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New iterative algorithms for signal approximation by using frames in a Hilbert space

Hassan Jamali* and Reza Pourkani

Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O. Box 518, Rafsanjan, Iran

Abstract. The aim of this paper is to improve the methods of reconstruction by frames. We present two iterative algorithms to approximate a signal f based on a given frame, by defining a recursive sequence that converges to the signal in a separable Hilbert space. To do so, we initiate the process by squaring the convergence rate of the classical frame algorithm. This, in turn, halves the number of required iterations while concurrently enhancing the overall speed of convergence. Then, by using the Chebyshev polynomials, we further improve the recently studied acceleration. Due to the special conditions of some signals, we may have to use frames with large condition numbers. The importance of these algorithms is better understood when the frame has a large condition number. Numerical results are presented to show that the established results are valid and the proposed algorithms are applicable. The results presented here represent a significant stride towards the advancement and acceleration of the existing frame algorithms. These findings introduce novel iterative techniques characterized by modified convergence rates, specifically designed for signal reconstruction and approximation.

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1. Introduction

Over the past few decades, frame theory has seen an impressive surge in its growth and development. This field of mathematics, which draws inspiration from linear algebra and signal processing, has undergone significant evolution, resulting in numerous exciting advancements in both its theoretical foundations and practical applications. The concept of frame is due to Duffin and Schaeffer, introduced in the context of non-harmonic Fourier series [12]. At first, frames were only used for irregular sampling problems (see [4, 22]). But, with the advent of wavelet theory, the real value and main importance of frames became clear [10, 15]. Nowadays, there exists a vast body of literature on their applications, and the current research landscape of frames is quite wide, including the construction of frames and their generalizations [3, 14, 20, 23]. Nevertheless, some other areas, such as applications to numerical analysis and approximation, have not been fully explored and require more attention [1, 2].

One of the attractions of frames is due to the representation of a function in a Hilbert space. In fact, frames have provided great flexibility in the approximation and representation of functions. In particular, frame decompositions are resilient against noise and quantization, and provide numerically stable reconstruction algorithms [5, 8].

In most signal processing problems, to reconstruct a signal in a separable Hilbert space, the information of the signal is stated as a sequence $\{c_n\}_n$ of scalars [10]. Therefore, it is crucial to know the way this sequence can be computed or approximated.

In this sense, due to various known factors of flexibility for frames, including their redundancy, stability, adaptivity, and diversity for the approximation of functions, we can use them instead of bases for problems in which it is not straightforward or even feasible to identify a basis.

An intuitive explanation can be given as to why frames play a key role in signal transmission. Let us assume that we aim to transmit a signal f, belonging to a Hilbert space $\mathcal{H}$, from a transmitter $\mathcal{A}$ to a receiver $\mathcal{R}$. Assume that both $\mathcal{A}$ and $\mathcal{R}$ know a frame $\{f_i\}_{i=1}^m$ for $\mathcal{H}$, and $\mathcal{A}$ transmits the frame coefficients

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Email addresses: jamali@mail.vru.ac.ir (H. Jamali), r.poorkani@gmail.com (R. Pourkani)
*Corresponding author.

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$\{< f, S^{-1} f_i >\}$, where S is the frame operator that is symmetric, invertible and positive-definite. Then, based on the knowledge of these numbers, the receiver $\mathcal{R}$ can reconstruct the signal f using the frame decomposition. Now, assume that $\mathcal{R}$ receives a noisy signal, that is, a perturbation $\{\langle f, S^{-1} f_i\rangle + c_i\}_{i=1}^m$ of the correct frame coefficients. Based on the received coefficients, $\mathcal{R}$ claims the transmitted signal to be

$$\sum_{i=1}^m \left(\langle f, S^{-1} f_i\rangle + c_i\right) f_i = \sum_{i=1}^m \langle f, S^{-1} f_i\rangle f_i + \sum_{i=1}^m c_i f_i = f + \sum_{i=1}^m c_i f_i,$$

which differs from the correct signal f by the noise $\sum_{i=1}^m c_i f_i$. If the frame is overcomplete, then the parts of noise contribution might add up to zero and cancel, what will never happen if $\{f_i\}_{i=1}^m$ is an orthonormal basis [8].

This has led to significant applications in signal analysis [1, 17, 9, 11, 16]. In this direction, there have been some efforts in the framework of the frame algorithm and its convergence acceleration [13, 18, 19, 21]. The acceleration methods are indispensable for frames having bad conditions. The aim of this work is to design some algorithms for approximating a signal by using frames, those which perform better in terms of the convergence rate and the speed of convergence compared to the existing algorithms in the literature. The novelty of this work lies in establishing a connection between numerical analysis and frame theory to improve the usual reconstruction technique based on the concept of frame. It is hoped that this paper can facilitate the study and applications of frames in the fields of approximation theory and numerical analysis.

