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Terminal digits under Zeckendorf expansion

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Abstract. The normality, Benford's Law, and Dirichlet's Theorem on primes in arithmetic progressions are well-known examples of topics on the digit distributions of a sequence underthe base-q expansion. In this paper, we introduce a result on the distribution of the terminal digits of linear sequences under the Zeckendorf expansion.

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1. Introduction

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $K_n \to \infty$ as $n \to \infty$. The digit distribution of the value of K_n under the base-q expansion, where $q > 1$ is an integer, has been the source of numerous mathematical studies. The normality and digit sums concern the entire digits of K_n [3, 7, 11, 12]. Benford's Law [1, 10] concerns the leading digits of K_n. Dirichlet's Theorem [4] on primes in arithmetic progressions concerns the terminal digit of the nth prime number ρ_n. For example, if $\rho_n = \sum_{k=0}^{m} c_k 4^k$ is the base-4 expansion, then by Dirichlet's Theorem, the probability of the last base-4 digit of the nth prime being j is equal to $1/\varphi(4) = 1/2$ for both $j = 1$ and $j = 3$, where φ is the Euler totient function.

What about the digits under Zeckendorf expansions? There are works such as [2, 8, 9, 11] on the digit distribution under the Zeckendorf expansion when $K_n = n$. For example, it is shown in [9] that the average number of summands in the Zeckendorf expansion of $n \in [F_m, F_{m+1})$ tends to $m/(\phi + 2)$, where ϕ is the golden ratio. However, it does not seem that the literature contains works on the digit distribution of other sequences under the Zeckendorf expansion, other than the recent work [6]. For example, what is the average number of summands in the Zeckendorf expansion of $11n + 3 \in [F_m, F_{m+1})$? Distribution results on the number of summands in the Zeckendorf expansion of the prime number sequence ρ_n are proved in [6].

The terminal digits of some standard sequences such as $an + b$, $an^2 + bn + c$, and c^n are periodic under the base-q expansion, where $q > 1$ is an integer, and calculating the probability of having a certain terminal digit for these sequences is a straightforward task. On the contrary, we find that calculating the probability under the Zeckendorf expansion even for the linear sequences is not as straightforward as under the base-q expansions. In this paper, we introduce probability results on the terminal digits of the linear sequence $an+b$ under the Zeckendorf expansion (see Theorem 1). In Section 8, we discuss an approach to other sequences and other digit problems. Results on the distribution of Zeckendorf digits of arithmetic progressions are available in the recent work [6, Section 6.1], and our result may follow from their approach. However, our result is obtained by an independent and more elementary method.

Let us begin by clarifying the meaning of probabilities that will be considered in this paper. Given a conditional statement $P(m)$, where $m \in \mathbb{Z}$ and a non-empty subset A of $\mathbb{N}$, define

$$\operatorname{Prob}\left\{ \, n \in A : P(K_n) \, \right\} := \lim_{N \to \infty} \frac{\#\{1 \le n \le N : P(K_n)\}}{\#\{n \in A : n \le N\}}. \tag{1}$$

For example, if A is finite, the limit always exists.

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2. Zeckendorf expansion

Let $\{F_k\}_{k=1}^{\infty}$ be the shifted Fibonacci sequence, i.e., $(F_1, F_2) = (1, 2)$ and $F_{k+2} = F_{k+1} + F_k$ for $k \in \mathbb{N}$. By Binet's formula

$$F_k = \frac{1}{\sqrt{5}}\left(\phi^{k+1} - (-\phi)^{-(k+1)}\right) = \frac{1}{\sqrt{5}}\phi^{k+1} + O(\omega^k), \tag{2}$$

where $\omega := 1/\phi$. By Zeckendorf's Theorem [13], given an integer $n \geq 0$, there is a unique (finite) subset A of $\mathbb{N}$ such that A does not contain consecutive integers and $n = \sum_{k\in A} F_k$. This expansion is called *the Zeckendorf expansion* of n.

