AN ACCELERATED BENDERS DECOMPOSITION ALGORITHM FOR THE MULTI-ITEM FIXED-CHARGE TRANSPORTATION PROBLEM

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Abstract:

In today's industrial and service sectors the role of transportation is unavoidable. Due to this importance, an optimized transportation plan with minimum transportation costs can be a favour for the managers. In this study a multi-item fixed-charge transportation problem with capacitated multiple transportation mode is considered. As such problem is of high degree of complexity, we focus on the Benders decomposition approach to solve it. For this aim, first the classical Benders decomposition approach is developed for the problem. This is the first time in the literature that the Benders decomposition algorithm is developed for this problem. In continue, as another novelty, an accelerated benders decomposition algorithm is developed for the problem by adding some valid inequalities to the classical Benders decomposition algorithm. These valid inequalities can effectively influence the performance of the classical Benders decomposition algorithm. Several test problems with various sizes are generated to test the proposed solution approaches. The test problems are solved by the classical branch and bound algorithm and the proposed classical and accelerated Benders decomposition algorithms. According to the obtained results the accelerated Benders decomposition algorithm performs better than others in terms of reporting optimal solution and CPU running time.

1 Introduction

In today's competitive world, due to technology and environment advancements, organizations face different environment where the managers know that the efforts for implementing the new methods for improving efficiency and effectiveness and also satisfying the customers' demands has resulted in unexpected outcomes [1]. On the other hand, success of an organization depends on its ability to provide outputs according to the expectations of its owners and managers. Therefore, organizations need to use new methods and approaches in order to reach the goals and increase their efficiency and effectiveness [2].

A good and regular relationship between organizations is an important issue to overcome the difficulties of products distribution and customers related services. Such relationships can help organizations to improve the level of customers related services and their performances. It can be claimed that, in today's world, the competition is among the supply chains instead of the organizations [3], [4]. In a supply chain, all organizations

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that contributed to a product are connected to each other like a chain in order to present better service to the final customer [5].

In order to make good relationships among the parts of a supply chain, the transportation of materials and products among the suppliers, manufacturers, and customers is important. Such transportation should be done in a safe and less costly plan. This plan can be obtained by considering the transportation problem. In this problem, the amount of product sent from each origin to each destination (there exists some origins and some destinations) is determined such that each destination receives its demand amount, and the total transportation cost of the network is minimized. This problem is very applicable and useful for the managers of a supply chain and can be used to plan transportation among two consecutive stages of a supply chain [6], [7], [8]. In this problem there may be several transportation modes such as aerial transport, railway transport, trucks, etc., where each mode has a finite capacity and fixed and variable cost per unit of distance.

The transportation problem has gain attention of the researchers as it needs to be solved efficiently. Consequently, there are many valuable studies on the transportation problem in the literature of optimization theory and operation research. The transportation problem in its classical version was introduced by Hitchcock [9] for the first time. In this problem there was a set of origins and a set of destinations, and the items had to be sent from the origins to the destinations just by a transportation mode with no limited capacity. Later in the study of Haley [10] the solid transportation problem with three dimensions was introduced. In this problem, in addition to the origins and destinations, the transportation mode was also considered. Mollanoori et al. [11] extended the solid transportation problem by considering step fixed charges between the routes. In such problems, the transportation costs are of very important issues. Thomas and Griffin [12] stated that 11 percent of the gross income of the United States is related to the logistics costs. Eskigun et al. [13] also claimed that about 30 percent of the total production costs are related to the logistics costs. This means that optimization of the transportation costs between the origins and destinations can be of critical issues in a supply chain. In such area, the transportation models with fixed or step-fixed costs are introduced and solved in the literature [14], [15], [16]. Also, route optimization may significantly reduce the transportation costs [17].

Another extension of the transportation problem was made by considering multiple product types. This is something that has gained significant attention of the researchers. In such a problem, there are several types of products, and each destination may demand all or some of the products with certain values. Akgün et al. [18] proposed a multimodal, multicommodity, and multiperiod planning problem for coal distribution to poor families. Jogunola et al. [19] focused on the multi-commodity optimization of peer-to-peer energy trading resources in smart grid. Babaei et al. [20] introduced a new model for production and distribution planning based on data envelopment analysis with respect to traffic congestion, blockchain technology, and uncertain conditions. Aggarwal et al. [21] introduced a novel optimisation model for sustainable multi-commodity transportation planning.

