

# FRACTAL DIMENSION OF THE RAILWAY NETWORKS OF HUNGARIAN COUNTIES

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## ABSTRACT

From a transportation point of view, one of the defining networks is the railway network. The railway network connects important points in terms of traffic or the transport of goods in a given area. Therefore, one of the characteristics of the railway network is how well it covers the points of the given area. The aim of the research was to use a quantitative method to examine how well the railway networks of the Hungarian counties involved in passenger traffic cover the areas of the counties, and how well a given county can be traveled by rail. During the research, the fractal dimensions of the railway networks of the Hungarian counties were determined with the help of chaotic mathematics, including the box counting method. The fractal dimension can be used to show the extent to which a line-like shape can cover a flat area. With the help of the values obtained for each county, the extent of the railway coverage of the county can be examined, i.e. the length of the railway network of the given county and the proportion of the area covered by it. Based on the results, the railway network of the counties of Hungary can be ranked in terms of railway coverage, which can create an opportunity to use the fractal dimension as a quantitative analysis method in railway development strategies. The research showed that the railway network of the counties of Hungary shows a fractal nature, and to what extent the fractal dimensions of the individual counties differ from each other. The results can be used during railway network development in the areas of safety, economy and optimization.

## KEY WORDS

railway networks, chaotic mathematics, fractal dimension, box counting method

## CLASSIFICATION

JEL: C69, R41

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## INTRODUCTION

After the industrial revolution, rail transport became one of the defining segments of passenger and freight transport, as a result of which railway networks began to develop by leaps and bounds. However, after the Second World War, railway network development took a backseat to road network development, which had both military and economic reasons. The road networks soon covered the geographical areas better than the railway networks, as a result of which short- and medium-distance goods and passenger traffic increasingly shifted towards road traffic. One of the harmful consequences of the increased road traffic is that today the emissions of road traffic have become significant in terms of total emissions. The other problem is caused by the increase in the number of cars, which regularly results in traffic jams in large cities and on their entrance roads. In order for people to choose rail transport instead of cars, a cheaper, faster and wider rail transport alternative must be provided. The connection between the development of the railway network and economic growth is now clear [1]. In intelligent railway developments, the structure of the railway network must also be taken into account, because it also affects the design of the e-ticket system. [2]. One of the directions of the developments is to increase the covering property of the railway network, therefore, one of the possible methods of quantitative investigation is the determination of the fractal dimension of the given railway network.

## ABOUT FRACTALS

The Polish mathematician Benoit Mandelbrot, the creator of fractal geometry, from the Latin word *fractus* (meaning: irregular, broken), coined the term fractal itself. Mandelbrot examined the mathematical objects found in natural forms [3], and showed that the shapes created by nature, which at first glance seem disordered, can actually be modeled using mathematical methods and follow specific laws. One such regularity is self-similarity. Although mathematicians had already created some of the methods for the mathematical characterization of planar and spatial shapes, Mandelbrot, by giving these shapes an independent name and classifying them in an independent scientific category, opened new horizons in the structural analysis methods of natural and artificial forms.

## THE CANTOR SET

In 1883, the German mathematician Georg Cantor published the method of generating the Cantor set, which was later named after him. Take an interval  $[0, 1]$ , from which we cut out the middle third in the first step. This results in two sections of one-third length. The Cantor set is obtained by repeating this procedure in each subsequent step, on each obtained section.

