Math. Commun. **30** (2025), 161–169
DOI: 10.64785/mc.30.2.1

Mathematical
Communications

Orthogonality relations for Poincaré series

Sonja Žunar

Faculty of Geodesy, University of Zagreb, Kačićeva 26, HR-10 000 Zagreb, Croatia

---

Abstract. Let G be a connected semisimple Lie group with finite center. We prove a formula for the inner product of two cuspidal automorphic forms on G that are given by Poincaré series of K-finite matrix coefficients of an integrable discrete series representation of G. As an application, we give a new proof of a well-known result on the Petersson inner product of certain vector-valued Siegel cusp forms. In this way, we extend results previously obtained by Muić for cusp forms on the upper half-plane, i.e., in the case when $G = \mathrm{SL}_2(\mathbb{R})$.

AMS subject classifications: 11F70, 11F46
Keywords: Poincaré series; Petersson inner product; orthogonality relations; automorphic forms; Siegel cusp forms
Received January 27, 2025; accepted March 14, 2025

---

1. Introduction

In [5], Muić gave a representation-theoretic proof of a formula for the Petersson inner product of two cusp forms in $S_m(\Gamma)$, where $m \in \mathbb{Z}_{\geq 3}$, Γ is a discrete subgroup of finite covolume in $\mathrm{SL}_2(\mathbb{R})$, and one of the cusp forms is given by a Poincaré series of polynomial type that lifts in a standard way to a Poincaré series of a K-finite matrix coefficient of an integrable discrete series representation of $\mathrm{SL}_2(\mathbb{R})$. Let G be a connected semisimple Lie group with finite center. In this paper, by generalizing the methods of [5], we prove a formula for the inner product of two cuspidal automorphic forms on G that are given by Poincaré series of K-finite matrix coefficients of an integrable discrete series representation of G. As an application, we give a new proof of a well-known result on the Petersson inner product of certain vector-valued Siegel cusp forms.

To explain our results in more detail, let us fix a maximal compact subgroup K of G and a Haar measure dg on G. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Let (π, H) denote a discrete series representation π of G acting on a complex Hilbert space H, and let H_K denote the $(\mathfrak{g}, K)$-module of K-finite vectors in H.

It is well known that for all $h, h' \in H$, the matrix coefficient $c_{h,h'} : G \to \mathbb{C}$,

$$c_{h,h'}(g) = \langle \pi(g)h, h' \rangle_H\,, \tag{1}$$

belongs to $L^2(G)$ [12, §4.5.9]. Moreover, by the Schur orthogonality relations [12, Theorem 4.5.9.3], we have

$$\left\langle c_{h,h'}, c_{v,v'} \right\rangle_{L^2(G)} = \frac{1}{d(\pi)} \langle h, v \rangle_H \langle v', h' \rangle_H\,, \qquad h, h', v, v' \in H, \tag{2}$$

where $d(\pi) \in \mathbb{R}_{>0}$ is the formal degree of π, i.e., the Plancherel measure of π [1, Proposition 18.8.5]. On the other hand, if (σ, V) is a discrete series representation of G that is not equivalent to π, then we have

$$\left\langle c_{h,h'}, c_{v,v'} \right\rangle_{L^2(G)} = 0, \qquad h, h' \in H,\ v, v' \in V.$$

Next, suppose that (π, H) belongs to the integrable discrete series of G, i.e., we have $c_{h,h'} \in L^1(G)$ for all $h, h' \in H_K$. Let Γ be a discrete subgroup of G. Then, given $h, h' \in H_K$, the Poincaré series

$$\left(P_\Gamma c_{h,h'}\right)(g) = \sum_{\gamma \in \Gamma} c_{h,h'}(\gamma g)$$

Email address: szunar@geof.hr

https://www.mathos.unios.hr/mc

converges absolutely and uniformly on compact subsets of G and defines a smooth, left Γ-invariant, $\mathcal{Z}(\mathfrak{g})$-finite, right K-finite function that belongs to $L^p(\Gamma\backslash G)$ for every $p \in [1, \infty]$ (see, e.g., [8, Lemma 3.2]). If $G = \mathcal{G}(\mathbb{R})$ for some Zariski connected semisimple algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$, and Γ is an arithmetic subgroup of $\mathcal{G}(\mathbb{Q})$, then the function $P_\Gamma c_{h,h'}$ is a cuspidal automorphic form for Γ [6, Lemma 4-2(i)]. Miličić proved that

$$\operatorname{span}_{\mathbb{C}} \left\{ P_\Gamma c_{h,h'} : h, h' \in H_K \right\}$$

is dense in the π-isotypic component $L^2(\Gamma\backslash G)_{[\pi]}$ of the right regular representation of G on $L^2(\Gamma\backslash G)$ [7, Lemma 6.6].

The main result of this paper is the following generalization of Schur orthogonality relations.

