ISSN 1846-6168 (Print), ISSN 1848-5588 (Online)
https://doi.org/10.31803/tg-20240517182356

Subject review
Received: 2024-05-17, Accepted: 2024-05-22

Dynamic Model of Spur Gear with Friction and Crack in Tooth Root

Aleš Belšak*, Matej Ozebek, Mario Hirz

Abstract: A mathematical model developed to support investigation of pairs of gears is presented. It includes the effects of actuation, the loads, the changing tooth stiffness, and the changing friction between tooth flanks during engagement. Friction between the tooth flanks generates an alternating non-symmetrical periodic function and the behaviour during the contact. The system actuation is modelled for different sources and the tooth stiffness. The mathematical model of the mechanical system of gear has the form of a system of two non-homogeneous nonlinear second-order differential equations.

Keywords: dynamic analysis; gear fatigue crack; gears; surface friction

1 INTRODUCTION

A number of measurements and numerical analyses is necessary for the diagnostics of mechanical systems prior to developing advanced algorithms for predicting damages and the remaining life-time of the system. This applies exemplarily for gear sets. Here, it is required to use mathematical models for analyses and optimization, which depend to a great extent on the quality of the applied diagnostics and the perception of the mechanical system.

This paper deals with a mathematical model that is developed to support investigation of pairs of gears. The accuracy of the model depends on the modelling requirements and the quality of observation, and available mathematical tools and software play an important role when developing relevant models.

In the past, various mathematical models have been developed by a number researchers, primarily focusing on gear dynamics, considering a multitude of different influencing quantities.

Exemplarily, a nonlinear time-varying dynamic model of a spur gear pair is studied by [1] for the prediction of modulation sidebands, which are caused by different modulation internal excitation. They pay attention to backlash, modulation time-varying mesh stiffness, and modulation transmission error.

The impact of dynamic backlash and rotational speed on the six-degrees-of-freedom model of the gear system with the time-varying meshing stiffness is studied by [2]. Here, it is possible to determine the relationship between dynamic backlash and centre distance.

A dynamic model of three shafts and two pairs of gears in mesh, with 26 degrees of freedom, with the effects of variable tooth stiffness, pitch and profile errors, friction and a localized tooth crack on one of the gears is presented by [3].

A two-stage planetary gearbox is presented by [4]. They establish a system coupling torsional dynamical model, which takes into account the time-varying mesh stiffness, friction forces and interstage coupling factors. The friction and lubrication states are classified in order to analyse the calculation of friction coefficients under various conditions.

The dynamic performance of the gear is dealt with by [5]. In the dynamic model, the profile deviation between involute gear and microsegment gear is considered as a displacement excitation.

An analytical model for helical gears is proposed by [6]; their model characterizes the contact plane dynamics and captures, owing to the sliding friction, the velocity reversal at the pitch line.

A dynamic wear prediction methodology is described by [7] to research the coupling effects between surface wear and dynamics of spur gear systems. Here, a quasi-static wear model and a translational-rotational-coupled nonlinear dynamic model are combined.

On the basis of the Coulomb model, [8] deal with an analytical analysis of tooth friction excitations in errorless spur and helical gears.

In their paper, [9] deal with a new gear surface roughness induced noise source model, with the sliding contacts between meshing gear teeth being considered. They use a linear time-varying model of a spur gear pair (with sliding friction) for the calculation of the instantaneous sliding velocity between pinion and gear teeth.

The dynamics of the gear pair is connected with sliding friction and a crack in the tooth root is associated with different loading level by [10].

As an enhancement of the topics treated in the different cited works, in our work we address the impact of wear upon the vibrations of gears. For this purpose, a mechanical model with reduced masses is developed. In this model, the function of friction is defined by taking into consideration the knowledge of the engagement conditions and theories of friction in the Hertz contact. Besides, function of stiffness changes along the line-of action at the gear tooth surface and an error function in the form of a fatigue crack in the tooth, by which the stiffness is reduced, are added to the model..

2 GEAR PAIR MODEL

The engaged pinion and gear can be substituted by a variety of alternative mechanical models (oscillating models), thus allowing observations of responses by the dynamic system. Different approaches and forms of alternative mechanical models can be found in literature [11-14].