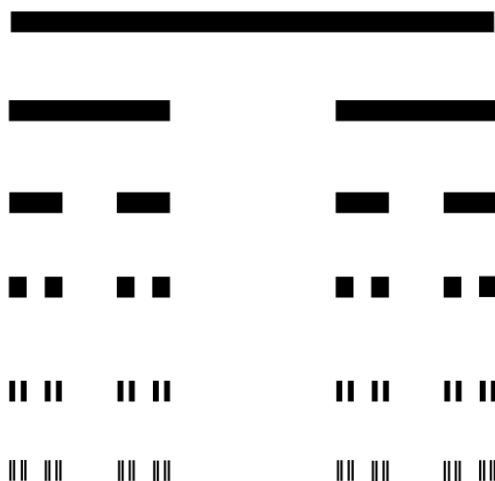


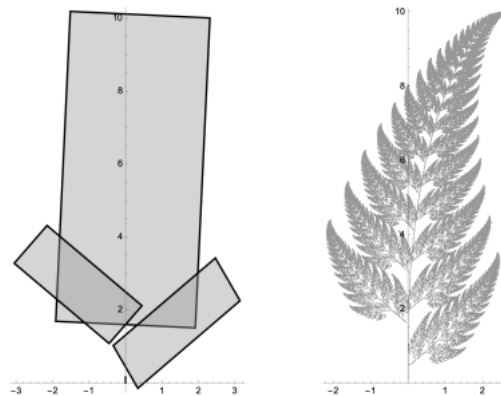
Figure 1. The first five steps of the Cantor-set [4].

Applying the Cantor set not to the number line, but to plane figures, and slightly modifying its algorithm, it is possible to create fractal objects similar to the patterns of plants. If we look at a fern, we can see that the shape of the whole plant is similar to the shape of the larger branches, the larger branches are similar to the smaller ones, and the smaller branches are similar to the leaves. As we examine the fern more and more closely, we find that it is self-similar to its larger-scale form in the examined scale. This allows us to use a mathematical model to produce a mathematical object very similar to a real fern. Figure 2 shows a real fern and a Barnsley fern [5], named after its creator, the British mathematician Michael Barsley.



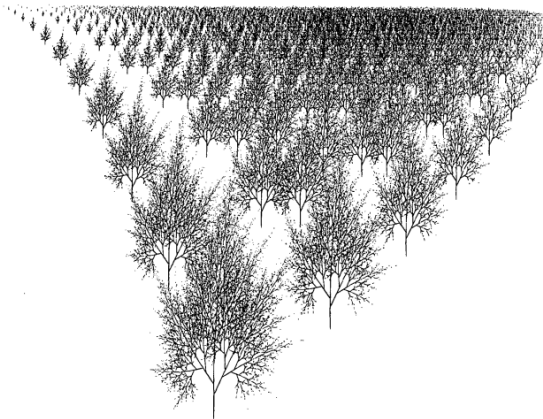
**Figure 2.** Real fern and the computer generated fractal fern [6].

The Barnsley fern is actually a Cantor set [7], of which we start from a rectangle, and at each step we create copies of it that are not equally reduced, rotated, and shifted in space during the iterations, Figure 3.



**Figure 3.** The first step of the Barnsley-fern and the result fractal [8].

This process can be used to create not only fern-like fractals, but also other plant-like fractals, such as the trees in Figure 4.



**Figure 4.** Forest of fractal trees [5].

## THE SIERPINSKI CARPET

A two-dimensional version of the Cantor set is the Sierpinski carpet, which was created by the Polish mathematician Waclaw Sierpinski in 1916. A Sierpinski carpet is created by taking a square and dividing it into nine congruent smaller squares, then erasing the middle square. Then we do the same with the remaining eight smaller squares and continue this procedure  $n$  times, Figure 5. Mathematical objects obtained in this way are miniature copies of themselves at all scales, so they are self-similar.

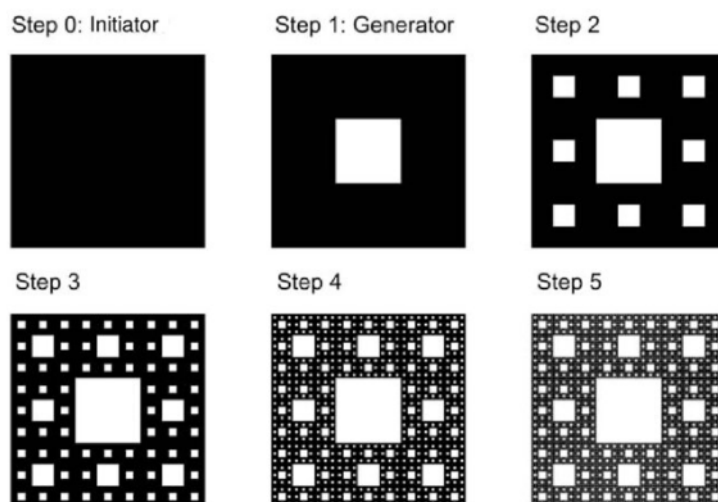


Figure 5. Generations of the Sierpinski carpet [9].