Given a Zeckendorf expansion $n = \sum_{k\in A} F_k \in \mathbb{N}_0$ and an integer $s \geq 1$, define $\mathrm{TB}_s(n) = (d_s, d_{s-1}, \ldots, d_1)$, where $d_k = 1$ if $k \in A$, and $d_k = 0$ if $k \notin A$ for $1 \leq k \leq s$. The tuple $\mathrm{TB}_s(n)$ is called *the terminal block of the s Zeckendorf digits of n*. We denote $\mathrm{TB}_1(n) = (d)$ simply by $\mathrm{TB}(n) = d$. For example, $\mathrm{TB}_s(0)$ is the tuple $(0, \ldots, 0)$ of length s, $\mathrm{TB}_5(F_3 + F_1) = (0, 0, 1, 0, 1)$, and

$$\mathrm{TB}(20) = \mathrm{TB}(F_6 + F_4 + F_2) = 0,$$
$$\mathrm{TB}(21) = \mathrm{TB}(F_7) = 0,\ \mathrm{TB}(22) = \mathrm{TB}(F_7 + F_1) = 1.$$

Notice that if we consider the terminal block $(1, 0, 1)$, then we know that the terminal block of its four Zeckendorf digits must be $(0, 1, 0, 1)$. So, we might as well consider $(0, 1, 0, 1)$ instead of $(1, 0, 1)$. For Theorem 1 below, we consider these terminal blocks. Let $\mathbf{d} := (d_s, d_{s-1}, \ldots, d_1)$ for $s \geq 1$ be a tuple of length s with entries in $\{0, 1\}$ such that $d_s = 0$, and $d_k = 1$ for $1 \leq k \leq s - 1$ implies $d_{k+1} = 0$. We call it *a standard Zeckendorf digit block of length s*.

Throughout the paper, let a and b be integers such that $a > 0$ and $b \geq 0$. Our main result is introduced below, which will be proven in Section 7.

Theorem 1. *Let $\mathbf{d}$ be a standard Zeckendorf digit block of length s. Then,*

$$\mathrm{Prob}\left\{\ n \in \mathbb{N} : \mathrm{TB}_s(an + b) = \mathbf{d}\ \right\} = \omega^s. \tag{3}$$

In particular,

$$\mathrm{Prob}\left\{\ n \in \mathbb{N} : \mathrm{TB}(an + b) = 0\ \right\} = \omega, \quad \mathrm{Prob}\left\{\ n \in \mathbb{N} : \mathrm{TB}(an + b) = 1\ \right\} = \omega^2.$$

If the terminal digit is 1, then the last two digits must be $(0, 1)$, and hence, by the main result of the theorem for $s = 2$, $\mathrm{Prob}\left\{\ n \in \mathbb{N} : \mathrm{TB}(an + b) = 1\ \right\} = \omega^2$. Notice that for the case of $s = 1$, the probabilities add up to 1, i.e., $\omega + \omega^2 = 1$, and it is gratifying to see that they add up to 1 for all s. For example, let us consider all three terminal blocks of length 2, including $(1, 0)$. Then, we have $\omega^2 + \omega^2 + \omega^3 = 1$, which also follows from the relation $\omega^{n+2} + \omega^{n+1} = \omega^n$ for all $n \in \mathbb{Z}$.

3. The simplest case

Obviously, the simplest case is $K_n = n$. There could be several different ways to approach this case, e.g., [5]. In this paper, we present one approach that extends to the linear sequences $an + b$.

Let $\mathbf{d}$ be a standard Zeckendorf digit block, which will be fixed throughout the paper. Let $Z_0 := \{0\}$ if $\mathbf{d} = (0, \ldots, 0)$, and $Z_0 := \varnothing$, otherwise. Let $z_0 := \#Z_0 = 1$. For $n \in \mathbb{N}$,

$$Z_n := \{m \in [F_n, F_{n+1}) : \mathrm{TB}_s(m) = \mathbf{d}\}, \quad z_n := \#Z_n.$$

If $\mathbf{d} = (d_s, d_{s-1}, \ldots, d_1) = (0, \ldots, 0)$, define $\delta := 0$. Otherwise, define δ to be the largest index such that $d_\delta = 1$. Hence, $\delta \leq s - 1$. If $\mathbf{d} = (0, \ldots, 0)$, then $z_0 = 1$, $z_1 = \cdots = z_s = 0$, and $z_{s+1} = 1$. If $\mathbf{d} \neq (0, \ldots, 0)$, i.e., $\delta > 0$, then $z_0 = \cdots = z_{\delta-1} = 0$, $z_\delta = 1$, $z_{\delta+1} = z_{\delta+2} = \cdots = z_s = 0$, and $z_{s+1} = 1$.