The remainder of this paper is organized as follows. The basic concepts and some properties of frames are presented in Section 2. In Section 3, we introduce a recursive formula to approximate any element of a Hilbert space based on a given frame, and then design an algorithm based on the recursive formula. Section 4 deals with an optimization problem and a modification of the algorithm introduced in Section 3. In Section 5, we examine our algorithms by some numerical examples. Finally, a summary of the conclusions of this paper is given in Section 6.

2. Preliminaries

Throughout this paper, $\mathcal{H}$ stands for a separable Hilbert space, I is a countable index set, and $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. First, following Cassazza [6], we briefly review the basics of frame theory.

Definition 1. A sequence $\{f_i\}_{i\in I}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \textstyle\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2$$

holds for all $f \in \mathcal{H}$. The constants A and B are called the lower bound and the upper bound, respectively. The smallest upper bound and the largest lower bound are called the optimal bounds of the frame. If $A = B$, it is called an A**-tight frame**.

The *frame operator* of a frame $\{f_i\}_{i\in I}$ is defined by

$$S : \mathcal{H} \longrightarrow \mathcal{H},\ Sf = \textstyle\sum_{i\in I}\langle f, f_i\rangle f_i.$$

The operator S is invertible, positive-definite and self-adjoint [8]. In fact,

$$AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}.$$

Moreover, the family $\{S^{-1} f_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, called the *canonical dual* of $\{f_i\}_{i\in I}$. More precisely, in view of the aforementioned properties of S, the reconstruction formula of a signal $f \in \mathcal{H}$ is given by

$$f = SS^{-1} f = \sum_{i\in I} \langle f, S^{-1} f_i\rangle f_i,$$

or

$$f = S^{-1} S f = \sum_{i\in I} \langle f, f_i\rangle S^{-1} f_i.$$

However, in order to apply these formulas, we need to compute the inverse of the frame operator, which can be complicated or even impossible when $\mathcal{H}$ is of high dimension or infinite-dimensional. The classical frame algorithm [8, 13] provides a way to approximate a signal f based on a given frame.

Basically, this algorithm is designed based on Richardson's iterative method:

$$g_0 = 0, \qquad g_j = g_{j-1} + \frac{2}{A+B}\left(f - Sg_{j-1}\right), \qquad j = 1, 2, 3, \dots .$$

The following lemma ensures the convergence of the classical frame algorithm.

Lemma 1. *[8] Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds A and B and the frame operator S. Then,*

$$\left\| I_{\mathcal{H}} - \frac{2}{A+B}S \right\| \leq \frac{B-A}{B+A}.$$

The content of the above discussion is summarized in the following algorithm.

The classical frame algorithm

1. Set $\rho = \left(\frac{B-A}{B+A}\right)$ and choose $\epsilon > 0$.
2. Let $j = 0$ and $g_j = 0$.
3. a) Set $j = j + 1$ and $r_{j-1} := f - Sg_{j-1}$.
   b) Let $g_j = g_{j-1} + \frac{2}{A+B}\left(r_{j-1}\right)$.
4. If $\rho^j \|f\| < \epsilon$, then stop and set g_j as an approximation for f; otherwise, go to Step 3.

In the case of a large condition number (the ratio of the upper bound to the lower bound), applying the Chebyshev method leads to another algorithm with convergence rate $\dfrac{2\sigma^j}{1+\sigma^{2j}}$, where $\sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$. For more details, see [8].

In this paper, by extending Richardson's method, we present an algorithm with convergence rate $\left(\frac{B-A}{B+A}\right)^2$, which is the square of the above convergence rate. Insignificantly, compared to the frame algorithm, this modification halves the number of iterations.

3. A modification of the classical frame algorithm

In this section, we propose a new recursive formula for approximating any element f based on a given frame in a Hilbert space $\mathcal{H}$. Then, we design an algorithm based on this formula whose convergence rate is the square of that of the frame algorithm.

Given $f \in \mathcal{H}$, consider the recursive formula

$$g_0 = 0, \qquad g_j = g_{j-1} + \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right)S\left(f - g_{j-1}\right). \tag{1}$$

In this case, the following lemma holds.

Lemma 2. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds A and B and frame operator S. Then,*

$$\left\| I_{\mathcal{H}} - \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right)S \right\| \leq \left(\frac{B-A}{B+A}\right)^2.$$

Proof. In view of Lemma 1,

$$\begin{aligned} \left\| I_{\mathcal{H}} - \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right)S \right\| &= \left\| I_{\mathcal{H}} - \frac{4}{A+B}S + \frac{4}{(A+B)^2}S^2 \right\| = \left\| \left(I_{\mathcal{H}} - \frac{2}{A+B}S\right)^2 \right\| \\ &\leq \left\| I_{\mathcal{H}} - \frac{2}{A+B}S \right\|^2 \leq \left(\frac{B-A}{B+A}\right)^2, \end{aligned}$$

where the last inequality can be deduced from Lemma 1. □

The following theorem shows that the recursive formula (1) squares the convergence rate of the classical frame algorithm.

