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On the Diophantine equation $F_n^2 + F_m^2 = 2^a$

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Abstract. Let F_n be the n-th Fibonacci number. In this paper, we show that the only nonnegative solutions (n, m, a) of the Diophantine equation $F_n^2 + F_m^2 = 2^a$ with $n \geq m \geq 0$ are $(1, 0, 0)$, $(2, 0, 0)$, $(3, 0, 2)$, $(6, 0, 6)$, $(1, 1, 1)$, $(2, 1, 1)$, $(2, 2, 1)$, $(3, 3, 3)$, $(6, 6, 7)$.

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1. Introduction

Let $(F_n)_{\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, for all $n \geq 0$. Lots of mathematicians have studied the interesting properties of the Fibonacci sequence.

Bravo and Luca [3] showed that the Diophantine equation $F_n + F_m = 2^a$ in nonnegative integers n, m, a has finitely many solutions. Bravo and Bravo [1] extended it so that the Diophantine equation $F_n + F_m + F_\ell = 2^a$ also has finitely many solutions. The number of Fibonacci numbers on the LHS has been studied up to five [13, 14].

The equation $F_n \pm F_m = y^a$ has also been studied [6, 8, 9, 10, 11, 5, 15]. Recently, Bravo, Díaz, and Luca [2] have found all nonnegative solutions of $(F_n + F_m)/(F_r + F_s) = 2^a$. As such, various forms of the equations are being studied.

In the present paper, we investigate the Diophantine equation

$$F_n^2 + F_m^2 = 2^a$$

in nonnegative integers n, m, a. As in the previous papers, we use the Matveev theorem to show the finiteness of the solutions in §2 and the Dujella-Pethő theorem to reduce the bound for n in §3. The result is as follows.

Theorem 1. *The only solutions of the Diophantine equation*

$$F_n^2 + F_m^2 = 2^a$$

in nonnegative integers n, m and a with $n \geq m$ are given by

$$(n, m, a) \in \left\{ \begin{array}{c} (1,0,0), (2,0,0), (3,0,2), (6,0,6), (1,1,1), \\ (2,1,1), (2,2,1), (3,3,3), (6,6,7) \end{array} \right\}.$$

2. Finiteness

To find a bound for n we apply Matveev's theorem (Theorem 2) to $\Lambda = \Lambda_1, \Lambda_2$ in two steps.

Step 1. Compute $\Lambda_1 = 2^a \cdot \sqrt{5}^2 \cdot \alpha^{-2n}$ to obtain $(n - m) \log \alpha <$ (an expression with $\log 2n$).

Step 2. Compute $\Lambda_2 = 2^a \cdot \sqrt{5}^2 \alpha^{-2m} (1 + \alpha^{2n-2m})^{-1}$ to obtain $n \log \alpha <$ (an expression with $(n-m) \log \alpha$).

Combining two inequalities, we obtain a bound of n. In this process, we need the *logarithmic height*.

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Definition 1. Let η be an algebraic number of degree d with minimal polynomial

$$a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d \prod_{i=1}^{d}(x - \eta^{(i)}),$$

where a_i's are relatively prime integers with $a_d > 0$, and $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_d + \sum_{i=1}^{d}\log\Big(\max\{|\eta^{(i)}|, 1\}\Big)\right)$$

is called the *logarithmic height* of η.

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$, then $h(\eta) = \log\max\{|p|, |q|\}$. The following properties holds:

- $h(\eta \pm \gamma) \le h(\eta) + h(\gamma) + \log 2$.
- $h(\eta\gamma^{\pm 1}) \le h(\eta) + h(\gamma)$.
- $h(\eta^s) = |s|h(\eta)$.

Theorem 2 (Matveev [12, Corollary 2.3], [4, Theorem 9.4])**.** *Assume that $\gamma_1, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field K of degree D, $b_1, \ldots, b_t \in \mathbb{Z}$, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t),$$

when the following holds:

- $B \ge \max\{|b_1|, \ldots, |b_t|\}$
- $A_i \ge \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\}$ *for all* $i = 1, \ldots, t$.