Consider $n \geq s + 1$. Then,

$$m \in Z_n \ \Rightarrow \quad m = F_n + \sum_{k\in A} F_k, \quad \ell := \max(A) \leq n - 2.$$

For convenience, we define $\max(A) := 0$ if $A = \varnothing$. Notice that $0 \le \ell \le n-2$. Then, we have an injection: $Z_\ell \to Z_n$ given by $q \mapsto F_n + q$. Thus,

$$z_n = z_{n-2} + z_{n-3} + \cdots + z_2 + z_1 + z_0.$$

If $n \ge s+2$,

$$z_n = z_{n-2} + (z_{n-3} + \cdots + z_2 + z_1 + z_0) = z_{n-2} + z_{n-1}. \tag{4}$$

Thus, $(z_{s+2}, z_{s+3}) = (1,2)$, and hence, $z_n = F_{n-s-1}$ for $n \ge s+2$. Therefore,

$$\#\{0 \le m < F_{n+1} : \mathrm{TB}_s(m) = \mathbf{d}\} = \sum_{k=0}^{n} z_k = z_{n+2} = F_{n-s+1} = \omega^s F_{n+1} + O(\omega^n). \tag{5}$$

For the probability defined in (1), we also need to consider an upper bound U of m that is not equal to a Fibonacci term. This will be done later for the general linear sequence.

4. Recursions

Our goal is to apply the idea of Section 3 to $K_n = an+b$. For this goal, we slice up Z_n as follows. For $j \in \mathbb{Z}$, let $X_0(j) = \{0\}$ if $j \equiv 0 \pmod{a}$ and $\mathbf{d} = (0,\dots,0)$, and $X_0(j) = \varnothing$, otherwise. Let $x_0(j) := \#X_0(j)$. For $n \in \mathbb{N}$,

$$X_n(j) := \{m \in [F_n, F_{n+1}) : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\}, \quad x_n(j) := \#X_n(j).$$

For example, since $X_n(j) \subset Z_n$, if $\delta > 0$, then

$$x_0(j) = \cdots = x_{\delta-1}(j) = x_{\delta+1}(j) = \cdots = x_s(j) = 0. \tag{6}$$

Then, we have recursions similar to (4).

Proposition 1. *Let $n \ge s+2$ be an integer. Then,*

$$x_n(j) = x_{n-1}(j - F_{n-2}) + x_{n-2}(j - F_n), \tag{7}$$

$$\sum_{j=1}^{a} x_n(j) = F_{n-s-1}.$$

Proof. If $m = F_n + \sum_{k\in A} F_k \in X_n(j)$ is a Zeckendorf expansion, then m can be reconstructed with F_n and numbers in $X_\ell(j - F_n)$, where $\ell := \max(A) \le n-2$, since $m \equiv j \pmod{a}$ implies that $\sum_{k\in A} F_k \equiv j - F_n \pmod{a}$. By the principle used in (4), we have

$$\begin{aligned} x_n(j) &= x_{n-2}(j-F_n) + x_{n-3}(j-F_n) + \cdots + x_0(j-F_n), \\ &= x_{n-2}(j-F_n) + x_{n-1}(j - F_n + F_{n-1}) \\ &= x_{n-1}(j-F_{n-2}) + x_{n-2}(j-F_n). \end{aligned}$$