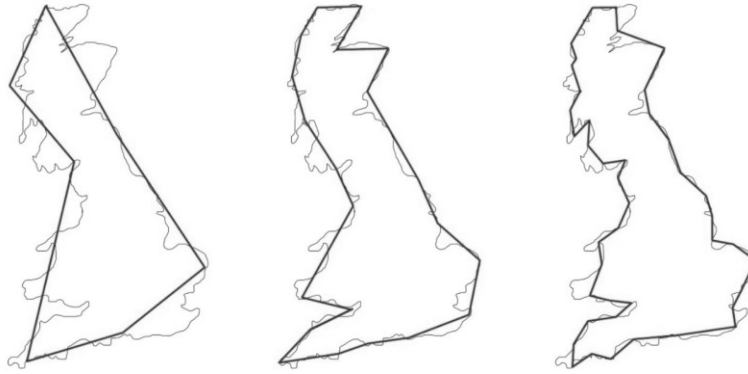
The algorithm of the Sierpinski carpet can be used for both planar and spatial shapes. One of its best-known forms is the Sierpinski triangle, which can be used to create fractal shapes similar to plants and mountainous landscapes by applying the rotations and ratio changes used in the case of the Barnsley fern, Figure 6.



Figure 6. Generations of the Sierpinski triangle [10] (left), and a Sierpinski mountain [11] (right).

## THE KOCH CURVE

One of the most well-known examples of the relationship between shapes created by nature and fractals is the *coastline problem*. English mathematician, physicist and meteorologist Lewis Fry Richardson became aware of an interesting problem. While investigating the possible causes of the war between the countries, he noticed that in the case of some neighboring countries, the countries give different lengths for the common border section [12]. Richardson showed that there are two reasons for the discrepancy. One is that it does not matter what precision measuring device is used. The more accurate the measuring device, the higher the value. The other is that even with a measuring device of the same accuracy, we get different values depending on where we start the measurement. This problem became known as the *coastline problem* [13]. The task is as follows: use a map and a measuring ruler to determine the length of Great Britain's coastline. The problem is caused by the fact that the obtained result is affected by the resolution of the map and the length of the ruler used in the measurement. In Figure 7, it is clear that different results are obtained on the same map, depending on the size of the ruler used.



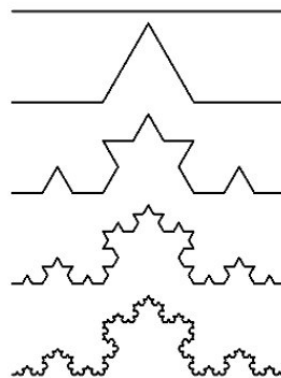
**Figure 7.** The coastal problem at the measurement [13].

Table 1, which presents the results of the measurements carried out by Peitgen [14], shows numerically how much the measurement accuracy affects the deviation of the results in a certain measurement range.