Theorem 1. *Let* $\{f_i\}_{i\in I}$ *be a frame for* $\mathcal{H}$ *with frame bounds* A *and* B *and frame operator* S*. Given* $f \in \mathcal{H}$*, the sequence* $\{g_j\}_{j=1}^{\infty}$ *defined in* (1) *converges to* f*, and*

$$\|f - g_j\| \leq \left(\tfrac{B-A}{B+A}\right)^{2j} \|f\|.$$

Proof. By our definition of the terms g_j, we conclude that

$$\begin{aligned} f - g_j =& f - g_{j-1} - \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right) S\left(f - g_{j-1}\right) \\ =& \left[I_{\mathcal{H}} - \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right) S\right]\left(f - g_{j-1}\right). \end{aligned}$$

Thus

$$\begin{aligned} \|f - g_j\| = & \quad \| \left(I_{\mathcal{H}} - \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right) S\right)\left(f - g_{j-1}\right) \| \\ \leq & \quad \| \left(I_{\mathcal{H}} - \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right) S\right) \| \|f - g_{j-1}\|, \end{aligned}$$

and the desired result follows from Lemma 2. □

The above theorem shows that the sequence $\{g_j\}_{j=1}^{\infty}$ converges to f with the convergence rate $\left(\frac{B-A}{B+A}\right)^2$.

We summarize these results in the following algorithm, which generates an approximation for $f \in \mathcal{H}$ with an arbitrary accuracy $\epsilon > 0$.

Algorithm 1

1. Set $\rho = \left(\frac{B-A}{B+A}\right)^2$ and choose $\epsilon > 0$.
2. Let $j = 0$ and $g_j = 0$.
3. a) Set $j = j + 1, r_{j-1} := f - g_{j-1}, v_{j-1} := Sr_{j-1}$.
   b) Let $g_j = g_{j-1} + \frac{4}{A+B}\left(v_{j-1} - \frac{1}{A+B}Sv_{j-1}\right)$.
4. If $\rho^j\|f\| < \epsilon$, then stop and set g_j as an approximation for f; otherwise go to Step 3.

It is easy to check that the convergence rate of this algorithm is the square of that of the classical frame algorithm. This feature can be better seen when A and B are far away from each other, that is, when the condition number of the frame, namely $\frac{A}{B}$, is big. Actually, the closer the condition number is to 1, the faster this algorithm converges.

4. Acceleration by the Chebyshev polynomials

As observed in the previous section, the convergence of Algorithm 1 depends on the upper and lower bounds of the frame, in other words, on the condition number. Unfortunately, in most available frames, just a rough upper bound and the existence of a lower bound $A > 0$ are given. Therefore, in order to improve Algorithm 1, we try to increase the speed of this algorithm by using the properties of Chebyshev polynomials.

In this section, we use these properties to present an algorithm whose convergence rate is $\dfrac{2\sigma^j}{1+\sigma^{2j}}$, where $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$. This convergence rate is relatively smaller than that obtained in Algorithm 1.

To begin with, for $f \in \mathcal{H}$, consider the sequence $\{h_n\}_{n=1}^{\infty}$ defined by $h_n = \sum_{j=1}^{n} a_{n_j} g_j$, with the property that $\sum_{j=1}^{n} a_{n_j} = 1$, and let $\{g_j\}_{j=1}^{\infty}$ be as in (1). Note that by the condition $\sum_{j=1}^{n} a_{n_j} = 1$, if

$g_1 = g_2 = \cdots = g_n = f$, then $h_n = f$.
Hence,

$$\begin{aligned} f - h_n &= \sum_{j=1}^{n} a_{n_j} f - \sum_{j=1}^{n} a_{n_j} g_j = \sum_{j=1}^{n} a_{n_j}(f - g_j) \\ &= \sum_{j=1}^{n} a_{n_j} \left(I_{\mathcal{H}} - \frac{2}{A+B} S \right)^{2j} (f - g_0). \end{aligned}$$

Letting $R = \left(I_{\mathcal{H}} - \dfrac{2}{A+B} S \right)^2$ and $Q_n(x) = \sum_{j=1}^{n} a_{n_j} x^j$ we obtain

$$f - h_n = Q_n(R)(f - g_0). \tag{2}$$

Thus, the error of this approximation is the consequence of applying a polynomial in terms of R to the initial error $f - g_0$.

Remark 1. Since

$$-\left(\frac{B-A}{B+A} \right)^2 \|f\|^2 \leq \left\langle \left(I_{\mathcal{H}} - \frac{2}{A+B} S \right)^2 f, f \right\rangle \leq \left(\frac{B-A}{B+A} \right)^2 \|f\|^2,$$

the spectrum of R lies in the interval $[-\rho, \rho]$, where $\rho = \left(\frac{B-A}{B+A} \right)^2$.