Theorem 1. *Let G be a connected semisimple Lie group with finite center and a fixed Haar measure dg, and let Γ be a discrete subgroup of G. Let (π, H) be an integrable discrete series representation of G. Then, we have the following:*

(i) **(Inner product formula)** *For all $h, h', v, v' \in H_K$, we have*

$$\left\langle P_\Gamma c_{h,h'}, P_\Gamma c_{v,v'} \right\rangle_{L^2(\Gamma\backslash G)} = \frac{1}{d(\pi)} \langle h, v \rangle_H \left(P_\Gamma c_{v',h'} \right)(1_G), \tag{3}$$

where 1_G denotes the identity element of G, and $d(\pi)$ denotes the formal degree of π.

(ii) *Let (σ, V) be an integrable discrete series representation of G that is not equivalent to π. Then, for all $h, h' \in H_K$ and $v, v' \in V_K$, we have*

$$\left\langle P_\Gamma c_{h,h'}, P_\Gamma c_{v,v'} \right\rangle_{L^2(\Gamma\backslash G)} = 0.$$

Our proof of Theorem 1 is inspired by [5, proof of Theorem 2.3] and it relies on the results on the Poincaré series $P_\Gamma c_{h,h'}$ proved in [7].

As an application of the inner product formula (3), in Section 3, we give a representation-theoretic proof of a well-known formula for the Petersson inner product of certain vector-valued Siegel cusp forms (see, e.g., [10, Lemma 1.2]). To explain this in more detail, let $n \in \mathbb{Z}_{>0}$, and let (ρ, V) be an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$, where $\omega_1 \geq \ldots \geq \omega_n > 2n$. We equip V with the Hermitian inner product $\langle \cdot\,, \cdot \rangle_V$ with respect to which the restriction $\rho|_{\mathrm{U}(n)}$ is unitary. Let Γ be a discrete subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ that is commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$.

A Siegel cusp form of weight ρ for Γ is a V-valued holomorphic function ϕ on the Siegel upper half-space

$$\mathcal{H}_n = \left\{ z = x + iy \in M_n(\mathbb{C}) : z^\top = z \text{ and } y > 0 \right\}$$

with the following two properties:

1. $\phi((Az + B)(Cz + D)^{-1}) = \rho(Cz + D)\, \phi(z)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ and $z \in \mathcal{H}_n$.
2. $\sup_{z \in \mathcal{H}_n} \left\| \rho\left(y^{\frac{1}{2}} \right) \phi(z) \right\|_V < \infty$.

Siegel cusp forms of weight ρ for Γ constitute a finite-dimensional complex vector space $S_\rho(\Gamma)$, which we equip with the Petersson inner product

$$\langle \phi_1, \phi_2 \rangle_{S_\rho(\Gamma)} = \frac{1}{|\Gamma \cap \{\pm I_{2n}\}|} \int_{\Gamma\backslash\mathcal{H}_n} \langle \rho(y)\phi_1(z), \phi_2(z) \rangle_V \det y^{-n-1} \prod_{1 \leq r \leq s \leq n} dx_{r,s}\, dy_{r,s}.$$

It is well known (see, e.g., [10, Theorem 1.1]) that the space $S_\rho(\Gamma)$ is spanned by the absolutely and locally uniformly convergent Poincaré series

$$\begin{aligned} \left(P_{\Gamma,\rho} f_{\mu,v} \right)(z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma} & \mu\left(\Big(Az + B - i(Cz + D) \Big) \Big(Az + B + i(Cz + D) \Big)^{-1} \right) \\ & \times \rho\left(\frac{1}{2i} \Big(Az + B + i(Cz + D) \Big) \right)^{-1} v, \end{aligned}$$

where v runs over V and μ runs over the ring $\mathbb{C}[X_{r,s} : 1 \le r, s \le n]$ of polynomials with complex coefficients in n^2 variables $X_{r,s}$ with $r, s \in \{1, \ldots, n\}$. In [10], we showed that these Poincaré series lift in a standard way to Poincaré series of K-finite matrix coefficients of an integrable antiholomorphic discrete series representation π_ρ^* of $\mathrm{Sp}_{2n}(\mathbb{R})$. By using a variant of this lift to transfer the inner product formula (3) from $L^2(\Gamma\backslash\mathrm{Sp}_{2n}(\mathbb{R}))$ to $S_\rho(\Gamma)$, we obtain the following well-known result that extends [5, Corollary 1.2] and [9, Corollary 1.4] and illuminates the connection of the Siegel cusp forms $P_{\Gamma,\rho} f_{1,v}$ with the reproducing kernel function for $S_\rho(\Gamma)$ (cf. [2]), as described in detail in [10, Lemma 1.2 and §7].