The classical transportation problem and its advanced models, in some cases, are with high degree of complexity. Therefore, use of heuristic and meta-heuristic algorithms can be useful to solve them [22]. According to the degree of complexity for this problem, the solution approaches such as the branch and bound algorithm, the Benders decomposition algorithm, the Lagrangean relaxation algorithm, the classical and hybrid meta-heuristic algorithms can be used as solution approaches. For application of these methods the studies such as Benders [23], Klose [24], Oliveira et al. [25], Wong and Üster [26], Song and Cheng [27], Fragkogios et al. [28], Bacci et al. [29], Yurek [30], etc. can be referred to.

This paper addresses the fixed-charge transportation problem with multiple items. A capacitated multiple transportation mode policy will be employed to obtain an efficient solution. The intrinsic property of the considered problem is its high degree of complexity and the need to resolve the problem optimally. For those reasons, a novel approach based on the Benders decomposition is selected to tackle the problem at hand and, to the best of the authors' knowledge, this is the first reported study that aims to use the mentioned approach for the considered type of problem. Furthermore, the idea is to modify the classical Benders decomposition approach by considering certain valid inequalities according to the nature of the problem and thus accelerate the solution procedure. Finally, carefully selected test problems will be considered with both the classical and modified approach in order to analyse and compare their performances.

2 The multi-item fixed-charge transportation problem (MIFCTP)

Consider the graph of $G=(V,E)$ as a distribution network, where $V=N+M$ is the number of nodes, N is the number of origins (indexed by i), and M is the number of destinations (indexed by j). In this network, K types of products (indexed by k) are to be transported from the origins to the destinations with L transportation modes (indexed by l). Each destination has a certain demand for each product type (say d_{jk} for demand of destination j for product type k). Each origin has a certain capacity of each type of product (say s_{ik} for demand of origin i for product type k). In order to transport an amount of a product type from an origin to a destination with a certain transportation mode, two types of fixed and variable costs are charged. Therefore, f_{ijkl} is defined for fixed cost of transporting an amount of product k from origin i to destination j by transportation mode l and v_{ijkl} is defined for variable cost of transporting one unit of product k from origin i to destination j by transportation mode l. Total weight capacity of transportation mode l in the arc of (i,j) is shown by c_{ijl}. The weight of product type k is shown by w_k. Non-negative variable X_{ijkl} is defined for the amount of product type k sent from origin i to destination j by transportation mode l. Binary variable Y_{ijkl} is defined to say whether or not an amount of product type k is sent from origin i to destination j by transportation mode l. Also M_{ijk} is an enough large value where its appropriate value is obtained as $M_{ijk}=\min\{s_{ik},d_{jk}\}$. This problem is formulated as given below [31]:

$$\min\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M}f_{ijkl}Y_{ijkl}+\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M}v_{ijkl}X_{ijkl} \tag{1}$$

subject to:

$$\sum_{l=1}^{L}\sum_{j=1}^{M}X_{ijkl}\le s_{ik}\quad \forall i,k \tag{2}$$

$$\sum_{l=1}^{L}\sum_{j=1}^{N}X_{ijkl}\ge d_{jk}\quad \forall j,k \tag{3}$$

$$X_{ijkl}\le M_{ijk}Y_{ijkl}\quad \forall i,j,k,l \tag{4}$$

$$\sum_{k=1}^{K}w_kX_{ijkl}\le c_{ijl}\quad \forall i,j,l \tag{5}$$

$$X_{ijkl}\ge 0, Y_{ijkl}\in\{0,1\}\quad \forall i,k \tag{6}$$

In the above formulation, objective function (1) minimizes total transportation costs between all origin-destination pairs including fixed and variable transportation costs. Constraints set (2) respects to the capacity of each origin for each product type. Constraints set (3) guarantee that the demand of each destination for each product type is fulfilled. Constraints set (4) guarantee that if a certain value of a product type is sent from an origin to a destination, the associated variable Y_{ijkl} takes value of 1. Constraints set (5) respects to the capacity of each transportation mode. Constraints set (6) is sign constraints.

As the transportation problems with fixed charge are of NP-hard type optimization problems [22], only the small size instances of such problems can be solved by optimization solvers. In such cases, by increasing the problem size, the solution time could be increased dramatically, and no optimal solution may be obtained. This issue happens because of the limited memory of the optimization solvers which is not enough to solve medium and large size instances of such problems. In the next section, a Benders decomposition approach is developed to solve the formulation (1)-(6).

3 Solution methodology

As the multi-item fixed-charge transportation problem (1)-(6) has a high degree of complexity, in this section a typical Benders decomposition approach [23] is proposed to solve it. According to this approach we can solve larger instances of this problem comparing to the traditional solution approaches such as branch and bound which is applied by the classical solvers such as CPLEX, LINGO, etc. In the rest of this section, first the classical form of the Benders decomposition approach is explained, and then, the classical and an accelerated forms of this approach are developed for the formulation (1)-(6).