First, we compute Step 1. By the Binet formula

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \text{ for } n \ge 0,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

Note that $1 < \alpha < 2$ and

$$\alpha^{n-2} < F_n < \alpha^{n-1} < \alpha^n.$$

Thus, since

$$2^a = F_n^2 + F_m^2 < \alpha^{2n} + \alpha^{2m} < 2\alpha^{2n} < 2 \cdot 2^{2n} = 2^{2n+1},$$

we have $a \le 2n$. Arranging the equation

$$\left(\frac{\alpha^n - \beta^n}{\sqrt{5}}\right)^2 + \left(\frac{\alpha^m - \beta^m}{\sqrt{5}}\right)^2 = 2^a, \tag{1}$$

we obtain

$$\begin{aligned}\left|\frac{\alpha^{2n}}{5} - 2^a\right| &= \left|\frac{2\alpha^n\beta^n}{5} - \frac{\beta^{2n}}{5} - \frac{\alpha^{2m}}{5} + \frac{2\alpha^m\beta^m}{5} - \frac{\beta^{2m}}{5}\right| \\ &< \frac{2\alpha^{n+m}}{5} + \frac{\alpha^{n+m}}{5} + \frac{\alpha^{m+n}}{5} + \frac{2\alpha^{m+n}}{5} + \frac{\alpha^{n+m}}{5} \\ &= \frac{7}{5}\alpha^{n+m},\end{aligned}$$

since $\alpha > 1$ and $|\beta| < 1$. Dividing by $\frac{\alpha^{2n}}{5}$, we obtain an inequality

$$\left|1 - 2^a \cdot \frac{5}{\alpha^{2n}}\right| < \frac{7}{\alpha^{n-m}}. \tag{2}$$

We apply the Matveev theorem to

$$\Lambda_1 = 2^a \cdot \frac{5}{\alpha^{2n}} - 1.$$

It is clear that $\Lambda_1 \neq 0$. Assume that $t = 3$ and

$$\gamma_1 = 2, \gamma_2 = \sqrt{5}, \gamma_3 = \alpha; \quad b_1 = a, b_2 = 2, b_3 = -2n.$$

Then

$$B := 2n \geq \max\{|b_1|, |b_2|, |b_3|\}$$

and

$$\begin{aligned} A_1 := 1.4 \geq & \max\{Dh(\gamma_1), |\log \gamma_1|, 0.16\} \\ = & \max\{2\log 2, |\log 2|, 0.16\} \\ = & 2\log 2 = 1.386\ldots, \end{aligned}$$

where $D = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$. Next, we choose A_2. Note that the minimal polynomial of $\gamma_2 = \sqrt{5}$ is $x^2 - 5$ and

$$h(\gamma_2) = \frac{1}{2}(\log 1 + \log\sqrt{5} + \log|-\sqrt{5}|) = \frac{1}{2}\log 5.$$

Then

$$\begin{aligned} A_2 := 1.7 \geq & \max\{Dh(\gamma_2), |\log \gamma_2|, 0.16\} \\ = & \max\{2 \cdot \frac{1}{2}\log 5, |\log\sqrt{5}|, 0.16\} \\ = & \log 5 = 1.609\ldots. \end{aligned}$$

Finally, let us choose A_3 as follows:

$$\begin{aligned} A_3 := 0.49 \geq & \max\{Dh(\gamma_3), |\log \gamma_3|, 0.16\} \\ = & \max\{2 \cdot \frac{1}{2}\log\alpha, |\log\alpha|, 0.16\} \\ = & \log\alpha = 0.4812\ldots. \end{aligned}$$

Let

$$\begin{aligned} C_1 = & 1.4 \times 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D) \cdot A_1 \cdot A_2 \cdot A_3 \\ = & 1.4 \times 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2) \times 1.4 \times 1.7 \times 0.49 \\ = & 1.13091\ldots \times 10^{12}. \end{aligned}$$

Then, by Theorem 2 and inequality (2), we obtain the inequality

$$\exp(-C_1 \cdot (1 + \log 2n)) < |\Lambda_1| < \frac{7}{\alpha^{n-m}}.$$

Thus,

$$-C_1 \cdot (1 + \log 2n) < \log 7 - (n - m)\log\alpha$$

and then

$$\begin{aligned} (n - m)\log\alpha < & \log 7 + C_1(1 + \log 2n) \\ < & \log 7 + 1.131 \times 10^{12}(1 + \log 2n) \\ < & 1.14 \times 10^{12}(1 + \log 2n). \end{aligned} \tag{3}$$

Now, we compute Step 2. By arranging (1), we obtain

$$\frac{\alpha^{2n}}{5} + \frac{\alpha^{2m}}{5} - 2^a = \frac{2\alpha^n\beta^n}{5} - \frac{\beta^{2n}}{5} + \frac{2\alpha^m\beta^m}{5} - \frac{\beta^{2m}}{5}.$$

Since $|\beta| < 1$, $\alpha > 1$, and $n \geq m$, the inequality

$$\left|\frac{\alpha^{2n}}{5} + \frac{\alpha^{2m}}{5} - 2^a\right| < \frac{2}{5}\alpha^n + \frac{1}{5}\alpha^n + \frac{2}{5}\alpha^n + \frac{1}{5}\alpha^n = \frac{6}{5}\alpha^n$$

holds. Dividing both sides by $\frac{1}{5}\left(\alpha^{2m} + \alpha^{2n}\right)$, we obtain an inequality

$$\left|1 - 2^a \cdot \frac{5}{\alpha^{2m} + \alpha^{2n}}\right| < \frac{6}{\alpha^n}. \tag{4}$$