When it comes to modelling gears, an approach based on reduced masses and displacements is often used. The reduced masses are calculated on the basis of the moment of inertia of the gear and the rotations are transformed into translations. The gear pair is modelled as a system of non-homogeneous, non-linear second-order differential equations – a system of equations of motion. Different forms of excitation, damage to the tooth and the mechanism of friction at the contact point between the tooth flanks in the system's pinion and gear geometry are represented in the selected model. The stiffness of the tooth during engagement and the engagement conditions at a rotational frequency are changed to excite the model internally. The first model involves linear contact stiffness of tooth flanks, whereas the improved model involves non-linear stiffness and it depends on the point of contact of the tooth flanks. The connection between the discs is modelled as a shock absorber with stiffness K and damping C that transmit the drive torque in the form of tangential force from pinion to gear. Both masses are rotated by the drive torque within the selected time-varying form. The model of pinion and gear is divided into two free masses with all the external influences, Fig. 1. The translation x of the spring and damper and the rotation θ of the gear are included in the differential equations. The equilibrium Euler's dynamic equation is defined for each gear. Two non-homogeneous second-order differential equations form the system of equations of motion for the gear pair. Drive torque M_{p1} and the output torque M_{p2} provide excitation. Below is the system of equations of motion [11]:

$$\begin{aligned} J_1\ddot{\theta}_1 + M_{\text{tr1}} + C(\dot{x}_1 + \dot{x}_2)R_i + K(x_1 + x_2)R_i = M_{\text{p1}} \\ J_2\ddot{\theta}_2 + M_{\text{tr2}} + C(\dot{x}_1 + \dot{x}_2)R_i + K(x_1 + x_2)R_i = M_{\text{p2}} \end{aligned} \tag{1}$$

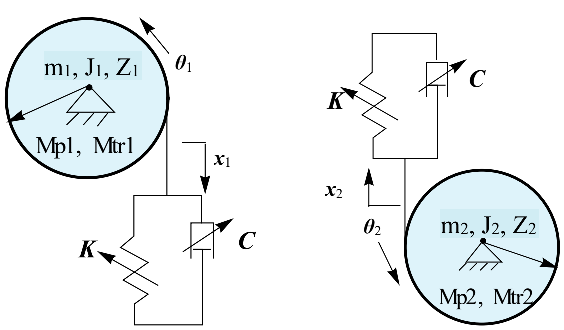

Figure 1 The model of gears

It is possible to divide the model of the pinion and gear set, Fig. 2, into two single masses, each of them as a body with rotational inertia J_{i} and with all external influences, depending on the translational displacements $x_1=\partial\theta_1 R_1$ and $x_2=\partial\theta_2 R_2$, damping C and the stiffnesses of springs K, acting on the radius R_i. In polar coordinates, the second derivative of the rotation is the angular acceleration of the rotational body. M_{tr} refers to the friction in the bearing.

For each body with external forces and torques acting on it, it is possible to define Equilibrium Euler dynamic Eq. (2). Considering small rotations in the same rotational directions as the two bodies, Fig. 2, it is possible to write as follows [10]

$$\begin{aligned} J_1\ddot{\theta}_1 + M_{\text{tr1}} + C(\partial\dot{\theta}_1 R_1 - \partial\dot{\theta}_2 R_2)R_1 + K(\partial\theta_1 R_1 \pm \partial\theta_2 R_2)R_1 = M_{\text{p1}} \\ J_2\ddot{\theta}_2 + M_{\text{tr2}} + C(\partial\dot{\theta}_2 R_2 - \partial\dot{\theta}_1 R_1)R_2 + K(\partial\theta_2 R_2 \mp \partial\theta_1 R_1)R_2 = M_{\text{p2}} \end{aligned} \tag{2}$$

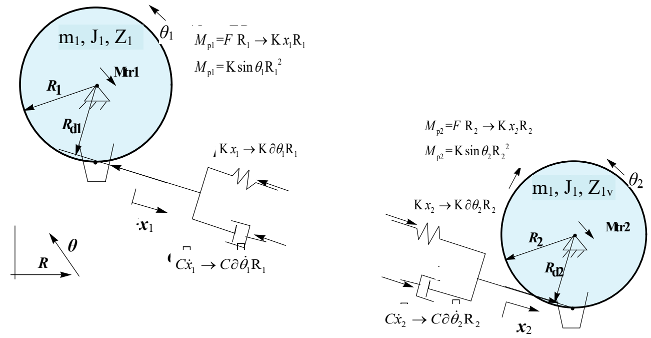

Figure 2 Forces and torques acting on gears

The gear pair is modelled precisely when the following factors are taken into consideration: rotation, the time-dependent changes of the drive torque M_{p1}, the output torque M_{p2}, damping C and the stiffness K of the tooth pair, as a real gear pair. If the tooth pair's stiffness, area pressure, Hertz contact relationship and the friction model during the contact are taken into consideration, the level of the model is improved.