Corollary 1. *Let $\phi \in S_\rho(\Gamma)$ and $v \in V$. Then, we have*

$$\left\langle \phi, P_{\Gamma,\rho} f_{1,v} \right\rangle_{S_\rho(\Gamma)} = \frac{\dim_{\mathbb{C}} V}{d(\pi_\rho)} \left\langle \phi(iI_n), v \right\rangle_V .$$

2. Proof of the inner product formula

Let G be a connected semisimple Lie group with finite center, and let Γ be a discrete subgroup of G. We fix a Haar measure dg on G, which induces the G-invariant Radon measure on $\Gamma\backslash G$ such that

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g)\, dg = \int_G f(g)\, dg \tag{4}$$

for every compactly supported, continuous function $f : G \to \mathbb{C}$ [4, Theorem 8.36]. Given a finite-dimensional complex Hilbert space V and $p \in [1, \infty]$, let $L^p(\Gamma\backslash G, V)$ denote the L^p space of functions $\Gamma\backslash G \to V$. We write $L^p(\Gamma\backslash G) = L^p(\Gamma\backslash G, \mathbb{C})$. Given an irreducible unitary representation π of G, let $L^2(\Gamma\backslash G)_{[\pi]}$ denote the π-isotypic component of the right regular representation $\left(R, L^2(\Gamma\backslash G)\right)$, i.e., the closure of the sum of closed irreducible G-invariant subspaces of $L^2(\Gamma\backslash G)$ that are equivalent to π.

Given a unitary representation (π, H) of G and a function $f \in L^1(G)$, a continuous linear operator $\pi(f) : H \to H$ is standardly defined by the condition

$$\langle \pi(f)h, h' \rangle_H = \int_G f(g) \langle \pi(g)h, h' \rangle_H \, dg, \qquad h, h' \in H,$$

where $\langle \cdot, \cdot \rangle_H$ denotes the inner product on H [3, (1.8b)]. It is well known that for all $f \in L^1(G)$ and $\varphi \in L^2(\Gamma\backslash G)$, we have

$$(R(f)\varphi)\,(g) = \int_G f(g')\, \varphi(gg')\, dg' \qquad \text{for a.a. } g \in G. \tag{5}$$

Moreover, by the dominated convergence theorem, we have the following lemma.

Lemma 1. *Let $f \in L^1(G)$, and let $\varphi : \Gamma\backslash G \to \mathbb{C}$ be a bounded, continuous function that belongs to $L^2(\Gamma\backslash G)$. Then, the right-hand side of (5) defines a continuous function $G \to \mathbb{C}$.*

In what follows, we use the notation $c_{h,h'}$ (resp., $d(\pi)$) introduced in (1) (resp., (2)) for a matrix coefficient (resp., the formal degree) of a discrete series representation π of G. The following lemma is an immediate corollary of (2).

Lemma 2. *Let (π, H) be a discrete series representation of G. Let $h' \in H \setminus \{0\}$. Then, the assignment*

$$h \mapsto \frac{d(\pi)^{\frac{1}{2}}}{\|h'\|_H} c_{h,h'}$$

defines a unitary G-equivalence $\Phi_{h'}$ from H to the irreducible closed G-invariant subspace

$$c_{H,h'} := \left\{ c_{h,h'} : h \in H \right\}$$

of the right regular representation $\left(R, L^2(G)\right)$.

Let $\mathfrak{g}$ denote the Lie algebra of G, and let K be a maximal compact subgroup of G. Given a unitary representation (π, H) of G, we denote by H_K the $(\mathfrak{g}, K)$-module of K-finite vectors in H.

Lemma 3. *Let (π, H) be an integrable discrete series representation of G. Let $h, h' \in H$ and $v, v' \in H_K$. Then, we have*

$$R(\overline{c_{v,v'}})c_{h,h'} = \frac{1}{d(\pi)} \langle h, v\rangle_H \, c_{v',h'}. \tag{6}$$

Proof. By (5) and Lemma 1, the continuous representative of the equivalence class $R\left(\overline{c_{v,v'}}\right) c_{h,h'} \in L^2(G)$ is given by

$$\begin{aligned}\left(R(\overline{c_{v,v'}})c_{h,h'}\right)(g) &= \int_G \overline{c_{v,v'}(g')}\, c_{h,h'}(gg')\, dg' \\ &= \int_G \overline{c_{v,v'}(g')}\, c_{h,\pi(g)^{-1}h'}(g')\, dg' \\ &= \left\langle c_{h,\pi(g)^{-1}h'}, c_{v,v'}\right\rangle_{L^2(G)} \\ &\overset{(2)}{=} \frac{1}{d(\pi)} \langle h, v\rangle_H \left\langle v', \pi(g)^{-1}h'\right\rangle_H \\ &= \frac{1}{d(\pi)} \langle h, v\rangle_H \, c_{v',h'}(g), \qquad g \in G. \qquad \square\end{aligned}$$

Let $\mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We recall that a function $\varphi : G \to \mathbb{C}$ is said to be:

1. right K-finite if $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}} \{\varphi(\,\cdot\, k) : k \in K\} < \infty$
2. $\mathcal{Z}(\mathfrak{g})$-finite if φ is smooth and we have $\dim_{\mathbb{C}} \mathcal{Z}(\mathfrak{g})\varphi < \infty$.