The second result follows from $\sum_{j=1}^{a} x_n(j) = z_n$. □

Notice that the arguments of x_n can be replaced with different representatives of the residue classes mod a. This leads us to consider *the so-called (shifted) Pisano cycle*, which is the cycle that is repeated when the Fibonacci sequence is reduced mod a. The length of this cycle is called *the Pisano period*. Let p be the Pisano period mod a and denote the Pisano cycle by

$$t_1, t_2, \dots, t_p,$$

where $0 \le t_k < a$. Thus, $F_k \equiv t_k \pmod{a}$ for all $1 \le k \le p$, $t_{p-1} = 0$, and $t_p = 1$. Examples of the Pisano cycles $(t_1, \dots, t_p)$ are listed below:

$$a = 2: \ (1,0,1), \ a = 3: \ (1,2,0,2,2,1,0,1), \ a = 4: \ (1,2,3,1,0,1).$$

We extend the definition of t_k for $k \in \mathbb{Z}$ such that $t_i = t_j$ if $i \equiv j \pmod{p}$. Hence, if $v \equiv w \pmod{p}$, then $F_v \equiv t_w \pmod{a}$ and $t_n + t_{n+1} \equiv t_{n+2} \pmod{a}$ for all $n \in \mathbb{Z}$. Thus, if $n \equiv k \pmod{p}$, then (7) is equivalent to

$$x_n(j) = x_{n-1}(j - t_{k-2}) + x_{n-2}(j - t_k). \tag{8}$$

5. Unfolding

Notice that $z_n = z_{n-1} + z_{n-2}$ can be unfolded as follows:

$$z_n = 2z_{n-2} + z_{n-3} = F_u z_{n-u} + F_{u-1} z_{n-u-1} = F_{n-s-1} \cdot 1 + F_{n-s-2} \cdot 0. \tag{9}$$

From (8), we obtain a recursion that is similar to (9), which will be used to reduce the estimates of the error terms when calculating the global count as in (5). This will allow us to find an asymptotic formula of $x_n(j)$ that is similar to that of F_k; see Theorem 2 below. We postpone the demonstration of unfolding (7) until the proof of Lemma 1.

Recall that x_n is a sequence of functions with integer arguments j as in (7). Recursion (8) will be repeatedly unfolded. To keep track of the arguments, we use tuples as follows. Let T_n for $n \in \mathbb{Z}$ be indeterminates, and define $T_n(t_k) := t_{k+n}$. The operator T_n is sometimes called the shift operator. Given a tuple $A = (a_1, a_2, \ldots, a_n)$ of length n, where the entries are polynomials in T_i over $\mathbb{Z}$, define $\operatorname{len}(A) := n$ and

$$A + f := (a_1 + f, a_2 + f, \ldots, a_n + f),$$

where f is a polynomial in T_i over $\mathbb{Z}$. Given t_k from the Pisano cycle, let $A(t_k)$ be the tuple obtained by evaluating all T_i in each entry of A with t_k, e.g., $(T_1, T_0 + T_2, T_1T_3)(t_k) = (t_{k+1}, t_k + t_{k+2}, t_{k+1}t_{k+3})$. If $A(t_k) = (a_1, \ldots, a_n)$, then define

$$\sum_{i \in A(t_k)} x_m(j+i) := \sum_{k=1}^{n} x_m(j + a_k).$$

Let $(\ )$ denote the tuple of length 0 and $A \sqcup (\ ) := A$. If $B = (b_1, b_2, \ldots, b_m)$ is a tuple, then $A \sqcup B := (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$.

Definition 1. Let $A_0 = (0)$ and $B_0 = (\ )$ be tuples of length 1 and 0, respectively. Define

$$\begin{aligned} A_{u+1} &= (A_u - T_{-u-2}) \sqcup B_u, \\ B_{u+1} &= A_u - T_{-u}. \end{aligned}$$

Then, for $n \geq 1$,

$$A_{u+1} = (A_u - T_{-u-2}) \sqcup (A_{u-1} - T_{-u+1}). \tag{10}$$

So,

$$\begin{aligned} & A_1 = (-T_{-2}) \ \longrightarrow\ A_2 = (-T_{-2} - T_{-3}, -T_0) \\ \longrightarrow\ & A_3 = (-T_{-2} - T_{-3} - T_{-4}, -T_0 - T_{-4}, -T_{-2} - T_{-1}) \\ \longrightarrow\ & A_4 = (-T_{-2} - T_{-3} - T_{-4} - T_{-5}, -T_0 - T_{-4} - T_{-5}, -T_{-2} - T_{-1} - T_{-5}, \\ & \qquad - T_{-2} - T_{-3} - T_{-2}, -T_0 - T_{-2}). \end{aligned}$$

From the construction, it is clear that

$$\operatorname{len}(A_u) = F_u \text{ and } \operatorname{len}(B_u) = F_{u-1} \text{ for } u \geq 0. \tag{11}$$

Using these tuples, we can state the unfolded versions of (7).