3.1 Benders decomposition approach

The Benders decomposition approach, as the name reveals, was originally proposed by Benders in 1962 [23]. This algorithm is a suitable scheme for solving the combinatorial optimization problems with high degree of complexity. In the structure of such problems a column of variables (integer or continuous) form a block and prevents the problem to be solved effectively. Therefore, the solution time is increased dramatically by increasing the problem size. According to the Benders decomposition algorithm, a problem is converted to two problems called sub-problem and master problem. The master problem provides a lower bound for the main problem while the sub-problem provides an upper bound for the main problem. The Benders decomposition scheme is executed in several iterations and, in the consecutive iterations, the obtained upper and lower bounds are improved untill the optimal solution is reached where the upper and lower bounds become equal. In order to describe the master and sub-problems, consider the below given mixed-integer linear problem [32]:

$$\begin{gathered}\min c_1^T x + c_2^T y \\ \text{subject to} \\ Ax + By \geq b \\ x \geq 0 \\ y \in \{0,1\}\end{gathered} \tag{7}$$

In this problem $c_1, x \in \mathrm{R}^n$, $c_2, y \in \mathrm{R}^q$, $b \in \mathrm{R}^m$, $A = [a]_{m\times n}$, and $B = [b]_{m\times q}$. Considering the variable y as a constant value $(\overline{y})$, the model shown below is obtained:

$$\begin{gathered}\min c_1^T x + c_2^T \overline{y} \\ \text{subject to} \\ Ax \geq b - B\overline{y} \\ x \geq 0\end{gathered} \tag{8}$$

Considering the dual variable of u for the constrain of the problem (8), the dual form of this problem is obtained as given below. This problem is the Benders sub-problem in the procedure of Benders decomposition algorithm:

$$\begin{gathered}\max c_2^T \overline{y} + (b - B\overline{y})^T u \\ \text{subject to} \\ A^T u \leq c_1^T \\ u \geq 0\end{gathered} \tag{9}$$

In the following, the Benders master problem is constructed as:

$$\min c_2^T \overline{y} + z \tag{10}$$

subject to

$$z \geq (b - By)^T \overline{u}^k \,,\; \overline{u}^k \in P \subseteq U, k = 1,2,\dots,K$$
$$(b - By)^T \overline{u}^l \leq 0 \,,\; \overline{u}^l \in R \subseteq U, l = 1,2,\dots,L$$
$$y \in \{0,1\}$$

In the master problem (10), the first constraint is an optimality cut and the second constraint is a feasibility cut. These cuts are used in the procedure of Benders decomposition scheme to tie the optimality gap and obtain better solution. The notation U shows is the feasible region which is obtained by the first and second constraints (10). On the other hand, P is the set of extreme points of U , and R is the set of extreme directions of U .

In each iteration of the Benders decomposition scheme only one of the above-mentioned cuts is used. The extreme point $\overline{u}^k$ that is used in the i-th iteration of the problem (10), is obtained from the solution of the problem (9) in its i-the iteration using the value of $\overline{y}$ obtained by iteration $i-1$. On the other hand, in the case that the problem (9) is unbounded, the extreme direction $\overline{u}^l$ that is used in the i -th iteration of the problem (10), is obtained from the problem show below by using the value of $\overline{y}$ obtained in the iteration $i-1$.

$$\max_{u,e} e = 0 \tag{11}$$

subject to

$$(b - B\overline{y})^T u = 1$$
$$A^T u \geq 0$$
$$u \geq 0$$

The Benders decomposition scheme can be stopped when the relative gap of the lower and upper bound values in an iteration become zero or less than a predetermined value (say ε). The procedure for the Benders decomposition approach is summarized in Algorithm 1.

Algorithm 1: The Benders decomposition procedure for solving the problem (7)

Define Z_{LB} and Z_{UB} as the lower and upper bounds for the problem (7)

Set the value of ε , $Z_{LB} = -\infty$ and $Z_{UB} = +\infty$

Set an initial value for the binary variable such as $y = \overline{y}$ to form a feasible solution for the problem (9)

While $Z_{UB} - Z_{LB} \geq \varepsilon$

  Solve the sub-problem (9) with $\overline{y}$ values

  If the sub-problem (9) solved optimally

    Obtain $\overline{u}^k$ extreme point and add the related optimality cut to the master problem (10)

    Update Z_{UB}

  Else-If the sub-problem (9) is unbounded

    Obtain $\overline{u}^l$ extreme direction from the problem (11) and add the related feasibility cut to the master problem (10)

  End-If

  Solve the sub-problem (10) considering its new cuts to find $\overline{y}$

  Update Z_{LB}

End-While

Input the value of Z_{LB} and $\overline{y}$ to the problem (8) to obtain the final solution's variable values

According to the Benders decomposition algorithm if the relative gap of zero is obtained, the final solution is optimal. Otherwise, the solution with lower relative gap is favored.