**Table 1.** Peitgen's measurement results [14].

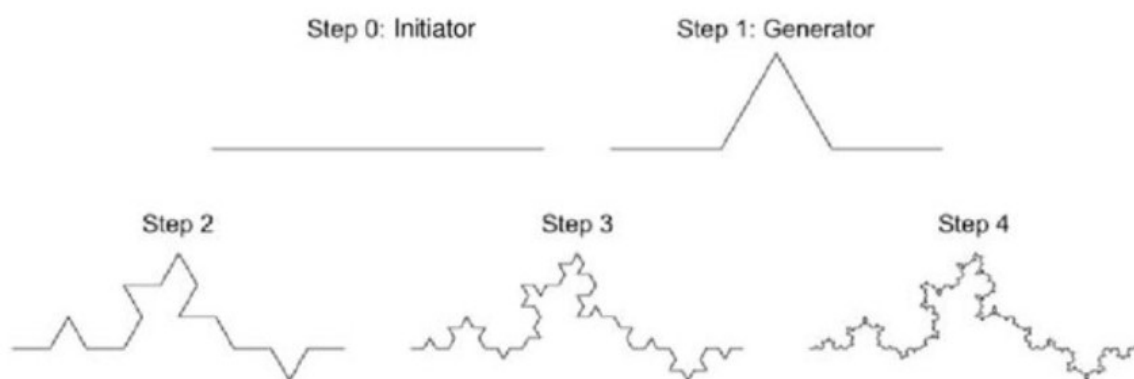
Measuring ruler length (km)	The length of the coast of Great Britain (km)
500	2600
100	3800
54	5770
17	8640

The shoreline problem is based on the Koch curve, the mathematical description of which was published by Helge von Koch in 1904 [15]. In the case of the Koch curve, we start from a section of unit length. In the first step, the section is divided into three equal parts, then the middle third is cut out, and the resulting gap is filled with two pieces of one-third length in such a way as to give a tent shape, Figure 8.



**Figure 8.** Generations of the Koch curve [16].

In the next step, the procedure at each stage is repeated, and then the iteration is continued at any step. In each step, the resulting shape is built from a reduced version of the previous curve, so it is self-similar at every scale, i.e. with increasing measurement accuracy, we encounter the same structure as with the previous measurement accuracy. It can be seen that the Koch curve is self-similar at any scale. At the same time, it is also easy to notice that a coast follows an irregular shape. In order to get a shape more similar to a coast, the regularity of the Koch curve must be reduced. This can be achieved by randomly supplementing the cut sections from above and below with tent shape, thus obtaining a random version of the Koch curve, Figure 9.

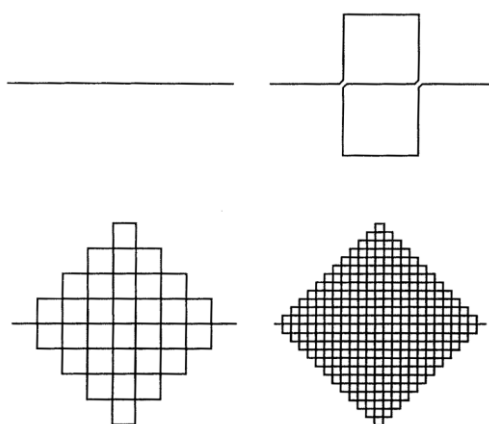


**Figure 9.** Generations of the random Koch curve [9].

During the iterations, the randomly generated Koch curve transforms from a straight line to a broken line, which is much more similar to a real coast.

### THE FRACTAL DIMENSION

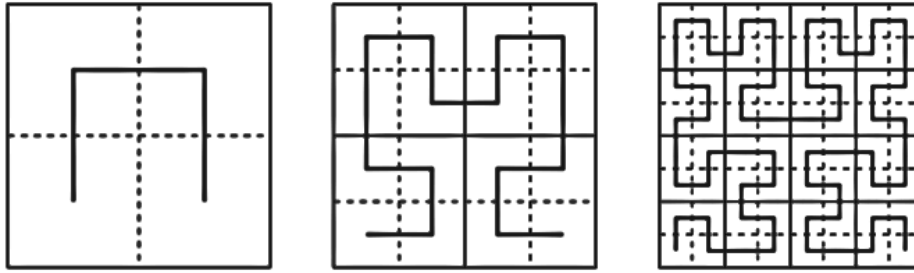
A straight line can be characterized by its length, i.e. that is a one-dimensional geometric shape. In the Euclidean approach, no matter how fragmented a curve is, it can only be characterized by its length, since its thickness does not exist, so it cannot have an area. In the same way, two-dimensional shapes can only have area, the concept of length cannot be interpreted for them. The Peano curve shown in Figure 10 consists of straight lines, but if the iterations were continued to infinity, it would pass through every point of a square. The area of the Peano curve cannot then be zero, in the case of infinite iteration it must be equal to the area of the square it passes through [7].