If we set $t = 4$ and

$$\gamma_1 = 2, \gamma_2 = \sqrt{5}, \gamma_3 = \alpha, \gamma_4 = 1 + \alpha^{2n-2m};$$
$$b_1 = a,\ b_2 = 2,\ b_3 = -2m,\ b_4 = -1,$$

then the LHS of (4) becomes

$$\Lambda_2 := 2^a \cdot \frac{5}{\alpha^{2m} + \alpha^{2n}} - 1 = \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3}\gamma_4^{b_4} - 1.$$

If $\Lambda_2 = 0$, by Galois conjugation $\overline{\alpha} = \beta$ in $\mathbb{Q}(\sqrt{5})$,

$$\alpha^{2n} \leq \alpha^{2m} + \alpha^{2n} = 2^a \cdot 4 = |\beta^{2m} + \beta^{2n}| \leq |\beta|^{2m} + |\beta|^{2n} \leq 2,$$

which yields a contradiction. So $\Lambda_2 \neq 0$. Note that

$$B := 2n \geq \max\{|b_1|, |b_2|, |b_3|, |b_4|\},$$

and we can set $A_1 = 1.4, A_2 = 1.7, A_3 = 0.49$ as in the previous step for Λ_1. Since

$$\begin{aligned} h(\gamma_4) =& h(1 + \alpha^{2n-2m}) \\ \leq& h(1) + h(\alpha^{2n-2m}) + \log 2 \\ =& |2n - 2m| h(\alpha) + \log 2 \\ =& (n - m)\log\alpha + \log 2, \end{aligned}$$

we have that

$$\begin{aligned} \max\{2h(\gamma_4), |\log\gamma_4|, 0.16\} \leq& \max\{2(n-m)\log\alpha + \log 4, \log(2\alpha^{2n-2m}), 0.16\} \\ =& 2(n-m)\log\alpha + \log 4 \\ =& 2(n-m)\log\alpha + 1.386\ldots, \end{aligned}$$

and thus we can set

$$A_4 := 2(n-m)\log\alpha + 1.4.$$

Then, by Theorem 2 and inequality (4), we obtain

$$\exp(-C_2 \cdot (1 + \log 2n) \cdot A_4) < |\Lambda_2| < \frac{6}{\alpha^n},$$

where

$$\begin{aligned} C_2 =& 1.4 \times 30^7 \times 4^{4.5} \times 2^2(1 + \log 2) \times 1.4 \times 1.7 \times 0.49 \\ =& 1.23815\ldots \times 10^{14}. \end{aligned}$$

Taking logarithms on both sides, we obtain

$$\begin{aligned} n \log\alpha &< \log 6 + C_2(1+\log 2n) \times A_4 \\ &< \log 6 + (1.3\cdot 10^{14})(1+\log 2n)\times(2(n-m)\log\alpha + 1.4) \\ &< \log 6 + (1.3\cdot 10^{14})(1+\log 2n) \\ &\quad \times (2\cdot 1.14\cdot 10^{12}\cdot(1+\log 2n)+1.4) \end{aligned} \tag{5}$$

by using (3). This inequality does not hold for $n > 3.18683\times 10^{30}$, i.e. the set of integer solutions of $F_n^2 + F_m^2 = 2^a$ is finite.

3. Reduction on n

To lower the upper bound of n we use Dujella–Pethő's reduction method. The following lemma is an immediate variation of the result due to Dujella and Pethő from [7].

Lemma 1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \le M \qquad \text{and} \qquad w \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

We use this reduction in two steps.

Step 1. Compute the upper bound of $w := n - m$.

Step 2. Compute the upper bound of $w := n$.

Now, we compute Step 1 of the reduction. Since

$$F_n{}^2 = \frac{\alpha^{2n}}{5} - \frac{2\alpha^n\beta^n}{5} + \frac{\beta^{2n}}{5},$$

we have that

$$\begin{aligned} \frac{\alpha^{2n}}{5} &= F_n{}^2 + \frac{2\alpha^n\beta^n}{5} - \frac{\beta^2 n}{5} \\ &< F_n{}^2 + \frac{2}{5} < F_n{}^2 + F_m{}^2 = 2^a. \end{aligned}$$