3.2 Benders decomposition approach for the MIFCTP

In this section the proposed Benders decomposition approach explained by Section 3.1 is used to solve the MIFCTP (the problem (1)-(6)) of Section 2. For this aim, the below steps are introduced to implement the Benders decomposition scheme for the MIFCTP (the problem (1)-(6)).

Step 1. Consider the binary variables of the problem (1)-(6) as constant values such as $Y_{ijkl} = \bar{Y}_{ijkl}$. Therefore, the following formulation is obtained.

$$\min \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{I}\sum_{j=1}^{M} v_{ijkl} X_{ijkl} \tag{12}$$

subject to

$$-\sum_{l=1}^{L}\sum_{j=1}^{M} X_{ijkl} \geq -s_{\text{ik}} \text{ , } \forall i,k \tag{13}$$

$$\sum_{l=1}^{L}\sum_{i=1}^{N} X_{ijkl} \geq -d_{jk} \quad \forall j,k \tag{14}$$

$$-X_{\text{ijkl}} \geq -M_{ijk}\bar{Y}_{ijkl} \quad \forall i,j,k,l \tag{15}$$

$$-\sum_{k=1}^{K} w_k X_{ijkl} \geq -c_{ijl} \quad \forall i,j,l \tag{16}$$

$$X_{ijkl} \geq 0 \ \forall i,j,k,l \tag{17}$$

Step 2. For the constraints (13)-(17) the dual variables α_{ik} , β_{jk} , γ_{ijkl} and δ_{ijl} are defined respectively. Then, the dual form of the problem (13)-(17) is obtained as:

$$\max_{\alpha,\beta,\gamma,\delta} \sum_{k=1}^{K}\sum_{i=1}^{N} -s_{ik}\alpha_{ik} + \sum_{k=1}^{K}\sum_{j=1}^{M} d_{jk}\beta_{jk} + \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} -M_{ijk}\bar{Y}_{ijkl}\gamma_{ijkl} + \sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{M} -c_{ijl}\delta_{ijl} \tag{18}$$

subject to

$$-\alpha_{ik} + \beta_{jk} - \gamma_{ijkl} - w_k\delta_{ijl} \leq v_{ijkl} \text{ , } \forall i,j,k,l \tag{19}$$

$$\alpha_{ik} \geq 0 \text{ , } \forall i,k \tag{20}$$

$$\beta_{jk} \geq 0 \text{ , } \forall j,k \tag{21}$$

$$\gamma_{ijkl} \geq 0 \text{ , } \forall i,j,k,l \tag{22}$$

$$\delta_{ijl} \geq 0 \text{ , } \forall i,j,l \tag{23}$$

Step 3. As the objective function of the problem (1)-(6) can be written in the following manner:

$$\min_{Y\in\{0,1\}} \left\{ \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} f_{ijkl}Y_{ijkl} + \min_{X\geq 0} \left\{ \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} v_{ijkl}X_{ijkl} \right\} \right\}, \tag{24}$$

considering the constant term $\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} f_{ijkl}\overline{Y}_{ijkl}$, the sub-problem of the Benders decomposition algorithm is obtained as:

$$\max_{\alpha,\beta,\gamma,\delta}\left\{\begin{array}{l}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} f_{ijkl}Y^{*}_{ijkl}+\sum_{k=1}^{K}\sum_{i=1}^{N}-s_{ik}\alpha_{ik}+\sum_{k=1}^{K}\sum_{j=1}^{M}d_{jk}\beta_{jk}\\ +\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M}-M_{ijk}\overline{Y}_{ijkl}\gamma_{ijkl}+\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{M}-c_{ijl}\delta_{ijl}\end{array}\right\} \tag{25}$$

subject to constraints (13)-(17).

Step 4. If the set of extreme points $\pi^{p}=\left(\alpha^{*p},\beta^{*p},\gamma^{*p},\delta^{*p}\right),\ p=1,2,\ldots,P,$ and the set of extreme directions $\rho^{r}=\left(\alpha^{*r},\beta^{*r},\gamma^{*r},\delta^{*r}\right),\ r=1,2,\ldots,R,$ for the feasible region of the dual problem is considered, then the following master problem is defined:

$$\min_{Y} Z \tag{26}$$

subject to

$$\begin{array}{c}Z\geq\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} f_{ijkl}Y_{ijkl}+\sum_{k=1}^{K}\sum_{i=1}^{N}-s_{ik}\alpha^{*p}_{ik}+\sum_{k=1}^{K}\sum_{j=1}^{M}d_{jk}\beta^{*p}_{jk}\\ +\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M}-M_{ijk}\gamma^{*p}_{ijkl}Y_{ijkl}+\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{M}-c_{ijl}\,\delta^{*p}_{ijl},\forall i,j,k,l,\ p=1,\ldots,P\end{array} \tag{27}$$