$$\begin{aligned} J_1\ddot{\theta}_1 = M_{\text{p1}} - M_{\text{tr1}} \pm F_{\text{trz1}}R_{\text{trz1}} - C\partial\dot{\theta}_1 R_1 R_1 + C\partial\dot{\theta}_2 R_2 R_1 - K\partial\theta_1 R_1 R_1 + K\partial\theta_2 R_2 R_1 \\ J_2\ddot{\theta}_2 = M_{\text{p2}} - M_{\text{tr2}} \pm F_{\text{trz2}}R_{\text{trz2}} - C\partial\dot{\theta}_1 R_1 R_2 + C\partial\dot{\theta}_2 R_2 R_2 - K\partial\theta_1 R_1 R_2 + K\partial\theta_2 R_2 R_2 \end{aligned} \tag{3}$$

The above system of equations of motion (Eq. 3) is edited so that excitations of the external torques M_{p1} and M_{p2} appear on the right, whereas all the other variables are on the left. In the model, K stands for the replacement stiffness coefficient and C for the total damping of the gear pair.

$$\begin{aligned} J_1\ddot{\theta}_1 = M_{\text{p1}} - M_{\text{tr1}} \pm F_{\text{trz1}}R_{\text{trz1}} - C\partial\dot{\theta}_1 R_1 R_1 + C\partial\dot{\theta}_2 R_2 R_1 - K\partial\theta_1 R_1 R_1 + K\partial\theta_2 R_2 R_1 \\ J_2\ddot{\theta}_2 = M_{\text{p2}} - M_{\text{tr2}} \pm F_{\text{trz2}}R_{\text{trz2}} - C\partial\dot{\theta}_1 R_1 R_2 + C\partial\dot{\theta}_2 R_2 R_2 - K\partial\theta_1 R_1 R_2 + K\partial\theta_2 R_2 R_2 \end{aligned} \tag{4}$$

For solving the system of equations of motion (Eq. 4), direct numerical integration is used. The torque transfer between the engaged teeth involves the force from tooth to tooth, which includes friction.

The friction force direction is tangential to the flanks of the tooth at the current point of engagement. On the basis of the engagement conditions it is clear that the tooth force is oriented in the direction of angle α with respect to the engagement curve. The force acting on the tooth is not constant anymore. It, however, varies in mode (single, double) and the position and direction on the tooth flank and the radius of the associated engagement point. If the position of associated tooth engagement point changes, the tooth stiffness K changes in accordance with the position of the engagement point. Simultaneously, the friction, representing the damping, becomes an additional exciting force of the mechanical system.

Normally, pinion and gear work have no backlash, and the bearings are not completely rigid. This forms additional sources of excitation. Inertia moments for both pinion and gear are necessary for the equations of motion of the developed system. The following data is considered when calculating the inertia moments: material density ρ, modulus m, the number of teeth Z and width b and the coefficient of profile shifting x [10].

$$J_i = \left(\frac{\pi Z_i^4}{32} + \frac{B_0 x_i}{10^3} \right) \left(\frac{m_t}{10^3} \right)^4 \cdot \rho \cdot b \tag{5}$$

$$B_0 = -2818{,}2 + 267{,}74 Z_i - 837.38 Z_i^2 + 0.8443 Z_i^3 \tag{6}$$

3 STIFFNESS OF THE TEETH PAIR

On the pitch diameter, the contact gear force can be split into tangential and normal components. In the mechanical model, the actual force acts on the spring K and the damper C on the pitch radius.