Therefore, in view of (2), the spectral theorem implies

$$\|f - h_n\| \leq \|Q_n(R)\| \|f - g_0\| \leq \max_{|x| \leq \rho} |Q_n(x)| \|f - g_0\|. \tag{3}$$

Now, to minimize this error, we should solve the minimization problem

$$\min_{Q \in \mathcal{Q}_n} \max_{|x| \leq \rho} |Q(x)|, \tag{4}$$

where $\mathcal{Q}_n := \{Q(x) : degQ \leq n, Q(1) = 1\}$.

This problem can be solved in terms of Chebyshev polynomials [7]. By using the properties of these polynomials, we design an algorithm with a better convergence rate. The polynomials have an important minimization property that makes them useful for convergence acceleration. The Chebyshev polynomials are defined by

$$c_n(x) = \begin{cases} \cos(n \arccos(x)) & |x| \leq 1 \\ \cosh(n \cosh^{-1}(x)) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1} \right)^n + \left(x + \sqrt{x^2 - 1} \right)^{-n} \right] & |x| > 1 \end{cases}$$

It is known that

$$c_0(x) = 1, c_1(x) = x, c_n(x) = 2x c_{n-1}(x) - c_{n-2}(x), \forall n \geq 2.$$

We refer the reader to [7] for more details.

The following lemma shows that these polynomials solve minimization problem (4).

Lemma 3. *[7] For $a < b < 1$, let $P_n(x) = \dfrac{c_n\left(\dfrac{2x - a - b}{b - a} \right)}{c_n\left(\dfrac{2 - a - b}{b - a} \right)}$. Then, for each $Q \in \mathcal{Q}_n$,*

$$\max_{a \leq x \leq b} |P_n(x)| \leq \max_{a \leq x \leq b} |Q(x)|.$$

Furthermore,

$$\max_{a \leq x \leq b} |P_n(x)| = \frac{1}{c_n\left(\dfrac{2 - a - b}{b - a} \right)}.$$

Corollary 1. *The polynomial* $\dfrac{c_n\left(\frac{x}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)}$ *minimizes the error* $\|f-h_n\|$.

Proof. It is enough to put $a=-b=-\rho$ in the previous lemma. Thus, the polynomial

$$P_n(x)=\frac{c_n\left(\dfrac{2x-a-b}{b-a}\right)}{c_n\left(\dfrac{2-a-b}{b-a}\right)}=\frac{c_n\left(\frac{x}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)} \tag{5}$$

minimizes the absolute error in (3). □

Now, to obtain an appropriate algorithm based on these polynomials, we first prove the following recursive relation for the approximate solutions h_n.

Proposition 1. *For each* $n\geq 2$*, the approximate solution* h_n*, with the smallest absolute error according to Corollary 1, satisfies*

$$h_n=\rho_n\left[h_{n-1}-h_{n-2}+\frac{4}{A+B}\left(I_{\mathcal{H}}-\frac{1}{A+B}S\right)S\left(f-h_{n-1}\right)\right]+h_{n-2},$$

where $\rho_n=\frac{\frac{2}{\rho}c_{n-1}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}$.

Proof. Combining the definition of $c_n(x)$ and (5) for $n\geq 2$, we obtain

$$\begin{aligned}c_n(\frac{1}{\rho})P_n(x)&=c_n(\frac{x}{\rho})\\&=\frac{2x}{\rho}c_{n-1}(\frac{x}{\rho})-c_{n-2}(\frac{x}{\rho})\\&=\frac{2x}{\rho}c_{n-1}(\frac{1}{\rho})P_{n-1}(x)-c_{n-2}(\frac{1}{\rho})P_{n-2}(x).\end{aligned}$$

Replacing x by R in this relation, it follows that

$$c_n(\frac{1}{\rho})P_n(R)=\frac{2R}{\rho}c_{n-1}(\frac{1}{\rho})P_{n-1}(R)-c_{n-2}(\frac{1}{\rho})P_{n-2}(R).$$

Now, by applying the two sides of this equality to $(f-g_0)$ and by considering (2), we deduce $f-h_n=P_n(R)(f-g_0)$ and conclude that

$$c_n(\frac{1}{\rho})(f-h_n)=\frac{2}{\rho}c_{n-1}(\frac{1}{\rho})R(f-h_{n-1})-c_{n-2}(\frac{1}{\rho})(f-h_{n-2}).$$

Hence, letting $R=\left(I_{\mathcal{H}}-\dfrac{2}{A+B}S\right)^2$ we obtain

$$c_n(\frac{1}{\rho})f-c_n(\frac{1}{\rho})h_n=\frac{2}{\rho}c_{n-1}(\frac{1}{\rho})\left(I_{\mathcal{H}}-\frac{2}{A+B}S\right)^2(f-h_{n-1})-c_{n-2}(\frac{1}{\rho})(f-h_{n-2}),$$

or equivalently,

$$\begin{aligned}c_n(\frac{1}{\rho})f-c_n(\frac{1}{\rho})h_n=&\frac{2}{\rho}c_{n-1}(\frac{1}{\rho})f+\frac{2}{\rho}c_{n-1}(\frac{1}{\rho})\left[-h_{n-1}-\left(\frac{4}{A+B}S-\frac{4}{(A+B)^2}S^2\right)(f-h_{n-1})\right]\\&-c_{n-2}(\frac{1}{\rho})f+c_{n-2}(\frac{1}{\rho})h_{n-2}.\end{aligned}$$