Lemma 1. *Given $n \geq s+1$, let $1 \leq k \leq p$ be the integer such that $n \equiv k \pmod{p}$. If $1 \leq u \leq n - s$, then*

$$x_n(j) = \sum_{i \in A_u(t_k)} x_{n-u}(j+i) + \sum_{i \in B_u(t_k)} x_{n-u-1}(j+i). \tag{12}$$

Proof. Let us use induction on u. The case for $u = 1$ is recursion (7). Assume that the statement is true for some $u \geq 1$. Then,

$$\begin{aligned} x_n(j) &= \sum_{i \in A_u(t_k)} x_{n-u}(j+i) + \sum_{i \in B_u(t_k)} x_{n-u-1}(j+i) \\ &= \sum_{i \in A_u(t_k)} x_{n-u-1}(j+i-t_{k-u-2}) + x_{n-u-2}(j+i-t_{k-u}) \quad \text{by (8)} \\ &\quad + \sum_{i \in B_u(t_k)} x_{n-u-1}(j+i) \\ &= \sum_{i \in A_{u+1}(t_k)} x_{n-u-1}(j+i) + \sum_{i \in B_{u+1}(t_k)} x_{n-u-2}(j+i). \end{aligned}$$

□

6. Local asymptotic formulas

In this section, we show that there is a value of u determined independently of n for which cancellations take place in the first term of the recursion in (12). This reveals a reduced estimate of the error term, as shown in Theorem 2 below. The real number β in the theorem will be identified later in Section 6.2.

Theorem 2. *Let* $\alpha = \omega^s/(a\sqrt{5})$. *Then,* $x_n(j) = \alpha\phi^n + O(\beta^n)$, *where* $1 < \beta < \phi$.

6.1. Cancellations

The argument of the function x_n factors through $\mathbb{Z}/a\mathbb{Z}$, i.e., $x_n(j) = x_n(j')$ if $j \equiv j' \pmod{a}$. We will show that the arguments $i \in A_u(t_k)$ in (12) eventually cover a complete residue system mod a as u grows. For $k \in \mathbb{N}$, let $\overline{A}_u(t_k)$ be the subset of $\mathbb{Z}/a\mathbb{Z}$ consisting of the residue classes $[y]$, where y runs over the entries of $A_u(t_k)$.

Theorem 3. *There is* $u \in \mathbb{N}$ *such that* $\#\overline{A}_u(t_k) = a$ *holds for all* $k \in \mathbb{Z}$, *i.e.,* $\overline{A}_u(t_k)$ *contains a complete residue system mod* a.

Proof. Let $1 \leq k \leq p$. By (10), $\#\overline{A}_u(t_k)$ is a non-decreasing sequence of u. Since $\#\overline{A}_u(t_k) \leq a$, there is $u \in \mathbb{N}$ such that $\#\overline{A}_u(t_k) = \#\overline{A}_v(t_k)$ for all $v \geq u$, i.e., it does not increase any more. If necessary, choose a larger value of u such that $t_{k-u-2} = 1$, which is possible since $t_{pn} = 1$ for all $n \in \mathbb{Z}$.

Let $\overline{A}_u(t_k) = \{d_1, \ldots, d_m\}$. Since $\#\overline{A}_{u+1}(t_k) = m$, recursion (10) implies

$$\overline{A}_{u+1}(t_k) = \{d_1, \ldots, d_m\} - [t_{k-u-2}],$$

which is defined to be $\{d_k - [t_{k-u-2}] : k = 1, \ldots, m\}$. By (10), $A_{u+2}(t_k) = (A_{u+1}(t_k) - t_{k-u-3}) \sqcup (A_u(t_k) - t_{k-u})$. Both tuples in $A_{u+2}(t_k)$ must have the same m residue classes; otherwise, $\overline{A}_{u+2}(t_k)$ will have more than m elements. Thus, we have the equality of sets:

$$\begin{aligned} \{d_1, \ldots, d_m\} - [t_{k-u-2} + t_{k-u-3}] &= \{d_1, \ldots, d_m\} - [t_{k-u}], \\ \text{i.e.,} \quad \{d_1, \ldots, d_m\} + [t_{k-u} - t_{k-u-2} - t_{k-u-3}] &= \{d_1, \ldots, d_m\} = \overline{A}_u(t_k). \end{aligned} \tag{13}$$