$$\begin{array}{c}\sum_{k=1}^{K}\sum_{i=1}^{N}-s_{ik}\,\alpha^{*r}_{ik}+\sum_{k=1}^{K}\sum_{j=1}^{M}d_{jk}\beta^{*r}_{jk}+\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M}\left(-M_{ijk}\gamma^{*r}_{ijkl}\right)Y_{ijkl}\\ +\sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{M}-c_{ijl}\,\delta^{*r}_{ijl}\leq 0,\quad \forall i,j,k,l,\ r=1,\ldots,R,\end{array} \tag{28}$$

$$Y_{ijkl}\in\{0,1\},\ \forall i,j,k,l. \tag{29}$$

In this master problem Z is a free of sign variable, and the constraints (27) and (28) are the optimality and feasibility cuts, respectively.

Step 5. Implementation of the Benders composition scheme, according to the sub-problem (25) with constraints (13)-(17) and the master problem (26)-(29), follows the procedure of Algorithm 1.

3.3 Accelerated Benders decomposition approach for the MIFCTP

In the procedure of the classical Benders decomposition approach, the issues such as fast improvement of the lower and upper bounds, obtaining less relative gap between the bounds, and less running time, are important. Therefore, in many studies, the focus is to improve the classical Benders decomposition algorithm based on these issues [21], [33]. An important way to improve the classical Benders decomposition algorithm for a problem is to add some valid inequalities to the problem. A valid inequality acts as a constraint in the problem and may influence the classical Benders decomposition algorithm to obtain a better solution in a shorter running time [34], [35].

A valid inequality is defined according to the nature of the problem and should not be defined in a way to violate the problem assumptions and main constraints. For the MIFCTP of this study, the valid inequalities are defined below:

$$\sum_{j=1}^{M}\sum_{l=1}^{L} c_{ijl} Y_{ijkl} \geq s_{ik} \ , \ \forall i,k \tag{30}$$

$$\sum_{i=1}^{N}\sum_{l=1}^{L} s_{ik} Y_{ijkl} \geq d_{jk} \ , \ \forall j,k \tag{31}$$

$$\sum_{i=1}^{M}\sum_{l=1}^{L} c_{ijl} Y_{ijkl} \geq d_{jk} \ , \ \forall j,k \tag{32}$$

These valid inequalities generally imply that all the transportation capacities are sufficient for the demand values and product capacities. In the procedure of the Benders decomposition algorithm, these inequalities are added to the master problem. Therefore, the master problem for the accelerated Benders decomposition algorithm for the MIFCTP is modified as shown here:

$$\min_{Y} Z$$

subject to

$$\begin{aligned} Z \geq & \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} f_{ijkl} Y_{ijkl} + \sum_{k=1}^{K}\sum_{i=1}^{N} - s_{ik} \alpha_{ik}^{*p} + \sum_{k=1}^{K}\sum_{j=1}^{M} d_{jk} \beta_{jk}^{*p} \\ & + \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} - M_{ijk} \gamma_{ijkl}^{*p} Y_{ijkl} + \sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{M} - c_{ijl} \delta_{ijl}^{*p} , \quad \forall i,j,k,l, \ p=1,\ldots,P , \end{aligned} \tag{33}$$

$$\begin{aligned} & \sum_{k=1}^{K}\sum_{i=1}^{N} - s_{ik} \alpha_{ik}^{*r} + \sum_{k=1}^{K}\sum_{j=1}^{M} d_{jk} \beta_{jk}^{*r} + \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{N}\sum_{j=1}^{M} \left(-M_{ijk} \gamma_{ijkl}^{*r}\right) Y_{ijkl} \\ & + \sum_{l=1}^{L}\sum_{i=1}^{N}\sum_{j=1}^{M} - c_{ijl} \delta_{ijl}^{*r} \leq 0 \ , \ \forall i,j,k,l, \ r=1,\ldots,R , \end{aligned} \tag{34}$$

$$\sum_{j=1}^{M}\sum_{l=1}^{L} c_{ijl} Y_{ijkl} \geq s_{ik} \tag{35}$$

$$\sum_{j=1}^{M}\sum_{l=1}^{L} c_{ijl} Y_{ijkl} \geq s_{ik} \ , \ \forall i,k \tag{36}$$

$$\sum_{i=1}^{N}\sum_{l=1}^{L} s_{ik} Y_{ijkl} \geq d_{jk} \ , \forall j,k \tag{37}$$

$$\sum_{i=1}^{M}\sum_{l=1}^{L} c_{ijl} Y_{ijkl} \geq d_{jk} \ , \forall j,k \tag{38}$$

$$Y_{ijkl} \in \{0,1\} \ , \forall i,j,k,l \tag{39}$$

The procedure for implementing the accelerated Benders decomposition algorithm follows Algorithm 1 presented above.