In view of (5), for $n \geq 2$,

$$c_n(\frac{1}{\rho})h_n = \frac{2}{\rho}c_{n-1}(\frac{1}{\rho})\left[h_{n-1} + \left(\frac{4}{A+B}S - \frac{4}{(A+B)^2}S^2\right)(f - h_{n-1})\right] - c_{n-2}(\frac{1}{\rho})h_{n-2}.$$

Therefore,

$$h_n = \frac{2}{\rho}\frac{c_{n-1}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}\left[h_{n-1} + \left(\frac{4}{A+B}S - \frac{4}{(A+B)^2}S^2\right)(f - h_{n-1})\right] - \frac{c_{n-2}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}h_{n-2}. \tag{6}$$

On the other hand, by relation (5) and the definition of ρ_n, we obtain

$$1 - \rho_n = 1 - \frac{\frac{2}{\rho}c_{n-1}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})} = -\frac{c_{n-2}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}.$$

In this case, we can rewrite (6) in the form

$$h_n = \rho_n\left[h_{n-1} + \left(\frac{4}{A+B}S - \frac{4}{(A+B)^2}S^2\right)(f - h_{n-1})\right] + (1-\rho_n)h_{n-2},$$

or equivalently,

$$\begin{aligned} h_n =& \rho_n\left[h_{n-1} - h_{n-2} + \left(\frac{4}{A+B}S - \frac{4}{(A+B)^2}S^2\right)(f - h_{n-1})\right] + h_{n-2} \\ =& \rho_n\left[h_{n-1} - h_{n-2} + \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right)S(f - h_{n-1})\right] + h_{n-2}. \end{aligned}$$

This completes the proof. □

Here, a simple calculation allows us to obtain the following recursive formula for the sequence $\{\rho_n\}_n$.

$$\rho_n = \left(1 - \frac{\rho^2}{4}\rho_{n-1}\right)^{-1}.$$

We are now ready to present another algorithm.

Algorithm 2

1. Set $\rho = \left(\frac{B-A}{B+A}\right)^2$, $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$ and choose $\epsilon > 0$.
2. Set $h_0 = 0$, $h_1 = \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right)f$, $\rho_1 = 2$, $n = 1$.
3. While $\frac{2\sigma^n}{1+\sigma^{2n}}\|f\| > \epsilon$, do:
   i) $n = n + 1$;
   ii) $\rho_n = \left(1 - \frac{\rho^2}{4}\rho_{n-1}\right)^{-1}$;
   iii) $h_n = \rho_n\left[h_{n-1} - h_{n-2} + \frac{4}{A+B}\left(I_{\mathcal{H}} - \frac{1}{A+B}S\right)(f - Sh_{n-1})\right] + h_{n-2}$.
4. $u_\epsilon = h_n$.

The following theorem investigates the convergence of Algorithm 2. In fact, we prove that the sequence $\{h_n\}_n$ converges to f with a better convergence rate compared to the sequence $\{g_n\}_n$.

Theorem 2. *Let* $\{f_i\}_{i\in I}$ *be a frame for a Hilbert space* $\mathcal{H}$ *with frame bounds* A *and* B *and frame operator* S*, and let* $f \in \mathcal{H}$*. Then, the sequence* $\{h_n\}_{n=1}^{\infty}$ *obtained from Algorithm 2 satisfies the inequality*

$$\|f - h_n\| \le \frac{2\sigma^n}{1+\sigma^{2n}}\|f\|,$$

where

$$\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}.$$

Proof. Using Lemma 3 and relation (3) we obtain

$$\|f - h_n\| \le \frac{1}{c_n(\frac{1}{\rho})}\|f - h_0\| = \frac{1}{c_n(\frac{1}{\rho})}\|f\|.$$

On the other hand, since $\frac{1}{\rho} > 1$,

$$\begin{aligned} c_n(\frac{1}{\rho}) =& c_n((\frac{B+A}{B-A})^2) \\ =& \frac{1}{2}\left[\left[(\frac{B+A}{B-A})^2 + \sqrt{(\frac{B+A}{B-A})^4 - 1}\right]^n + \left[(\frac{B+A}{B-A})^2 + \sqrt{(\frac{B+A}{B-A})^4 - 1}\right]^{-n}\right] \\ =& \frac{1}{2}\left[\left[\frac{\sqrt{A^2+B^2}+\sqrt{2AB}}{\sqrt{A^2+B^2}-\sqrt{2AB}}\right]^n + \left[\frac{\sqrt{A^2+B^2}+\sqrt{2AB}}{\sqrt{A^2+B^2}-\sqrt{2AB}}\right]^{-n}\right] \\ =& \frac{1}{2}\left(\frac{1}{\sigma^n} + \sigma^n\right) = \frac{1+\sigma^{2n}}{2\sigma^n}, \end{aligned}$$

where $\sigma = \frac{\sqrt{A^2+B^2}-\sqrt{2AB}}{\sqrt{A^2+B^2}+\sqrt{2AB}}$. This equality, together with (6), completes the proof. □

Straightforward computations show that $\frac{2\sigma^n}{1+\sigma^{2n}} < \rho_n$, which means that the convergence rate of the sequence $\{h_n\}_n$ is smaller than that of the sequence $\{g_n\}_n$. This convergence rate is also less than that of the sequence obtained by convergence acceleration of the frame algorithm presented in [13].