It is well known that, given an integrable discrete series representation (π, H) of G and $h, h' \in H_K$, the Poincaré series

$$\left(P_\Gamma c_{h,h'}\right)(g) = \sum_{\gamma\in\Gamma} c_{h,h'}(\gamma g)$$

converges absolutely and uniformly on compact subsets of G and defines a $\mathcal{Z}(\mathfrak{g})$-finite, right K-finite function on $\Gamma\backslash G$ that belongs to $L^p(\Gamma\backslash G)$ for every $p \in [1, \infty]$ (cf. [8, Lemma 3.2]).

The following proposition is a variation of [7, Theorem 6.4(i)].

Proposition 1. *Let (π, H) be an integrable discrete series representation of G, and let $h' \in H_K \setminus \{0\}$. We define the subspace*

$$P_\Gamma c_{H_K,h'} = \left\{P_\Gamma c_{h,h'} : h \in H_K\right\}$$

of $L^2(\Gamma\backslash G)$. Then, exactly one of the following holds:

(i) $P_\Gamma c_{H_K,h'} = 0$.

(ii) *The assignment*

$$h \mapsto \frac{\|h'\|_H}{\left\|P_\Gamma c_{h',h'}\right\|_{L^2(\Gamma\backslash G)}} P_\Gamma c_{h,h'}$$

defines a $(\mathfrak{g}, K)$-module isomorphism $H_K \to P_\Gamma c_{H_K,h'}$ that extends uniquely to a unitary G-equivalence

$$\Phi^\Gamma_{h'} : H \to \mathrm{Cl}_{L^2(\Gamma\backslash G)} P_\Gamma c_{H_K,h'}.$$

Proof. The assignment $h \mapsto P_\Gamma c_{h,h'}$ is obviously K-equivariant. To show that it is $\mathfrak{g}$-equivariant, we note that, given $h \in H_K$ and $X \in \mathfrak{g}$, we have

$$\begin{aligned}
\frac{d}{dt}\left(P_\Gamma c_{h,h'}\right)(g\exp(tX)) &= \sum_{\gamma\in\Gamma} \frac{d}{dt} c_{h,h'}(\gamma g\exp(tX)) \\
&= \sum_{\gamma\in\Gamma} \frac{d}{ds}\bigg|_{s=0} c_{h,h'}(\gamma g\exp((t+s)X)) \\
&= \sum_{\gamma\in\Gamma} \frac{d}{ds}\bigg|_{s=0} \langle \pi(\gamma g\exp(tX))\pi(\exp(sX))h, h'\rangle_H \\
&= \sum_{\gamma\in\Gamma} \langle \pi(\gamma g\exp(tX))\pi(X)h, h'\rangle_H \\
&= \left(P_\Gamma c_{\pi(X)h,h'}\right)(g\exp(tX)), \qquad g \in G,
\end{aligned}$$

where termwise differentiation in the first equality is valid since the resulting series converges absolutely and uniformly as t varies over any compact subset of $\mathbb{R}$. Evaluating at $t = 0$, we obtain

$$R(X)P_\Gamma c_{h,h'} = P_\Gamma c_{\pi(X)h,h'}.$$

Thus, the assignment $h \mapsto P_\Gamma c_{h,h'}$ defines a $(\mathfrak{g}, K)$-equivariant map $\Psi^\Gamma_{h'} : H_K \to P_\Gamma c_{H_K,h'}$.

Suppose that $P_\Gamma c_{H_K,h'} \neq 0$. Then, by the irreducibility of the $(\mathfrak{g}, K)$-module H_K, $\Psi^\Gamma_{h'}$ is an isomorphism of $(\mathfrak{g}, K)$-modules. Moreover, by [7, Theorem 6.4(i)], $\mathrm{Cl}_{L^2(\Gamma\backslash G)} P_\Gamma c_{H_K,h'}$ is an irreducible, closed, G-invariant subspace of $L^2(\Gamma\backslash G)$ that is unitarily equivalent to π. By Schur's lemma [11, §3.3.2], there exists $c \in \mathbb{R}_{>0}$ such that the inifinitesimal equivalence $c\,\Psi^\Gamma_{h'}$ of the G-representations (π, H) and $\mathrm{Cl}_{L^2(\Gamma\backslash G)} P_\Gamma c_{H_K,h'}$ extends to a unitary G-equivalence $\Phi^\Gamma_{h'}$. By the unitarity of $\Phi^\Gamma_{h'}$, we have

$$\|h'\|_H = \left\|\Phi^\Gamma_{h'}(h')\right\|_{L^2(\Gamma\backslash G)} = \left\|c\,\Psi^\Gamma_{h'}(h')\right\|_{L^2(\Gamma\backslash G)} = c\left\|P_\Gamma c_{h',h'}\right\|_{L^2(\Gamma\backslash G)},$$

hence $c = \left\|P_\Gamma c_{h',h'}\right\|^{-1}_{L^2(\Gamma\backslash G)} \|h'\|_H$. This implies (ii). □