**Figure 10.** Generations of the Peano curve [17].

In Euclidean geometry, shapes can only have an integer number of dimensions. A point is dimensionless, a line is one-dimensional, plane shapes are two-dimensional, and spatial shapes are three-dimensional shapes. However, the Peano curve and other space-filling fractal shapes [18] such as the Hilbert curve (Figure 5) seem to violate the dimensional rules of Euclidean geometry. Although they start as lines, they penetrate the plane more and more during the iterations.

In order to be able to determine how much each fractal covers the space available to them, it is necessary to examine how they behave during the iterations. The Cantor set starts out as a continuous section and then gets smaller and smaller in length. The Koch curve fits an infinite length into a finite area, but never fills the space and never becomes two-dimensional. The Peano curve, on the other hand, increasingly covers the available plane, and its dimension converges to the two.



**Figure 11.** The first three iterations of the Hilbert curve [18].

The starting point for determining the fractal dimension is self-similarity. If we examine the relationship between the reduction rate ( $r$ ) and the number of intervals needed to cover ( $N$ ) in the case of a geometric shape, then in the case of a section we get that

$$N_r = \frac{1}{r} = r^{-1} \quad (1)$$

because half of the sections of that length are twice as long, a third is three times as long, and a fifth is as long you need five times as much. In the case of a square, the number of squares required to cover it increases quadratically,

$$N_r = \frac{1}{r^2} = r^{-2} \quad (2)$$

and in the case of a cube, 8 of the half-sized cubes, 27 of the third-sized cubes, and 125 of the fifth-sized smaller cubes, required, therefore,

$$N_r = \frac{1}{r^3} = r^{-3} \quad (3)$$

relationship exists. Written in a general form, we get the relation

$$N_r = \frac{1}{r^{D_S}} = r^{-D_S} \quad (4)$$

where  $D_S$  is the similarity dimension of the examined shape. If we extend this to self-similar shapes, after some mathematical transformations, we get the formula

$$D_S = \frac{\log N_r}{\log \frac{1}{r}} \quad (5)$$

which can be used to determine the dimension of a shape [7]. Based on these, we obtain  $D_S(\text{Cantor}) = 0,63$  for the Cantor set,  $D_S(\text{Koch}) = 1,26$  for the Koch curve, and  $D_S(\text{Sierpinski}) = 1,89$  for the Sierpinski carpet. In the case of a fractional dimension value, the examined shape is a fractal, but in the case of an integer dimension value, we cannot say for sure that it is not a fractal. Such are space-filling fractals, such as the Peano curve, for which  $D_S(\text{Peano}) = 2$ . The existence of self-similarity is clear for basic geometric shapes, but real objects are not made up of completely identical shapes, so self-similarity exists only up to a certain scale. In order to be able to determine the dimension of an arbitrary shape, another method must be used.

## THE BOX COUNTING METHODE

If we want to determine the fractal dimension of a complex, irregular shape, we cannot use the similarity dimension method, because the shape is not made up of smaller copies of its own shape. In such a case, one of the applicable methods is the box counting method. In doing so, we place a square grid with smaller and smaller divisions on the examined shape, and then examine the relationship between the decrease in the size of the grid and the number of cells required to cover the shape. For example, if the shape is covered with a  $6 \times 8$  grid at the start,

and the grid size is halved in the next step, we get a  $12 \times 16$  grid. The number of cells needed to cover the shape increases, but not necessarily double. As the grid cell size decreases, there is a proportionality between the number of cells needed to cover and the grid cell size. Similar to regular shapes, this proportionality can be considered a dimension. Since 8, 64, 512, ... cells are needed to cover the  $1/3$ ,  $1/9$ ,  $1/27$ , ... cell size Sierpinski carpet, it can be concluded that in the case of the box counting method, the relationship