5. Numerical results

In this section, we examine the performance of our algorithms by providing some numerical experiments. We also aim to compare our algorithms with each other and with the classical frame algorithm in terms of the runtime. Note that all of the reported experiments were performed on a system with a 64-bit 2.4 GHz core i5 processor and 4.00 GB RAM using MATLAB version 2010.

The first example compares the convergence rate of Algorithm 1 with that of Algorithm 2, for a general 100-dimensional Hilbert space.

Example 1. Let $\mathcal{H}$ be a Hilbert space of dimension 100. Assume that $\{e_i\}_{i=1}^{100}$ is an orthonormal basis for $\mathcal{H}$, and let

$$\{f_i\}_{i=1}^{5050} = \{e_1, e_2, e_2, e_3, e_3, e_3, ..., e_{100}, ..., e_{100}\},$$

that is, the sequence in which each vector e_i is repeated i times. Thus, $\{f_i\}_{i=1}^{5050}$ is a frame for $\mathcal{H}$ with frame bounds $A = 1$ and $B = 100$, respectively [8]. Assume that $f \in \mathcal{H}$, and to simplify the calculations, suppose that $\|f\| = 1$. Then, by using Algorithm 1 with $\epsilon = 0.1$ in Step 1, we obtain $\rho = \left(\frac{B-A}{B+A}\right)^2 \simeq 0.961$. We observed that Algorithm 1 requires 58 steps to converge to numerical precision. With the stopping criterion $\frac{2\sigma^n}{1+\sigma^{2n}}\|f\| \le \epsilon$ such that $\sigma \simeq 0.382$ and $\epsilon = 0.1$, Algorithm 2 converges in 4 iterations.

In the following example, $\mathcal{H}$ is considered to be a finite-dimensional subspace of $\underline{L}^2(-1,1)$. Since the sequence $\{x^i\}_{i=0}^{\infty}$ of functions is a linearly independent set in $\underline{L}^2(-1,1)$, it follows that $\mathcal{H} = span\{1, x, x^2, x^3, \dots, x^{n-1}\}$ is a subspace of $\underline{L}^2(-1,1)$ of dimension n in which $\{x^i\}_{i=0}^{n-1}$ is a basis. Using the Gram–Schmidt orthonormalization algorithm, we obtain the orthonormal basis corresponding to this basis as $\{e_1(x), e_2(x), \dots, e_n(x)\}$. As shown in the following examples, we employ the orthonormal property of the basis to create a frame. As a result, the type of these orthonormal members as polynomials makes no difference while constructing the frame in the following example. Let

$$\{f_i\}_{i=1}^{2^{n+2}-2} = \{e_1(x), e_1(x), e_2(x), e_2(x), e_2(x), e_2(x), e_n(x), ..., e_n(x)\}, \tag{7}$$

so that each $e_i(x)$ is repeated 2^i times. It is easy to show that $\{f_i\}_{i=1}^{2^{n+2}-2}$ is a frame for $\mathcal{H}$ with lower bound $A = 2$ and upper bound $B = 2^n$.

Example 2. According to the previous statements, if we let $n = 15$, then $\mathcal{H} = span\{1, x, x^2, x^3, \dots, x^{14}\}$ and $\{e_1(x), e_2(x), ..., e_{15}(x)\}$ is the corresponding orthonormal basis obtained by applying the Gram–Schmidt algorithm to the basis $\{x^i\}_{i=0}^{14}$. As mentioned earlier, $\{f_i\}_{i=1}^{2^{17}-2}$ is a frame for $\mathcal{H}$, and it is easy to show that the frame operator S of this frame is $Sf(x) = \sum_{i=1}^{15} 2^i \langle f(x), e_i(x)\rangle e_i(x)$.
Assume that

$$f(x) = \begin{bmatrix} -2 & 10 & 3 & 11 & 10 & -15 & 5 & 15 & 36 & 92 & 20 & 14 & 2 & 1 & 5 \end{bmatrix} \times \begin{bmatrix} e_1(x) \\ e_2(x) \\ e_3(x) \\ \vdots \\ e_{15}(x) \end{bmatrix},$$

and let $\epsilon = 0.001$. Using the classical frame algorithm for this $f \in \mathcal{H}$, and using MATLAB codes, we find that the number of iterations required for the convergence of the algorithm is 94777, and the runtime is about 0.29 seconds. The convergence of Algorithm 1 to f is reduced to 47389 iterations, and the runtime is about 0.154 seconds. Applying Algorithm 2 for approximating f reduces the number of iterations needed to converge the algorithm to 524, which is much less than those of the other two algorithms. The runtime of this algorithm is about 0.0051 seconds. This information shows that our first algorithm is almost twice as fast as the classical frame algorithm. Also, our second algorithm is at least 56 times faster than the classical algorithm and 30 times faster than our first algorithm. In addition, our second algorithm converges faster in terms of the required number of iterations compared to the other two algorithms. Figure 1 shows the graph of $f(x)$ and the graphs of its final approximations obtained from these three algorithms.