The following proof of Theorem 1 is based on the techniques used in [5, proof of Theorem 2.3] (resp., [9, Proposition 7.1]) to compute the inner product of certain cuspidal automorphic forms on $\mathrm{SL}_2(\mathbb{R})$ (resp., $\mathrm{Sp}_{2n}(\mathbb{R})$).

Proof of Theorem 1. (i) Let $h, h', v, v' \in H_K$. Noting that the inner product formula (3) obviously holds if $P_\Gamma c_{H_K,h'} = 0$, we suppose that $P_\Gamma c_{H_K,h'} \neq 0$.

We have

$$\begin{aligned}
\left\langle P_\Gamma c_{h,h'}, P_\Gamma c_{v,v'}\right\rangle_{L^2(\Gamma\backslash G)} &= \int_{\Gamma\backslash G} \left(P_\Gamma c_{h,h'}\right)(g) \sum_{\gamma\in\Gamma} \overline{c_{v,v'}(\gamma g)}\, dg \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \left(P_\Gamma c_{h,h'}\right)(\gamma g)\, \overline{c_{v,v'}(\gamma g)}\, dg \\
&\overset{(4)}{=} \int_G \left(P_\Gamma c_{h,h'}\right)(g)\, \overline{c_{v,v'}(g)}\, dg \\
&\underset{\text{Lem. 1}}{\overset{(5)}{=}} \left(R(\overline{c_{v,v'}})P_\Gamma c_{h,h'}\right)(1_G),
\end{aligned} \tag{7}$$

where the right-hand side denotes the value at 1_G of the unique continuous function belonging to the equivalence class $R(\overline{c_{v,v'}})P_\Gamma c_{h,h'} \in L^2(\Gamma\backslash G)$.

On the other hand, by applying the G-equivalence

$$\Phi^\Gamma_{h'} \circ \Phi^{-1}_{h'} : c_{H,h'} \to \mathrm{Cl}_{L^2(\Gamma\backslash G)} P_\Gamma c_{H_K,h'}$$

to equality (6), we obtain

$$R(\overline{c_{v,v'}})P_\Gamma c_{h,h'} = \frac{1}{d(\pi)} \langle h, v\rangle_H\, P_\Gamma c_{v',h'},$$

from which it follows that

$$\left(R(\overline{c_{v,v'}})P_\Gamma c_{h,h'}\right)(1_G)=\frac{1}{d(\pi)}\left\langle h,v\right\rangle_H\left(P_\Gamma c_{v',h'}\right)(1_G). \tag{8}$$

Equalities (7) and (8) imply (3).

(ii) Let $h,h'\in H_K$ and $v,v'\in V_K$. By Lemma 1, we have

$$P_\Gamma c_{h,h'}\in L^2(\Gamma\backslash G)_{[\pi]}\qquad\text{and}\qquad P_\Gamma c_{v,v'}\in L^2(\Gamma\backslash G)_{[\sigma]}.$$

Since $L^2(\Gamma\backslash G)_{[\pi]}\perp L^2(\Gamma\backslash G)_{[\sigma]}$, this implies (ii). □

3. Application to vector-valued Siegel cusp forms

Let $n\in\mathbb{Z}_{>0}$. As in [10, Lemma 1.2], let (ρ,V) be an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight $\omega=(\omega_1,\dots,\omega_n)\in\mathbb{Z}^n$, where $\omega_1\geq\dots\geq\omega_n>2n$. Thus, ρ is the up to equivalence unique irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ such that there exists a unit vector $v^{top}\in V$ with the following two properties:

1. $\rho(B)v^{top}\subseteq\mathbb{C}v^{top}$, where B is the subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathbb{C})$.
2. $\rho(a)v^{top}=\left(\prod_{r=1}^n a_r^{\omega_r}\right)v^{top}$ for every diagonal matrix $a=\mathrm{diag}(a_1,\dots,a_n)\in\mathrm{GL}_n(\mathbb{C})$.

We equip V with a Hermitian inner product $\langle\,\cdot\,,\,\cdot\,\rangle_V$ with respect to which the restriction $\rho\big|_{\mathrm{U}(n)}$ is unitary.