$$N_r \propto r^{-D_b} \quad (6)$$

must be fulfilled in the case of a fractal. This is consistent with the general fractal dimension relation. With a simple mathematical rearrangement, we get the formula

$$D_b = \frac{\log N_r}{\log \frac{1}{r}} \quad (7)$$

where  $N_r$  is the number of cells that have a point in common with a point in the shape,  $r$  is the cell size, and  $D_b$  is the box counting dimension. Based on the above, with  $D_S = D_b$  from which it follows that it gives the same value for deterministic and randomized fractals, so randomization does not affect the fractal dimension. It follows from these that box counting can be used to determine the fractal dimension for any two-dimensional shape [7].

## MATERIALS AND METHODS

The research analyzed the networks of railway tracks that can be used for passenger traffic in Hungarian counties. For this, the “Railway network of Hungary by county” map was used, shown in Figure 12, published publicly by MÁV-START and valid at the time of the research. The map shows the railway tracks that can be used for passenger traffic with a national and county pass, broken down by county at a scale of 1 : 500 000.

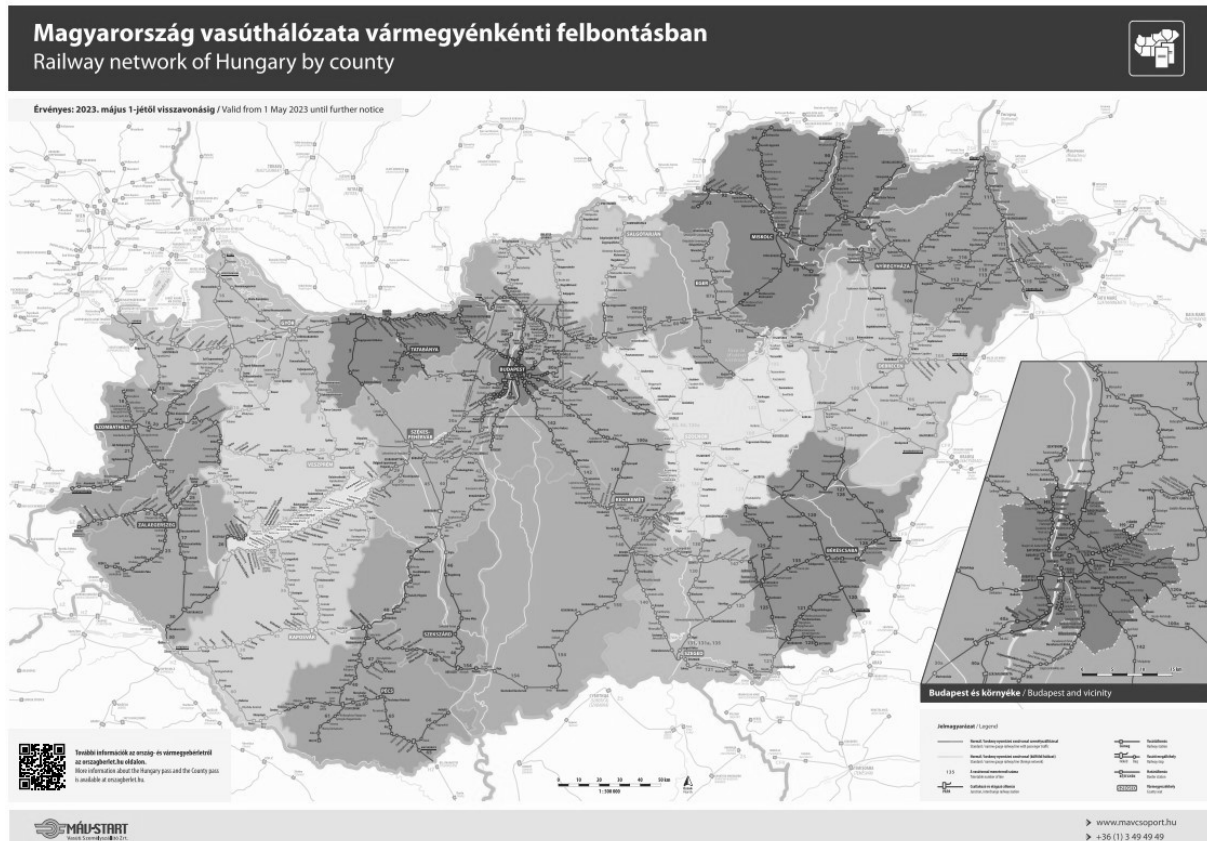


Figure 12. Railway network of Hungary by county [19].