To better compare, we provide more information about the runtime of these three algorithms by applying the algorithms to the same f and different tolerances. A summary of the obtained information is given in Table 1.

Table 2 shows the number of iterations that each of these three algorithms needs to converge to this fixed f.

Example 3. Let

$$\mathcal{H} = span\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx | n = 1, 2, 3, \dots, 250\right\}.$$

Then, $\mathcal{H}$ is a subspace of $\underline{L}^2(-\pi, \pi)$ and the set

$$\Gamma = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx | n = 1, 2, 3, \dots, 250\right\}$$

ϵ	0.1	0.01	0.001
The classical algorithm	0.884	0.233	0.29
Algorithm 1	0.118	0.123	0.154
Algorithm 2	0.0041	0.005	0.0051

Table 1: The runtime of the classical frame algorithm, Algorithm 1 and Algorithm 2 in seconds for $\epsilon = 0.1, 0.01$ and 0.001

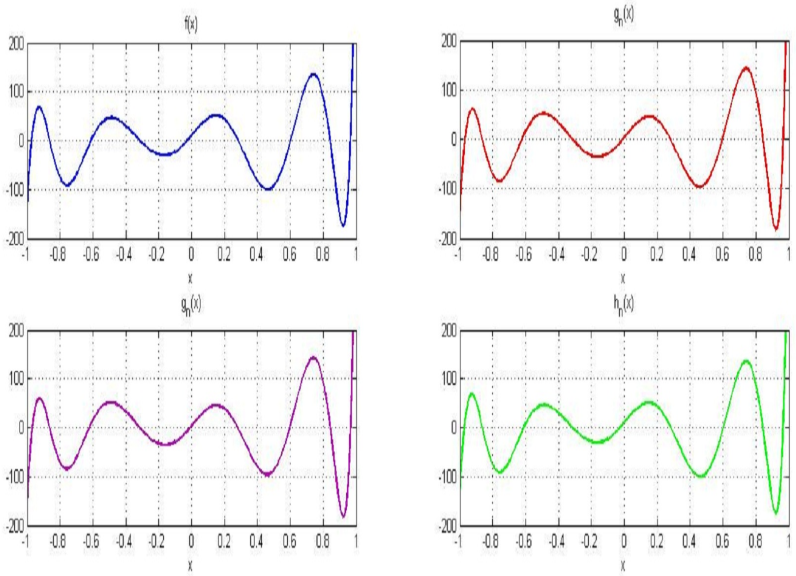

Figure 1: **Upper left:** The graph of $f(x)$. **Upper right:** The graph of the approximation of $f(x)$ obtained from the classical frame algorithm. **Lower left:** The graph of the approximation of $f(x)$ obtained from Algorithm 1. **Lower right:** The graph of the approximation of $f(x)$ obtained from Algorithm 2.

ϵ	0.1	0.01	0.001
The classical algorithm	57051	75914	94777
Algorithm 1	28526	37957	47389
Algorithm 2	316	420	524

Table 2: The required number of iterations for the convergence of the classical frame algorithm, Algorithm 1 and Algorithm 2 to $f(x)$ in Example 3 for $\epsilon = 0.1, 0.01$ and 0.001

ϵ	0.1	0.01	0.001
The classical algorithm	1511	1801	2091
Algorithm 1	756	901	1046
Algorithm 2	71	84	97

Table 3: The required number of iterations for the convergence of the classical frame algorithm, Algorithm 1 and Algorithm 2 to $f(x)$ in Example 2 for $\epsilon = 0.1, 0.01$ and 0.001

is an orthonormal basis for it. Assume that $e_1(x) = \frac{1}{\sqrt{2\pi}}$ and for each $n = 1, 2, \ldots, 250$, $e_{2n}(x) = \frac{1}{\sqrt{\pi}} \sin nx$ and $e_{2n+1}(x) = \frac{1}{\sqrt{\pi}} \cos nx$. We define the sequence $\{f_k\}_{k=1}^{32127}$ as follows:

$$f_k = \begin{cases} e_{2n}, & k = \frac{n(n+5)}{2}, n = 1, 2, \ldots, 250, \\ \\ e_{2n-1}, & \frac{(n-1)(n+4)}{2} < k < \frac{n(n+5)}{2}, n = 1, 2, \ldots, 251. \end{cases}$$