4 Computational experiments

In this section, some experiments are done to evaluate the performance of the proposed classical and accelerated Benders decomposition algorithms for the MIFCTP. Therefore, some test problems are generated with various sizes. Then the procedures of the algorithms are coded in GAMS and run on a PC with an Intel Core 2 Duo 2.53 GHz processor and 2.00 GB RAM. The details of the test problems and the obtained results are elaborated in this section.

4.1 Test problems

In order to evaluate the proposed Benders decomposition algorithms, some test problems of the MIFCTP are required. Here we randomly generate some test problems. The size of each test problem is determined by the number of origins $|N|$, the number of destinations $|M|$, the number of product types $|K|$, and the number of transportation modes $|L|$. According to the model (1)-(6), each test problem has the following characteristics:

- Number of continuous variables: $|N|\times|M|\times|K|\times|L|$,
- Number of binary variables: $|N|\times|M|\times|K|\times|L|$,
- Number of constraints: $|N|\times|K|+|M|\times|K|+3(|N|\times|M|\times|K|\times|L|)+|N|\times|M|\times|L|$.

The test problems used in this study and their characteristics are shown by Table 1.

Table 1. The test problems and their characteristics.

Test problems	$\|N\|$	$\|M\|$	$\|L\|$	$\|K\|$	Number of variables		Number of constraints
					Continuous	Binary	
TP1	3	7	2	2	84	84	314
TP2	5	10	2	2	200	200	730
TP3	10	15	2	3	900	900	3075
TP4	10	20	3	3	1800	1800	6090
TP5	15	25	3	5	5625	5625	18200
TP6	15	30	3	5	6750	6750	21825
TP7	20	35	4	5	14000	14000	45075
TP8	20	40	4	7	22400	22400	70820
TP9	25	45	5	7	39375	39375	124240
TP10	25	50	5	7	43750	43750	139025
TP11	30	55	6	7	69300	69300	218395
TP12	35	70	6	8	117600	117600	368340
TP13	40	75	7	8	168000	168000	525920
TP14	45	80	7	8	201600	201600	631000
TP15	50	90	8	10	360000	360000	1117400
TP16	70	100	8	10	560000	560000	1737700
TP17	90	120	8	10	864000	864000	2680500
TP18	100	130	10	12	1560000	1560000	4812760
TP19	120	160	10	12	2304000	2304000	7107360

On the other hand, in order to generate the parameters of the MIFCTP for the test problems, we use uniform distribution noted by $U(a,b)$ where a number is generated between the values a and b uniformly. The details for generating the values of the parameters are given by Table 2.

Table 2. The details for generating the values of the parameters of the test problems.

Parameter	Domain	Value
w_k	$k=1,\ldots,K$	1
s_{ik}	$i=1,\ldots,N,\ k=1,2,\ldots,K$	$U(90,200)$
d_{jk}	$j=1,\ldots,M,\ k=1,2,\ldots,K$	$U(10,20)$
c_{ijl}	$i=1,\ldots,N,\ j=1,\ldots,M, l=1,\ldots,L$	$U(50,200)$
f_{ijkl}	$i=1,\ldots,N,\ j=1,\ldots,M, k=1,\ldots,K,\ l=1,\ldots,L$	$U(1,4)$
v_{ijkl}	$i=1,\ldots,N,\ j=1,\ldots,M, k=1,\ldots,K,\ l=1,\ldots,L$	$U(2,8)$
M_{ijk}	$i=1,\ldots,N,\ j=1,\ldots,M, k=1,\ldots,K$	$\min\{s_{ik}, d_{jk}\}$

4.2 Results and comparative study

Here, the final results obtained for the test problems of the MIFCTP generated in Section 4.1 are reported and discussed. For comparison purposes we consider the below algorithms, and their performances are compared and discussed.

- The branch and bound (B&B) approach by CPLEX solver of the GAMS.
- The classical Benders decomposition algorithm proposed by Section 3.2.
- The accelerated Benders decomposition algorithm proposed by Section 3.3.

All of the experiments are performed by the GAMS and the obtained results are represented by Table 3 and Table 4. The obtained results are represented in terms of lower bound, upper bound, relative gap, and CPU time (for the Benders decomposition algorithms).

Table 3. The results obtained by the branch and bound (B&B) approach by CPLEX solver of the GAMS.