The position of the engagement point changes on the tooth flank resulting in changes of the combined tooth stiffness with time and position. In the substituted model, the spring K represents the stiffness of the teeth pair depending on the elasticity of the tooth flank and the local elasticity of the contact surface. Four serially connected springs are applied to model the serial set-up of the tooth pair during the engagement. The stiffness of this spring system is calculated as a harmonic mean.

$$\frac{1}{k_{\text{povp.}}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_{1\text{hz}}} + \frac{1}{k_{2\text{hz}}} \rightarrow k_{\text{povp.}} = \frac{k_1 k_2 k_{1\text{hz}} k_{2\text{hz}}}{\left(k_1 + k_2 + k_{1\text{hz}} + k_{2\text{hz}}\right) n} \tag{7}$$

The stiffness of the tooth at any radius between the root and the top land being k_1 and k_2 (n refers to the number of teeth engaged), whereas k_{hz} stands for the surface stiffness. Thus, the Hertz surface stiffness, which is given by [14], represents a part of the overall stiffness. The stiffness of a pair of contact surfaces is assumably constant during the entire period of engagement. The modulus E_y and the Poisson ratio ν as well as the shape of the contact define the surface stiffness h_{hz}.

$$k_{\text{hz}} = \frac{\pi \cdot E_{\text{y}}}{4(1 - \nu^2)} \tag{8}$$

The material, loads and geometries of the surfaces in contact additionally influence the shape of the contact.

4 FRICTION FORCE AND TORQUE

The transferred torque acts on the tooth with a pulsating rectangular variable wave force during the operation. The drive torque is represented by the dynamic forces.

$$M_{\text{p1}} = (F_{\text{dinA}} + F_{\text{dinB}}) r_{\text{ON2}} \tag{9}$$

The friction in the system includes the friction in the bearings of the shaft and the friction at the Hertz contact point produced between the tooth flanks. The mechanisms of friction and relative gliding, and the lubrication within the dynamic model of the gear pair are taken into consideration.

In Eq. (10) the average sliding velocity is calculated at any point of engagement Y on the radius r_Y. In the equation, the rotation speed v at the pitch radius, the base radius R_b, and the pressure angle Φ are considered.

$$v_{\text{sr-dsr}} = \frac{v_{\text{obodna}}}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) R_{\text{b1}} \beta_{\text{r1}} \tag{10}$$

A function for describing the friction μ_A at a single point of engagement was developed by [12]:

$$\mu_{\text{A}} = \frac{v_{2\text{A}} - v_{1\text{A}}}{F_{\text{A}} t_{\text{A}}} \int_{-a_0}^{+a_0} \eta_{\text{A}} \, \mathrm{d}x \tag{11}$$

F_A stands for the dynamic force transmission, t_A for the thickness of the oil film, v for the relative sliding velocity, a_0 for the radius of the Hertz contact and η_A for the dynamic viscosity of the lubricant.

We took the following factors into consideration when developing the dynamic model of gear and pinion. The change of the tooth force, sliding with direction of changing, rolling friction between the tooth flanks and in the bearings, and an unsatisfactory driving torque.

The model is primarily stimulated by the engagement and with the altering tooth stiffness; however, several facts need to be observed, otherwise a problem related to the determination of the damping occurs, which must be set up experimentally by observation. Friction forces for double (F_{trA}) and single (F_{trC}) engagements are:

$$\begin{aligned} F_{\text{trA}} &= \mu F_{\text{nA}} \\ F_{\text{trC}} &= \mu F_{\text{nC}} \end{aligned} \tag{12}$$

The torque of sliding friction can be as follows:

$$M_{\text{trY}} = F_{\text{trY}} r_{\text{N1Y}} \tag{13}$$

The friction torque distribution for typical engagement of a gear pair is shown in Fig. 3.

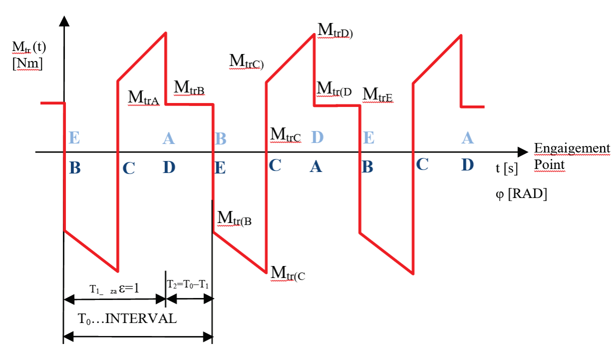

Figure 3 Friction torque at the intervals of engagement of tooth pair

A repetitive engagement behaviour for the gear calculated with Wolfram Mathematica is presented in Fig. 4. The frequency of engagement is taken into consideration and friction torque is calculated with a constant friction coefficient of 0.5 at constant torque transfer of 20 Nm. [10]