Based on the map, pictures of the individual counties were taken at the same scale, and in these pictures the fractal dimension was determined with the FrakOut software [20]. The software determines the fractal dimension of the images using the box counting method. In the first step, it places a grid consisting of a single cell on the image, and then reduces the size of the cell grid by half in each subsequent step. In the software, the cells covering the shape must be marked with a mouse or a digital drawing tool. If a given cell has at least one pixel point in common with the analyzed image, it must be marked as a covering cell in the image. In the box counting process, the resolution of the images used limited the number of steps in which the grid size could be refined, because the thickness of the drawn lines also determines the detail of the image [21]. On the railway map used, this could be done in six steps, because in the seventh step the thickness of the line became comparable to the cell size of the grid used. Thus, the resolution of the smallest grid in the analysis was 1,25 km.

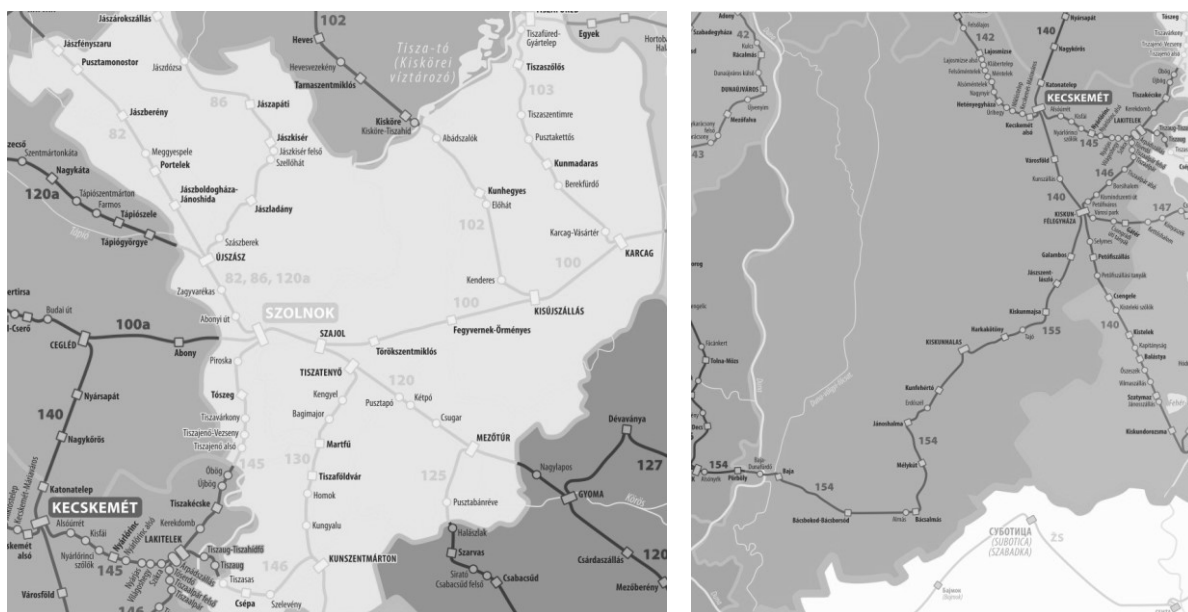
## RESULTS

The results of the fractal dimension analysis performed with the software are shown in Table 2. Among the railway networks of the counties, Pest county has the largest fractal dimension, which is understandable, since it is centrally located within Hungary and is located in the vicinity of the capital, Budapest. Due to its spatial location, the railway line starting from any county must eventually pass through Pest county and end in the capital. These railway lines cover the county more densely than in any other county. The fractal dimension of Budapest is the largest, which also results from the fact that within the capital there are even railway lines connecting the wing lines, which increase the density of the shape.