Hence, $\frac{\alpha^{2n}}{5} < 2^a$ or $1 < 2^a\cdot 5\cdot\alpha^{-2n}$. Taking logarithms on both sides, we obtain

$$z_1 := a\log 2 - 2n\log\alpha + \log 5 > 0.$$

Then, since

$$0 < z_1 \le e^{z_1} - 1 < \frac{7}{\alpha^{n-m}}$$

by inequality (2), we obtain an inequality

$$0 < a\log 2 - 2n\log\alpha + \log 5 < \frac{7}{\alpha^{n-m}}$$

and thus

$$0 < a\left(\frac{\log 2}{\log\alpha}\right) - 2n + \left(\frac{\log 5}{\log\alpha}\right) < \frac{7}{\log\alpha}\cdot\alpha^{-(n-m)} < 15\alpha^{-(n-m)}.$$

Now, let

$$u = a, \quad \gamma = \frac{\log 2}{\log \alpha} \notin \mathbb{Q}, \quad v = 2n, \quad \mu = \frac{\log 5}{\log \alpha},$$
$$A = 15 > 0, \quad B = \alpha > 1, \quad w = n - m.$$

Since $n < 3.18683 \times 10^{30}$ in the previous section and

$$u = a \leq 2n < 2 \cdot 3.18683 \times 10^{30} < 6.4 \times 10^{30},$$

we can set $M := 6.4 \times 10^{30}$. Then $q = q_{70} > 6M$ from the 70th convergent

$$\frac{p_{70}}{q_{70}} = \frac{2286663434222676088439108961099 13}{158749759840390984049158390593424}$$

of γ and

$$\begin{aligned}\epsilon =& \|\mu q\| - M\|\gamma q\| \\ =& 0.050737\ldots - 0.036496\ldots = 0.01424\ldots.\end{aligned}$$

Then, by Lemma 1

$$n - m < \frac{\log(Aq/\epsilon)}{\log B} < 168.542\ldots$$

and, by inequality (5) of Step 2 in the previous section,

$$\begin{aligned} n\log\alpha &< \log 6 + (1.3 \times 10^{14})(1 + \log 2n)(2(n-m)\log\alpha + 1.4) \\ &\leq \log 6 + (1.3 \times 10^{14})(1 + \log 2n)(2 \cdot 168 \cdot \log\alpha + 1.4). \end{aligned}$$

This inequality yields $1 \leq n < 1.92962 \times 10^{18}$. Now, we compute Step 2 of the reduction. By inequality (4),

$$|1 - e^{z_2}| < \frac{6}{\alpha^n}$$

holds if we set $\varphi(t) = 5\left(1 + \alpha^{2t}\right)^{-1}$ and

$$z_2 := a\log 2 - 2m\log\alpha + \log\varphi(n - m).$$

If $z_2 > 0$, then

$$0 < z_2 < e^{z_2} - 1 < \frac{6}{\alpha^n}.$$

On the other hand, if $z_2 < 0$, then

$$|1 - e^{z_2}| < \frac{6}{\alpha^n} < 6 \cdot \frac{1}{12} = \frac{1}{2}$$

for $n > 6$, and thus

$$0 < |z_2| \leq e^{|z_2|} - 1 = e^{|z_2|}|e^{z_2} - 1| < 2 \cdot \frac{6}{\alpha^n} = \frac{12}{\alpha^n}$$

holds. Hence we have

$$0 < |z_2| < \frac{12}{\alpha^n}.$$

Dividing both sides by $\log\alpha$, we obtain

$$0 < \left| a\left(\frac{\log 2}{\log\alpha}\right) - 2m + \left(\frac{\log\varphi(n-m)}{\log\alpha}\right)\right| < \frac{12}{\log\alpha} \cdot \alpha^{-n} < 25\alpha^{-n}$$

and set

$$u = a, \quad \gamma = \frac{\log 2}{\log\alpha} \notin \mathbb{Q}, \quad v = 2m, \quad \mu = \frac{\log\varphi(n-m)}{\log\alpha},$$
$$A = 25 > 0, \quad B = \alpha > 1, \quad w = n.$$

Since

$$u = a \leq 2n < 2 \cdot 1.92962 \times 10^{18} < 4 \cdot 10^{18},$$

we can set $M := 4 \cdot 10^{18}$. Then $q = q_{92} > 6M$ from the 92nd convergent

$$\frac{p_{92}}{q_{92}} = \frac{110504885181872237696754298142651378827960967}{767171229541946277425325161137983029443001 09}$$

of γ and $\epsilon = \|\mu q\| - M\|\gamma q\| > 0$ for each $n - m = 1, 2, \ldots, 168$. Then, by Lemma 1,

$$w = n < \frac{\log(Aq/\epsilon)}{\log B}$$

holds for each value of $n - m$. The largest value of the RHS is $230.46555\ldots$ when $n - m = 35$. So, we may check $F_n^2 + F_m^2 = 2^a$ for $0 \leq m \leq n \leq 230$. We can find all nonnegative solutions of the equation with the help of computers for this feasible bound.

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