Test problem	B&B by GAMS	
	Objective function value	Optimality gap
TP1	566	0
TP2	789	0
TP3	1621	0
TP4	1983	0
TP5	4013	0
TP6	4827	0
TP7	5480	0
TP8	8659	0
TP9	9724	0
TP10	11088	0
TP11	11907	0
TP12	17531	0
TP13	18652	0
TP14	19939	0
TP15	Not available	Not available
TP16	Not available	Not available
TP17	Not available	Not available
TP18	Not available	Not available
TP19	Not available	Not available

As can be seen from Table 4, test problem TP10 is solved by the accelerated Benders decomposition algorithm in only 11 iterations which is much less than the number of iterations needed for this test problem using the classical Benders decomposition algorithm (365 iterations).

Table 4. The results obtained by the proposed classical and accelerated Benders decomposition algorithms.

Test problem	The classical Benders decomposition approach					The accelerated Benders decomposition approach				
	Lower bound	Upper bound	Gap	Iterations	CPU time	Lower bound	Upper bound	Gap	Iterations	CPU time
TP1	566	566	0	26	7.79	566	566	0	8	3.28
TP2	79	789	0	40	21.03	79	789	0	8	7.06
TP3	1621	1621	0	66	44.52	1621	1621	0	8	14.83
TP4	1983	1983	0	80	61.85	1983	1983	0	9	10.96
TP5	4012	4012	0	173	4378.4	4012	4012	0	9	482.11
TP6	4827	4827	0	196	677.9	4827	4827	0	9	334.76
TP7	5480	5480	0	197	230.85	5480	5480	0	10	21.97
TP8	8659	8659	0	332	456.28	8659	8659	0	11	89.45
TP9	9724	9724	0	400	21881.99	9724	9724	0	11	1013.896
TP10	11088	11088	0	365	797.38	11088	11088	0	11	222.14
TP11	11907	11907	0	582	5999.66	11907	11907	0	13	1016.099
TP12	17531	17531	0	739	8778.8	17531	17531	0	15	3350.839
TP13	18652	18652	0	612	12514.57	18652	18652	0	15	2202.56
TP14	19939	19939	0	893	14384.97	19939	19939	0	18	2378.48
TP15	27837	27837	0	918	19946.34	27837	27837	0	25	3518.77
TP16	31129	31129	0	1006	33989.47	31129	31129	0	25	3743.38
TP17	37486	37486	0	1206	115247.91	37486	37486	0	48	7622.45
TP18	48346	48346	0	1565	277878.5	48346	48346	0	61	21540.65
TP19	59572	59572	0	1926	834129.36	59572	59572	0	71	45158.36

Based on the obtained results, performances of the proposed algorithms can be discussed. According to the results presented in Table 3, the CPLEX solver of GAMS can solve 14 test problems optimally. The remaining test problems cannot be solved by the CPLEX solver of GAMS because these test problems are of large size and the available memory is not enough to solve them exactly. On the other hand, the classical and accelerated Benders decomposition approaches are successful to solve all the considered test problems optimally. As can be seen from Table 4, both of the approaches report the same upper bound and lower bound in each test problem. Therefore, for all test problems a relative gap of zero is obtained by both of the approaches. The difference is that the accelerated benders decomposition approach results in much shorter CPU running time and less number of iterations than the classical Benders decomposition approach. This improvement is because of the valid inequalities added to the master problem. Some details of the iterations for the test problem TP10 obtained by the classical Benders decomposition approach are shown by Table 5. This table shows that the lower and upper bounds of 10665 and 120153 in the first iteration are moved to the value of 11088 in iteration 365 which terminates the Benders decomposition algorithm for this test problem.

Table 5. Some iterations and related bounds of the classical Benders decomposition approach obtained for test problem TP10.

Iteration	Lower bound	Upper bound	Relative gap	Iteration	Lower bound	Upper bound	Relative gap
1	10665	120153	0.91123	216	10880	120153	0.90944
9	10673	120153	0.91116	225	10889	120153	0.90936
18	10682	120153	0.91108	234	10898	120153	0.90929
27	10691	120153	0.91101	243	10907	120153	0.90921
36	10700	120153	0.91093	252	10916	120153	0.90914
45	10709	120153	0.91086	261	10925	120153	0.90906
54	10718	120153	0.91078	270	10934	120153	0.90899
63	10727	120153	0.91071	279	10943	120153	0.90891
72	10736	120153	0.91063	288	10952	120153	0.90884
81	10745	120153	0.91056	297	10961	120153	0.90876
90	10754	120153	0.91048	300	10964	120153	0.90874
99	10763	120153	0.91041	309	10973	120153	0.90866
108	10772	120153	0.91033	318	10982	120153	0.90859
117	10781	120153	0.91026	327	10991	120153	0.90851
126	10790	120153	0.91019	336	11000	120153	0.90844
135	10799	120153	0.91011	345	11009	120153	0.90836
144	10808	120153	0.91004	354	11016	120153	0.90831
153	10817	120153	0.90996	359	11016	27001	0.59164
162	10826	120153	0.90989	360	11026	14001	0.21169
171	10835	120153	0.90981	361	11037	12470	0.11355
180	10844	120153	0.90974	362	11054	12022	0.07794
189	10853	120153	0.90966	363	11085	11325	0.02092
198	10862	120153	0.90959	364	11088	11250	0.0144
207	10871	120153	0.90951	365	11088	11088	0