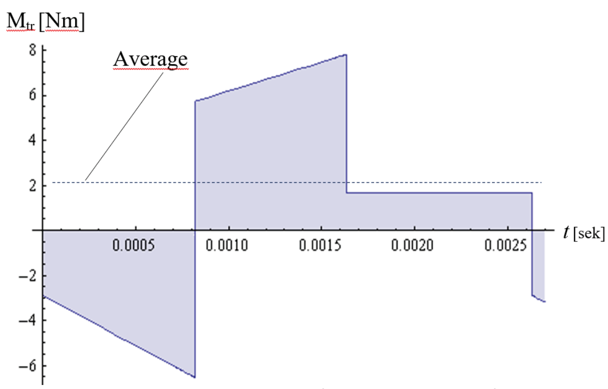

Figure 4 Friction torque distribution for one engagement of tooth pair

5 SOLVING THE MATHEMATICAL MODEL

A system of two second-order, non-linear, non-homogeneous differential equations forms equations of motion for the extended model of pinion and gear. The system of differential equations is prepared for solving, using numerical integration with software Wolfram Mathematica. By inserting numerical values for real pinion and gear from the data in Tab. 1 and the definition of the initial conditions.

Table 1 Data of the pinion and gear [10]

Parameters of the gear:	
$m_n = 4$	module
$z_1 = 19$	number of teeth for the pinion
$z_2 = 34$	number of teeth for the gear
$i = 1.789$	gear ratio
$n_1 = 20\ s^{-1}$	revolutions for the pinion shaft
$M_{p1}(t) = 20$ Nm	drive torque
$M_{p2}(t) = i \cdot M_{p1}(t)$	output torque
$b_1 = b_2 = 0{,}02$ m	tooth thickness
$r_1 = 0.03$ m	bearing radius
$l_1 = 0.02$ m	width of the bearing
$\alpha_n = 20°$	pressure angle
$\beta = 0°$	helix angle
$\varepsilon = 1.61$	overall degree of engagement
$a = 0.106$ m	wheel base
$c = 0.08$	damping coefficient (the tooth flanks)
$k = 462.1106$ N/m	average tooth pair stiffness
$\rho = 7800$ kg/m^3	material density
$Ey = 210000$ MPa	Yield point of the tooth material
$\nu = 0.3$	Poisson's ratio
$\sigma_{el} = 500$ MPa	allowed stress for the tooth material
$J_1 = 0.00051095054$ kgm^2	inertial moment of the cylinder – pinion
$J_2 = 0.00523938275$ kgm^2	inertial moment of the cylinder – gear
$m_1 = 0.708$ kg	mass of the pinion
$m_2 = 2.266$ kg	mass of the gear

In our model, the influence of a crack in a tooth root is used as a reduction of stiffness of one gear tooth of the pinion.

We acquired responses of gear rotation from the numerical integration of differential equations as presented in Fig. 5. The response of the convergent transition responding at each change and a semi-stationary form of repetition are presented in the diagram. An increase in rotation and change of response indicate the single change in the stiffness of the gear tooth with a crack in tooth root. The second square wave shows the response of a tooth pair with a crack in a tooth root with tooth stiffness reduced by 35 %, all other responses are from the tooth pair without cracks in tooth root.

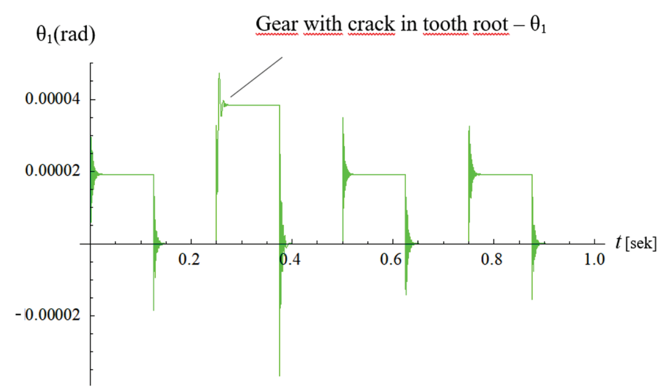

Figure 5 Detailed response of gear rotations of the considered model

Due to damping or energy dissipation of a non-conservative model, the amplitude is reduced. In accordance with the physical properties, any decreasing stiffness of the tooth causes an increase of the amplitude, whereas the oscillation frequency does not change.