Let $J_n=\begin{pmatrix} & I_n\\ -I_n & \end{pmatrix}\in\mathrm{GL}_{2n}(\mathbb{C})$, and let $i\in\mathbb{C}$ be the imaginary unit. The Lie group

$$\mathrm{Sp}_{2n}(\mathbb{R})=\left\{g\in\mathrm{SL}_{2n}(\mathbb{R}):g^\top J_n g=J_n\right\}$$

acts on the left on the Siegel upper half-space

$$\mathcal{H}_n=\left\{z=x+iy\in M_n(\mathbb{C}):z^\top=z\text{ and }y>0\right\}$$

by

$$g.z=(Az+B)(Cz+D)^{-1} \tag{9}$$

and on the right on the space $V^{\mathcal{H}_n}$ of functions $\mathcal{H}_n\to V$ by

$$\left(f\big|_\rho g\right)(z)=\rho(Cz+D)^{-1}f(g.z)$$

for all $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}_{2n}(\mathbb{R})$, $z\in\mathcal{H}_n$, and $f\in V^{\mathcal{H}_n}$. The maximal compact subgroup

$$K=\left\{k_{A+iB}:=\begin{pmatrix}A&B\\-B&A\end{pmatrix}\in\mathrm{GL}_{2n}(\mathbb{R}):A+iB\in\mathrm{U}(n)\right\}$$

of $\mathrm{Sp}_{2n}(\mathbb{R})$ is the stabilizer of iI_n with respect to action (9).

Let (π_ρ,H_ρ) be the integrable discrete series representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ defined on the complex Hilbert space H_ρ of holomorphic functions $f:\mathcal{H}_n\to V$ such that

$$\int_{\mathcal{H}_n}\left\|\rho\left(y^{\frac{1}{2}}\right)f(z)\right\|_V^2\,d\mathsf{v}(z)<\infty, \tag{10}$$

where $d\mathsf{v}(z)=\det y^{-n-1}\prod_{1\leq r\leq s\leq n}dx_{r,s}\,dy_{r,s}$. The Hilbert space norm on H_ρ is given by the square root of (10), and the action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on H_ρ is given by

$$\pi_\rho(g)f=f\big|_\rho g^{-1},\quad g\in\mathrm{Sp}_{2n}(\mathbb{R}),\ f\in H_\rho.$$

The $(\mathfrak{g}, K)$-module $(H_\rho)_K$ of K-finite vectors in H_ρ is spanned by the functions $f_{\mu,v} : \mathcal{H}_n \to V$,

$$f_{\mu,v}(z) = \mu\left((z - iI_n)(z + iI_n)^{-1}\right)\rho\left(\frac{1}{2i}(z + iI_n)\right)^{-1} v, \tag{11}$$

where v runs over V and μ runs over $\mathbb{C}[X_{r,s} : 1 \le r, s \le n]$, the ring of polynomials with complex coefficients in n^2 variables $X_{r,s}$ with $r, s \in \{1, 2, \ldots, n\}$ [10, (15)].

Given a complex Hilbert space H, let us denote its dual Hilbert space by H^*; thus, H^* is the space of linear functionals $h^* = \langle\,\cdot\,, h\rangle_H : H \to \mathbb{C}$, where $h \in H$. The contragredient representation of π_ρ is the discrete series representation (π_ρ^*, H_ρ^*) of $\mathrm{Sp}_{2n}(\mathbb{R})$ given by

$$\pi_\rho^*(g)f^* = \left(\pi_\rho(g)f\right)^*, \quad g \in \mathrm{Sp}_{2n}(\mathbb{R}),\ f \in H_\rho.$$

The representation π_ρ^* belongs to the integrable discrete series of $\mathrm{Sp}_{2n}(\mathbb{R})$ since for all $h, h' \in \left(H_\rho\right)_K$, we have the elementary equality

$$c_{h^*,(h')^*} = \overline{c_{h,h'}} \in L^1(\mathrm{Sp}_{2n}(\mathbb{R})).$$

Obviously, we have

$$(H_\rho^*)_K = \mathrm{span}_{\mathbb{C}}\left\{f_{\mu,v}^* : \mu \in \mathbb{C}[X_{r,s} : 1 \le r, s \le n],\ v \in V\right\}.$$

Given a function $f : \mathcal{H}_n \to V$, we define a function $F_f : \mathrm{Sp}_{2n}(\mathbb{R}) \to V$,

$$F_f(g) = \left(f\big|_\rho g\right)(iI_n). \tag{12}$$

By [10, Proposition 6.4(ii)], given $f \in (H_\rho)_K$ and $v \in V$, the K-finite matrix coefficient $c_{f_{1,v}^*, f^*}$ of the representation π_ρ^* is given by