**Table 2.** The fractal dimensions of Counties.

County	Fractal dimension ( $D_b$ )
Bács-Kiskun	1,24
Baranya	1,26
Békés	1,37
Borsod-Abaúj-Zemplén	1,27
Csongrád-Csanád	1,32
Fejér	1,37
Győr-Moson-Sopron	1,34
Hajdú-Bihar	1,34
Heves	1,28
Jász-Nagykun-Szolnok	1,38
Komárom-Esztergom	1,27
Nógrád	1,24
Pest	1,43
Somogy	1,36
Szabolcs-Szatmár-Bereg	1,33
Tolna	1,24
Vas	1,29
Veszprém	1,36
Zala	1,30
Budapest	1,48

Putting the fractal dimensions of each county in order, Jász-Nagykun-Szolnok county has the second highest value ( $D_b = 1,38$ ), and Bács-Kiskun, Nógrád and Tolna counties have the lowest value ( $D_b = 1,24$ ). If we look at the railway map of these counties, we can clearly see that the railway network of Jász-Nagykun-Szolnok county fills the available two-dimensional space better than, for example, the railway network of Bács-Kiskun county, Figure 13.



**Figure 13.** Railway network of Jász-Nagykun-Szolnok county (left), and Bács-Kiskun county (right).

The difference in individual values can be attributed to several factors. It is influenced by the topography and the historical development of the railway network, which is related to the location and economic role of the settlements of the time. It should also be taken into account that the territory of Hungary was larger when the railway network was established, and the location of the settlements belonging to Hungary at that time also had an impact on the structure of the current network.

It is important to consider that the present research only examined the passenger transport railway lines in Hungary that can be used with a national and county pass, but the entire railway network of Hungary has more lines than this. In the case of Hungary, the analysis gave a fractal dimension of 1,67, which is a slightly smaller value than Fokasz's value of 1,74, but he examined the fractal dimension of the entire Hungarian railway network, taking into account all lines [21]. The close results confirm that the findings of this research are relevant. The fact that the fractal dimension of the railway network of the counties is always lower than that of Hungary stems from the peculiarity of the box counting method, from the fact that the examined area must be covered with a square grid of  $n \times m$ . Since the grids covering individual counties also extend to areas outside the county, the unnecessarily covered areas add up and reduce the fractal dimension obtained for Hungary. In this case, the county-by-county box-counting method is only suitable for comparing the investigated counties with each other, but not for the entire national network.

## CONCLUSIONS

The research determined the fractal dimension of the railway networks of the Hungarian counties using a quantitative method. This created an opportunity to compare them from the point of view of how well they cover the territory of the given county, i.e. to what extent they affect the possible points of the county's territory. Based on the results, it can be concluded

that the territory of Pest county surrounding the capital is networked with the largest proportion of railway lines, which is not surprising, since the capital is the central structural element of the network. It has also been proven that the fractal dimension of Budapest is the largest, since the capital is geographically located within Pest county. One of the reasons is that the large number of railway lines are concentrated in an even smaller area due to its central location. The distribution of the counties with the largest fractal dimension is almost uniform in the eastern and western parts of the country, while the counties with the smallest fractal dimension are located in the central part of the country, in the north and south, such as the counties of Nógrád, Heves, Tolna and Baranya. Based on the results of the research, it can be said that the railway networks of the Hungarian counties involved in passenger transport show a fractal nature, and their fractal dimensions show a small standard deviation, which indicates that the national railway coverage is homogeneous. Further geological, economic and sociological research would be necessary to reveal the causes of the differences, which go beyond the purpose and possibilities of this research. Nevertheless, the obtained results can be used in the planning of the railway network development from the aspects of safety, economy and traffic optimization.

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