In Fig. 6, Fig. 7 and Fig. 8, the tooth gear rotation under the same operating parameters and different lengths of fatigue cracks is presented, reducing tooth stiffness by 10 %, followed by 20 % and finally by 35 %.

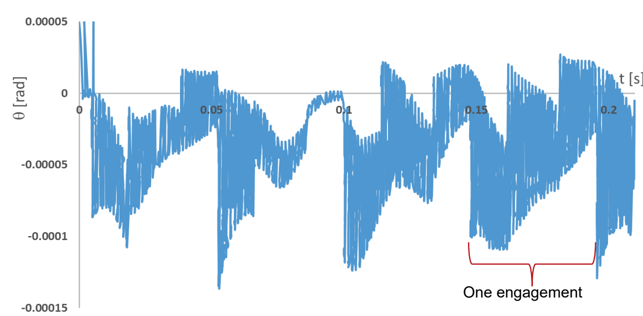

Figure 6 Tooth gear rotation with a 10 % reduction of tooth stiffness (M_{p1} = 130 Nm)

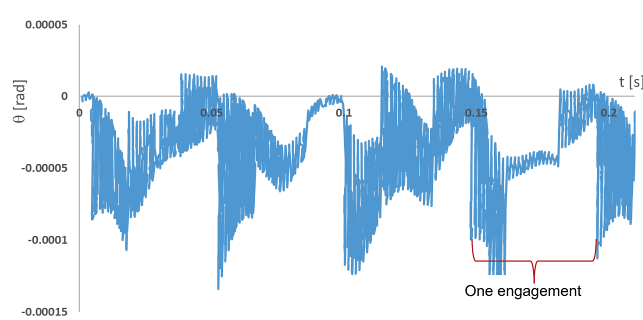

Figure 7 Tooth gear rotation with a 20 % reduction of tooth stiffness (M_{p1} = 130 Nm)

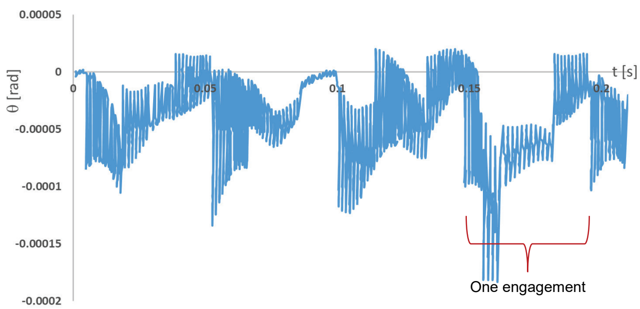

Figure 8 Tooth gear rotation with a 35 % reduction of tooth stiffness (M_{p1} = 130 Nm)

In Fig. 6 to Fig. 8 it is shown how a damaged tooth influences the tooth gear rotation – with the same parameters, being the drive torque, friction torque and variable stiffness, which represent additional excitation mechanisms applied in the mathematical model. If we observe the part of the diagram presenting the engagement of the gear pair with a crack in the tooth root, it is evident that in case of an increasing crack, which reduces tooth stiffness, gear rotation amplitude is redistributed in time domain. With reduction of tooth stiffness amplitude, the tooth gear rotation is increased.

6 CONCLUSION

The paper introduces the development of a mathematical model of a spur gear. The model includes a mathematically described variable excitation resulting from a friction model at the contact point of the tooth flanks. The shapes of the excitations make it possible to compare the model and the responses of the real gear drives. The excitation with input and output torque, variable stiffness of the gears, damage of a pinion with a fatigue crack in the tooth root, variable friction and damping are used in the replacement mechanical model of the gear pair. The nonlinear differential systems of equations are solved by numerical integration. As a result, the fluctuating nature of friction between the teeth flanks is confirmed by the developed and introduced excitation function of friction. The obtained results of the model represent the basis for comparing the results acquired with measurements, where it is difficult to locate a fault on the gear due to various influences. At the same time, the introduced model enables a more efficient search for suitable methods for noise reduction or signal filtering of measured signals in the course of gear investigation and optimization.