$$c_{f_{1,v}^*, f^*} = C_\rho\, v^* F_f, \tag{13}$$

where

$$C_\rho = \left\|f_{1,v^{top}}\right\|_{H_\rho}^2 \in \mathbb{R}_{>0}.$$

Lemma 4. *Let $v \in V$. Then, we have*

$$\left\|f_{1,v}^*\right\|_{H_\rho^*}^2 = C_\rho\, \|v\|_V^2. \tag{14}$$

Proof. We have

$$\begin{aligned}\left\|f_{1,v}^*\right\|_{H_\rho^*}^2 &= c_{f_{1,v}^*, f_{1,v}^*}(1_{\mathrm{Sp}_{2n}(\mathbb{R})}) \overset{(13)}{=} C_\rho \left\langle F_{f_{1,v}}(1_{\mathrm{Sp}_{2n}(\mathbb{R})}), v\right\rangle_V \\ &\overset{(12)}{=} C_\rho \left\langle f_{1,v}(iI_n), v\right\rangle_V \overset{(11)}{=} C_\rho \langle v, v\rangle_V = C_\rho\, \|v\|_V^2.\end{aligned}$$

□

Let Γ be a discrete subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ that is commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$. The space $S_\rho(\Gamma)$ of Siegel cusp forms of weight ρ for Γ is the finite-dimensional space of holomorphic functions $\phi : \mathcal{H}_n \to V$ with the following two properties (cf. [10, Lemma 4.1]):

(i) $\phi\big|_\rho \gamma = \phi$ for all $\gamma \in \Gamma$.

(ii) $\sup_{z \in \mathcal{H}_n} \left\|\rho\left(y^{\frac{1}{2}}\right)\phi(z)\right\|_V < \infty$.

We equip $S_\rho(\Gamma)$ with the Petersson inner product

$$\langle \phi_1, \phi_2\rangle_{S_\rho(\Gamma)} = \frac{1}{|\Gamma \cap \{\pm I_{2n}\}|}\int_{\Gamma\backslash\mathcal{H}_n} \langle \rho(y)\,\phi_1(z), \phi_2(z)\rangle_V\ d\mathrm{v}(z).$$

Proposition 2.

(i) *For every* $f \in (H_\rho)_K$*, the Poincaré series*

$$P_{\Gamma,\rho} f = \sum_{\gamma\in\Gamma} f\big|_\rho \gamma$$

converges absolutely and uniformly on compact subsets of $\mathcal{H}_n$.

(ii) *We have*

$$S_\rho(\Gamma) = \left\{ P_{\Gamma,\rho} f : f \in (H_\rho)_K \right\}.$$

(iii) *The assignment* $\phi \mapsto F_\phi$ *defines a unitary isomorphism* $\Phi_{\rho,\Gamma}$ *from* $S_\rho(\Gamma)$ *to a finite-dimensional subspace* $L_\rho(\Gamma)$ *of the Hilbert space* $L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}), V)$.

(iv) *For every* $f \in (H_\rho)_K$*, the Poincaré series*

$$P_\Gamma F_f = \sum_{\gamma\in\Gamma} F_f(\gamma\,\cdot\,)$$

converges absolutely and uniformly on compact subsets of $\mathrm{Sp}_{2n}(\mathbb{R})$*, and we have*

$$\Phi_{\rho,\Gamma}(P_{\Gamma,\rho} f) = F_{P_{\Gamma,\rho} f} = P_\Gamma F_f. \tag{15}$$

Proof. Claims (i) and (ii) follow from [10, Theorem 6.7], (iii) follows from [10, Lemma 4.4], and (iv) follows from (i) and [10, Lemma 6.6]. □

With the above results and the inner product formula (3) at hand, we are ready to give a representation-theoretic proof of Corollary 1, which is restated here for the reader's convenience as Corollary 2.

Corollary 2. *Let* $\phi \in S_\rho(\Gamma)$ *and* $v \in V$*. Then, we have*

$$\left\langle \phi, P_{\Gamma,\rho} f_{1,v} \right\rangle_{S_\rho(\Gamma)} = \frac{\dim_{\mathbb{C}} V}{d(\pi_\rho)} \left\langle \phi(iI_n), v \right\rangle_V .$$

Proof. Let us fix an orthonormal basis $(e_j)_{j=1}^{\dim_{\mathbb{C}} V}$ for V. We note that for all $\varphi_1, \varphi_2 \in L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}), V)$, we have

$$\left\langle \varphi_1, \varphi_2 \right\rangle_{L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)} = \sum_{j=1}^{\dim_{\mathbb{C}} V} \left\langle e_j^*\varphi_1, e_j^*\varphi_2 \right\rangle_{L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}))} . \tag{16}$$