The lower bounds, upper bounds and relative gap values of the iterations of the accelerated Benders decomposition algorithm for test problem TP10 are presented in more details in Figure 1 and Figure 2.

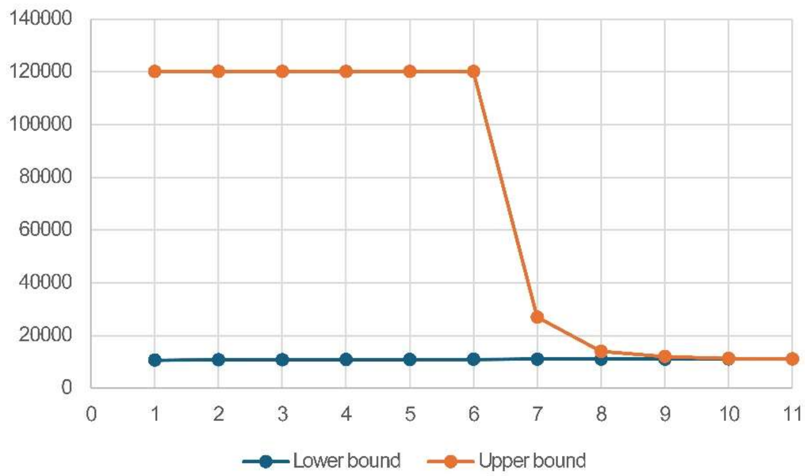

Figure 1. The graph of 11 iterations of the accelerated Benders decomposition algorithm for test problem TP10 in terms of lower and upper bound values.

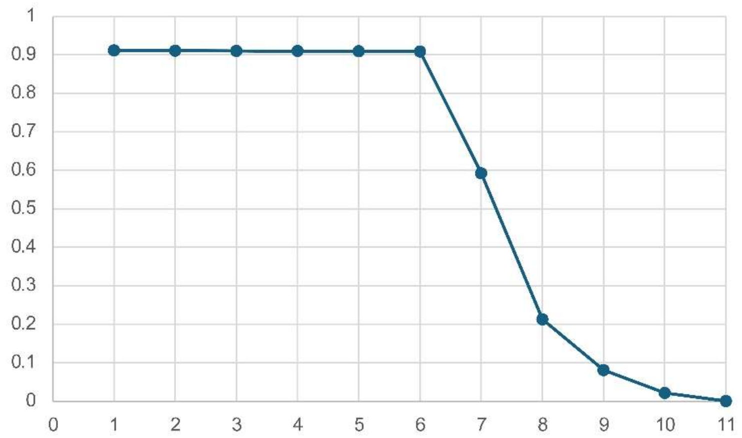

Figure 2. The graph of 11 iterations of the accelerated Benders decomposition algorithm for test problem TP10 in terms of relative gap values.

5 Concluding remarks

In this study, a multi-item fixed-charge transportation problem is considered. In order to solve it efficiently, multiple transportation mode policy is employed, where some capacitated transportation modes are available to transport multiple product types from the origins to the destinations. As such, the problem is of high degree of complexity, and the focus was on the Benders decomposition approach in order to solve it. Hence, first the classical Benders decomposition approach was developed. To the best of the authors' knowledge, this is the first time in the literature that the Benders decomposition algorithm is employed for this kind of problem. Furthermore, an accelerated Benders decomposition algorithm was developed for the problem at hand by adding some valid inequalities to the classical Benders decomposition algorithm. These inequalities can effectively improve the performance of the classical Benders decomposition algorithm. Several test problems were generated and solved by the classical branch and bound algorithm and the proposed classical and accelerated Benders decomposition algorithms. According to the obtained results, the accelerated Benders decomposition algorithm performs better than others in terms of optimal solution and the CPU running time.

Ini the future work, heuristic, meta-heuristic, and hyper-heuristic approaches shoud be applied to solve the MIFCTP considered here in Section 2 and thus produce a solid background for assessment of all those aproaches for the considered type of problem.

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