7 REFERENCES

[1] Wang, Z., Zhang, L., Luo, Y. & Chen, C. (2017). Dynamic Model of Spur Gear Pair with Modulation Internal Excitation. *International Journal of Rotating Machinery, 2017*, 1-8. https://doi.org/10.1155/2017/1264904

[2] Liu, Jie, Liu, S., Zhao, W. & Zhang, L. (2019). Dynamic characteristics of Spur Gear Pair with Dynamic Center distance and backlash. *International Journal of Rotating Machinery, 2019*(11), 1-9. https://doi.org/10.1155/2019/2040637

[3] Jia, S., Howard, I. & Wang, J. (2003). The dynamic modeling of multiple pairs of spur gears in mesh, including friction and geometrical errors. *International Journal of Rotating Machinery, 9*(6), 437-442. https://doi.org/10.1155/s1023621x03000423

[4] Xiao, Z., Chen, F. & Zhang, K. (2021). Analysis of dynamic characteristics of the multistage planetary gear transmission system with Friction Force. *Shock and Vibration*, *2021*, 1-10. https://doi.org/10.1155/2021/8812640

[5] Xiong, Y., Huang, K., Wang, T., Chen, Q. & Xu R. (2016). Dynamic Modelling and Analysis of the Microsegment Gear, *Shock and Vibration, 2016*, 1-13. https://doi.org/10.1155/2016/9691647

[6] He, S., Gunda, R. & Singh, R. (2007). Inclusion of Sliding Friction in Contact Dynamics Model for Helical Gears. *Journal of Mechanical Design, 129*(1), 48-57. https://doi.org/10.1115/1.2359474

[7] Yang, Y. & Zhang, J. (2016). Investigation on coupling effects between surface wear and dynamics in a spur gear system. *Tribology International, 101*, 383-394. https://doi.org/10.1016/j.triboint.2016.05.006

[8] Velex, P. & Sainsot, P. (2002). An analytical study of tooth friction excitations in errorless spur and helical gears. *Mechanism and Machine Theory, 37*(7), 641-658. https://doi.org/10.1016/S0094-114X(02)00015-0

[9] Kim, S. & Singh, R. (2007). Gear surface roughness induced noise prediction based on a linear time-varying model with sliding friction. *Journal of Vibration and Control, 13*(7), 1045-1063. https://doi.org/10.1177/1077546307078829

[10] Ozebek, M., Flašker, J. & Belšak, A. (2009). Diagnostika zobniške dvojice z uporabo mehanskega modela. *Magistrsko delo.* M. Ozebek, Maribor. (in Slovenian)

[11] Parker, R. G., Vijayakra, S. M. & Imajo, T. (2000). Non–linear dynamic response of a spur gear pair modelling and experimental comparisons, *Journal of sound and Vibration*, *237*(3), 435-455. https://doi.org/10.1006/jsvi.2000.3067

[12] Dowson, D. & Higginson, G. R. (1960). The effect of material properties on the lubrication of elastic rollers. *Journal of Mechanical Engineering Science, 2*(3), 188-194. https://doi.org/10.1243/JMES_JOUR_1960_002_028_02

[13] Kuang, J. H. & Lin, A. D. (2001). The effect of tooth wear on the vibration spectrum of a spur gear pair. *Journal of vibrations and acoustics, 123*(3), 311-317. https://doi.org/10.1115/1.1379371

[13] Kuang, J. H. & Lin, A. D. (2001). The effect of tooth wear on the vibration spectrum of a spur gear pair. *Journal of vibrations and acoustics, 123*(3), 311-317. https://doi.org/10.1115/1.1379371

[14] Yang, D. C. H. & Sun, Z. S. (1985). A rotary model for spur gear dynamics. *Journal of Mechanisms, Transmissions, and Automation in Design, 107*(4), 529-535. https://doi.org/10.1115/1.3260759

Authors' contacts:

Aleš Belšak, Assist. Prof. Dr.
(Corresponding author)
University of Maribor, Faculty of Mechanical Engineering,
Smetanova ulica 17, 2000 Maribor, Slovenia
+386 2 220 7710, ales.belsak@um.si

Matej Ozebek, MSc
Higher Vocational School,
Pot na Lavo 22, 3000 Celje, Slovenia
+386 3 428 5853, matej.ozebek@sc-celje.si

Mario Hirz, Assoc. Prof. Dipl.-Ing. Dr. techn.
University of Maribor, Faculty of Mechanical Engineering,
Inffeldgasse 11/II, 8010 Graz, Austria
+43 316 873 35220