By Proposition 2(ii), we have

$$\phi = P_{\Gamma,\rho} f \tag{17}$$

for some $f \in (H_\rho)_K$. By Proposition 2(iii) and (15), the Petersson inner product

$$\left\langle \phi, P_{\Gamma,\rho} f_{1,v} \right\rangle_{S_\rho(\Gamma)} = \left\langle P_{\Gamma,\rho} f, P_{\Gamma,\rho} f_{1,v} \right\rangle_{S_\rho(\Gamma)}$$

equals

$$\begin{aligned}
\left\langle P_\Gamma F_f, P_\Gamma F_{f_{1,v}} \right\rangle_{L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}),V)} &\overset{(16)}{=} \sum_{j=1}^{\dim_{\mathbb{C}} V} \left\langle e_j^* P_\Gamma F_f, e_j^* P_\Gamma F_{f_{1,v}} \right\rangle_{L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}))} \\
&= \sum_{j=1}^{\dim_{\mathbb{C}} V} \left\langle P_\Gamma e_j^* F_f, P_\Gamma e_j^* F_{f_{1,v}} \right\rangle_{L^2(\Gamma\backslash \mathrm{Sp}_{2n}(\mathbb{R}))} ,
\end{aligned}$$

which by formula (13) for matrix coefficients of π_ρ^* equals

$$C_\rho^{-2} \sum_{j=1}^{\dim_{\mathbb{C}} V} \left\langle P_\Gamma c_{f^*_{1,e_j}, f^*}, P_\Gamma c_{f^*_{1,e_j}, f^*_{1,v}} \right\rangle_{L^2(\Gamma \backslash \mathrm{Sp}_{2n}(\mathbb{R}))} . \tag{18}$$

Applying the inner product formula (3), we rewrite (18) as

$$C_\rho^{-2} \sum_{j=1}^{\dim_{\mathbb{C}} V} \frac{1}{d(\pi_\rho^*)} \left\| f^*_{1,e_j} \right\|^2_{H_\rho^*} \left(P_\Gamma c_{f^*_{1,v}, f^*} \right) \left(1_{\mathrm{Sp}_{2n}(\mathbb{R})} \right),$$

and then, noting that $d(\pi_\rho^*) = d(\pi_\rho)$ and applying (14) and, for the second time, formula (13) for matrix coefficients of π_ρ^*, as

$$\begin{aligned} \frac{\dim_{\mathbb{C}} V}{d(\pi_\rho)} \left(P_\Gamma v^* F_f \right) \left(1_{\mathrm{Sp}_{2n}(\mathbb{R})} \right) &\overset{(15)}{=} \frac{\dim_{\mathbb{C}} V}{d(\pi_\rho)} \left(v^* F_{P_{\Gamma,\rho} f} \right) \left(1_{\mathrm{Sp}_{2n}(\mathbb{R})} \right) \\ &\overset{(17)}{=} \frac{\dim_{\mathbb{C}} V}{d(\pi_\rho)} \left\langle F_\phi \left(1_{\mathrm{Sp}_{2n}(\mathbb{R})} \right), v \right\rangle_V \overset{(12)}{=} \frac{\dim_{\mathbb{C}} V}{d(\pi_\rho)} \langle \phi(iI_n), v \rangle_V , \end{aligned}$$

thus finishing the proof of the corollary. □

Acknowledgements

This work is supported by the Croatian Science Foundation under the project number HRZZ-IP-2022-10-4615.

References

[1] J. Dixmier, *C^*-algebras*, *North-Holland Mathematical Library, Vol. 15*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[2] R. Godement, *Série de Poincaré et Spitzenformen. Vol. 10, Séminaire Henri Cartan*, no. 1, exp. no. 10, 1957–1958, 1–38.

[3] A. W. Knapp, *Representation Theory of Semisimple Groups*, *Princeton Mathematical Series* **36**, Princeton University Press, Princeton, NJ, 1986.

[4] A. W. Knapp, *Lie Groups Beyond an Introduction*, second ed., *Progress in Mathematics* **140**, Birkhäuser Boston, Inc., Boston, MA, 2002.

[5] G. Muić, On the inner product of certain automorphic forms and applications, *J. Lie Theory* **22** (2012), no. 4, 1091–1107.

[6] G. Muić, Fourier coefficients of automorphic forms and integrable discrete series, *J. Funct. Anal.* **270** (2016), no. 10, 3639–3674.

[7] G. Muić, Smooth cuspidal automorphic forms and integrable discrete series, *Math. Z.* **292** (2019), no. 3-4, 895–922.

[8] S. Žunar, On the non-vanishing of Poincaré series on irreducible bounded symmetric domains, preprint. Available at `https://arxiv.org/abs/2501.06876`.

[9] S. Žunar, On a family of Siegel Poincaré series, *Int. J. Number Theory* **19** (2023), no. 9, 2215–2239.

[10] S. Žunar, Construction and non-vanishing of a family of vector-valued Siegel Poincaré series, *J. Number Theory* **268** (2025), 95–123.

[11] N. R. Wallach, *Real Reductive Groups. I*, *Pure and Applied Mathematics* **132**, Academic Press, Inc., Boston, MA, 1988.

[12] G. Warner, *Harmonic Analysis on Semi-simple Lie Groups. I*, *Die Grundlehren der mathematischen Wissenschaften, Band 188*, Springer-Verlag, New York-Heidelberg, 1972.