorporating interval-valued probabilistic linguistic information. The proposed methods - an aggregation-based approach and a TODIM-based ranking model - demonstrate enhanced decision robustness in complex, uncertain environments.

Key findings include: Improved Decision Accuracy: IVPULPFSs offer greater flexibility in capturing decision-maker hesitancy and uncertainty, outperforming existing fuzzy decision models.

Robust Aggregation Operators: The newly developed IVPULPFWA and IVPULPFWG operators provide accurate multi-criteria decision evaluations.

Real-World Applicability: The model successfully applies to human resource selection and doctoral thesis evaluation, showing potential for financial modeling, smart city planning, and AI-driven decision systems.

5.2 Future Research

Directions Scalability for Large-Scale Decision-Making [22, 23]: Future work should explore how IVPULPFSs can be extended to large-scale group decision-making (LSGDM) scenarios.

Integration with AI and Machine Learning [24]: Investigating deep learning models for automating linguistic probability assessments.

Advanced Aggregation Operators [25]: Developing nonlinear aggregation methods (e.g., power Bonferroni mean, power Heronian mean) to refine decision-making accuracy.

This study contributes to uncertainty-aware decision science, offering a novel, scalable framework for complex decision-making applications.

Acknowledgements

This research was supported by Funds for First-class Discipline Construction in Beijing University of Chemical Technology (XK1802-5).

6 REFERENCES

[1] Abdullah, S., Qiyas, M., Naeem, M., & Liu, Y. (2022). Pythagorean cubic fuzzy Hamacher aggregation operators and their application in green supply selection problem. *AIMS Mathematics*, *7*(3), 4735-4766. https://doi.org/10.3934/math.2022263

[2] Li, L., Ji, C., & Wang, J. (2021). A novel multi-attribute group decision-making method based on q-rung dual hesitant fuzzy information and extended power average operators. *Cognitive Computation*, *13*, 1345-1362. https://doi.org/10.1007/s12559-021-09932-8

[3] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[4] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*, 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3

[5] Bellman, R. E. & Zadeh, L. A. (1970). Decision-making in a fuzzy environment. *Management Science*, *17*(4), B-141. https://doi.org/10.1287/mnsc.17.4.B141

[6] Torra, V. (2010). Hesitant fuzzy sets. *International Journal of Intelligent Systems*, *25*(6), 529-539. https://doi.org/10.1002/int.20418

[7] Zeng, W., Ma, R., Li, D., Yin, Q., Xu, Z., & Khalil, A. M. (2022). Novel operations of weighted hesitant fuzzy sets and their group decision making application. *AIMS Mathematics*, *7*(8), 14117-14138. https://doi.org/10.3934/math.2022778

[8] Li, J., Ye, J., Niu, L. L., Chen, Q., & Wang, Z. X. (2022). Decision-making models based on satisfaction degree with incomplete hesitant fuzzy preference relation. *Soft Computing*, *26*(7), 3129-3145. https://doi.org/10.1007/s00500-021-06635-y

[9] Zhang, S., Xu, Z., & He, Y. (2017). Operations and integrations of probabilistic hesitant fuzzy information in decision making. *Information Fusion*, *38*, 1-11. https://doi.org/10.1016/j.inffus.2017.02.001

[10] Jiang, F. & Ma, Q. (2018). Multi-attribute group decision making under probabilistic hesitant fuzzy environment with application to evaluate the transformation efficiency. *Applied Intelligence*, *48*, 953-965. https://doi.org/10.1007/s10489-017-1041-x

[11] Song, H. & Chen, Z. C. (2021). Multi-attribute decision-making method based distance and COPRAS method with probabilistic hesitant fuzzy environment. *International Journal of Computational Intelligence Systems*, *14*(1), 1229-1241. https://doi.org/10.2991/ijcis.d.210318.001