It is straightforward to show that $\{f_k\}_{k=1}^{32127}$ is a frame with the lower and the upper bound $A = 1$ and $B = 252$, respectively. Let

$$f(x) = 1 + \sum_{n=1}^{250} 2\sqrt{\pi} n \sin nx + \sum_{n=1}^{250} \sqrt{\pi} n \cos nx.$$

Applying these three algorithms to approximate $f(x)$ we obtain $t_1 \simeq t_c \simeq t_2$, in which t_1 is the runtime of Algorithm 1, t_2 is the runtime of Algorithm 2, and t_c is the runtime of the classical frame algorithm. We see that the runtime of Algorithm 2 is higher compared to the other two algorithms, and also, the required number of iterations that Algorithm 2 needs to convergence is much less than the other two algorithms. Table 3 shows the number of iterations of each algorithm per different accuracy values.

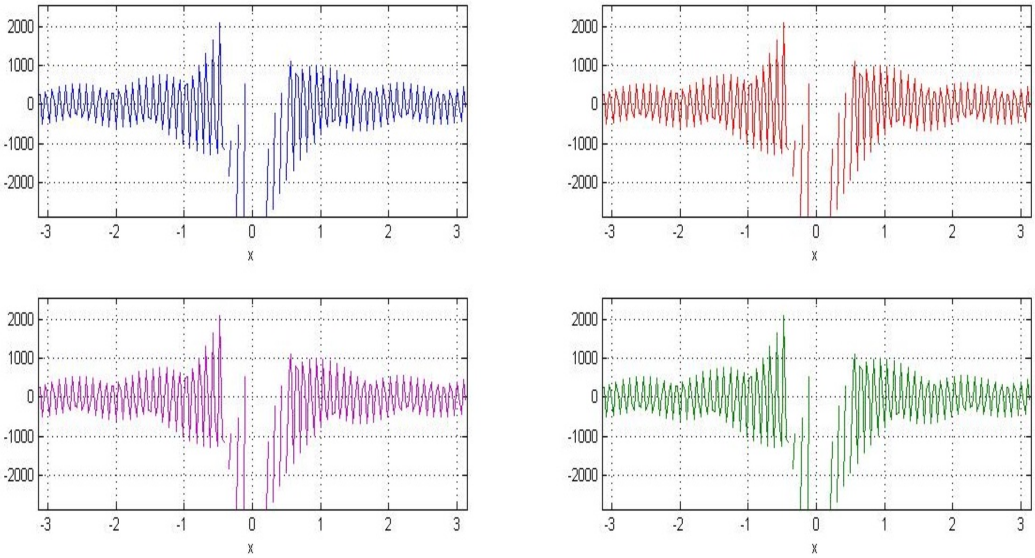

Figure 2: **Upper left:** The graph of $f(x)$ in Example 3. **Upper right:** The graph of the approximation of $f(x)$ obtained from the classical frame algorithm. **Lower left:** The graph of the approximation of $f(x)$ obtained from Algorithm 1. **Lower right:** The graph of the approximation of $f(x)$ obtained from Algorithm 2.

Figures 1 and 2 depict the accuracy of these three algorithms. They show that our two algorithms are as accurate as the classical frame algorithm.

Example 4. In view of Example 3, let

$$\mathcal{H} = span\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx | n = 1, 2, 3, \ldots, 15\right\}$$

and

$$f(x) = 1 + \sum_{n=1}^{15} 2\sqrt{\pi}n \sin nx + \sum_{n=1}^{15} \sqrt{\pi}n \cos nx.$$

By using frame (7) with $n = 31$ and applying the desired algorithms to approximate the function $f(x)$ with $\epsilon = 0.01$, it can be seen that the runtime of Algorithm 2 is significantly higher than that of Algorithm 1 and the classical frame algorithm.

t_c	t_1	t_2
3844	2118	2.6113

Table 4: The runtime of the classical frame algorithm, Algorithm 1 and Algorithm 2 in seconds for approximating $f(x)$ in Example 4 with $\epsilon = 0.01$ in terms of the notations used in Example 3

As shown in Table 4, and Examples 2 and 3, it is better to use Algorithm 1 when the difference between the upper and the lower bounds of the frame is not very large. However, when the difference between the bounds is very large, the use of the classical frame algorithm does not seem reasonable. In this case, Algorithm 2 can be helpful.

6. Conclusion

By introducing new iterative methods, we developed two algorithms to accelerate the convergence speed of the frame algorithm for approximating and reconstructing a signal f belonging to a separable Hilbert space. The convergence rates of the algorithms were much better than the convergence rate of the classical frame algorithm. The convergence speed of our first algorithm was twice as that of the classical algorithm. However, by using the Chebyshev polynomials, Algorithm 2 had a better convergence rate compared to Algorithm 1 and the classical algorithm. When the condition number of the desired frame was large, Algorithm 2 performed faster than the classical algorithm and even Algorithm 1 for signal reconstruction and approximation.

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