[12] Wang, J., Shang, X., Xu, W., Ji, C., & Feng, X. (2021). Extensions of linguistic Pythagorean fuzzy sets and their applications in multi-attribute group decision-making. *Pythagorean Fuzzy Sets: Theory and Applications*, 367-405. https://doi.org/10.1007/978-981-16-1989-2_15

[13] Liu, P. & Qin, X. (2017). An extended VIKOR method for decision making problem with interval-valued linguistic intuitionistic fuzzy numbers based on entropy. *Informatica, 28*(4), 665-685. https://doi.org/10.15388/Informatica.2017.151

[14] Chen, Z., Liu, P., & Pei, Z. (2015). An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *International Journal of Computational Intelligence Systems*, *8*(4), 747-760. https://doi.org/10.1080/18756891.2015.1061394

[15] Song, C., Zhao, H., Xu, Z., & Hao, Z. (2019). Interval-valued probabilistic hesitant fuzzy set and its application in the Arctic geopolitical risk evaluation. *International Journal of Intelligent Systems*, *34*(4), 627-651. https://doi.org/10.1002/int.22069

[16] Liu, P. & Cheng, S. (2019). Interval-valued probabilistic dual hesitant fuzzy sets for multi-criteria group decision-making. *International Journal of Computational Intelligence Systems*, *12*(2), 1393-1411. https://doi.org/10.2991/ijcis.d.191119.001

[17] Gomes, L. F. A. M. & Lima, M. M. P. P. (1991). TODIMI: Basics and application to multicriteria ranking. *Foundations of Computing and Decision Sciences*, *16*(3-4), 1-16.

[18] Kahneman, D., & Tversky, A. (2013). Prospect theory: An analysis of decision under risk. *Handbook of the Fundamentals of Financial Decision Making: Part I*, 99-127.

[19] Li, J., Bi, C., Gao, F., & He, W. (2024). An integrated linguistic Pythagorean fuzzy decision-making approach for risk analysis of offshore wind turbine. *Ocean Engineering*, *291*, 116450. https://doi.org/10.1016/j.oceaneng.2023.116450

[20] Alolaiyan, H., Kalsoom, U., Shuaib, U., Razaq, A., Baidar, A. W., & Xin, Q. (2024). Precision measurement for effective pollution mitigation by evaluating air quality monitoring systems in linguistic Pythagorean fuzzy Dombi environment. *Scientific Reports*, *14*(1), 31944. https://doi.org/10.1038/s41598-024-83478-1

[21] Wang, H., Qin, Y., Liu, Y., Liu, H., & Rong, Y. (2024). Multiple attribute group decision-making based on the ExpTODIM method and linguistic Pythagorean operators. *Granular Computing*, *9*(3), 60. https://doi.org/10.1007/s41066-024-00485-3

[22] Camuffo, A., Gambardella, A., Messinese, D., Novelli, E., Paolucci, E., & Spina, C. (2024). A scientific approach to entrepreneurial decision-making: Large-scale replication and extension. *Strategic Management Journal*, *45*(6), 1209-1237. https://doi.org/10.1002/smj.3580

[23] Cheng, X., Zhang, K., Wu, T., Xu, Z., & Gou, X. (2024). An opinions-updating model for large-scale group decision-making driven by autonomous learning. *Information Sciences*, *662*, 120238. https://doi.org/10.1016/j.ins.2024.120238

[24] Kou, G., Eti, S., Yüksel, S., Dinçer, H., Ergün, E., & Gökalp, Y. (2025). Innovative solution suggestions for financing electric vehicle charging infrastructure investments with a novel artificial intelligence-based fuzzy decision-making modelling. *Artificial Intelligence Review*, *58*(1), 1-32. https://doi.org/10.1007/s10462-024-11012-w

[25] Tang, H. H. & Ahmad, N. S. (2024). Fuzzy logic approach for controlling uncertain and nonlinear systems: A comprehensive review of applications and advances. *Systems Science & Control Engineering*, *12*(1), 2394429. https://doi.org/10.1080/21642583.2024.2394429

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