Notice that for each d_i, equality (13) implies

$$d_i + [t_{k-u} - t_{k-u-2} - t_{k-u-3}] = d_i + [t_{k-u-2}] = d_i + [1] \in \overline{A}_u(t_k).$$

Thus, we have a function π on $\overline{A}_u(t_k)$ given by $d_i \mapsto d_i + [1]$, which is clearly a permutation on $\overline{A}_u(t_k)$. Notice that $\{\pi^r(d_1) : r = 0, 1, \ldots, a-1\} = \mathbb{Z}/a\mathbb{Z}$, which implies $m = a$. Since there are only finitely many k to consider, there is u such that $\#\overline{A}_u(t_k) = a$ for all $1 \leq k \leq p$. □

For the rest of the paper, let u be the smallest positive integer described in Theorem 3 such that $F_u > a$. By Theorem 3, we can pull out $\sum_{i=1}^{a} x_{n-u}(i)$ from $\sum_{i \in A_u(t_k)} x_{n-u}(j+i)$, and it is the consequence of the initial values (6) and recurrence (7) that $\sum_{i=1}^{a} x_{n-u}(i) = F_{n-u-s-1}$. That is,

$$x_n(j) = F_{n-u-s-1} + \sum_{i \in A'_u(t_k)} x_{n-u}(j+i) + \sum_{i \in B_u(t_k)} x_{n-(u+1)}(j+i), \tag{14}$$

where $\operatorname{len}(A'_u) = F_u - a$. Notice that the error term of $F_{n-u-s-1}$ is $O(\omega^n)$. In Section 6.2 below, we use expression (14) to prove that its error term is $o(\phi^n)$.

6.2. The error estimates

Let us identify the real number β described in Theorem 2. Let $f(x) = x^{u+1} - (F_u - a)x - F_{u-1}$ be the polynomial, which is related to recursion (14). By Descarte's Rule of Signs, f has at most one positive real zero. Then, $f(1) < 0 < f(\phi)$, which implies that it has precisely one positive real zero. Define $\beta > 1$ to be the positive real zero of $f(x)$, so that

$$\beta \cdot (F_u - a) + F_{u-1} = \beta^{u+1}. \tag{15}$$

The big-O notation is common, but it is worth keeping track of the constants involved in the big-O notation when proving things via induction. For a real number $C > 0$, if $|g(x)| < C\,|f(x)|$, we write

$$g(x) = O(f(x))_C.$$

We are ready to prove Theorem 2. The main work is to control the error term $O(\beta^n)$. In fact, we shall prove via induction on n that

$$x_n(j) = \alpha\phi^n + O(\beta^n)_{C_n} \tag{16}$$

for $n \geq 0$, where $C_n = C_{n-1}(1 + \phi^{u+s}(\omega/\beta)^n)$ for $n > s+u$ and $C_k = C_0$ for all $0 \leq k \leq s+u$. We may choose C_0 to be the least positive integer such that (16) works for all $0 \leq n \leq s+u$.

Assume that estimate (16) is true for x_{n-u} and x_{n-u-1} for some $n \geq s+u$. We use recursion (12) to find an estimate of $x_n(j)$. First, notice that for all integers m and u,

$$F_u\phi^{m-u} + F_{u-1}\phi^{m-u-1} = \phi^m. \tag{17}$$

By (14) and the induction hypothesis, we have

$$\begin{aligned} x_n(j) = F_{n-u-s-1} &+ \sum_{i \in A'_u(t_k)} \alpha\phi^{n-u} + O(\beta^{n-u})_{C_{n-u}} \\ &+ \sum_{i \in B_u(t_k)} \alpha\phi^{n-u-1} + O(\beta^{n-u-1})_{C_{n-u-1}}. \end{aligned} \tag{18}$$

Calculated below are the dominant terms of the RHS of (18), where we use $\alpha = \omega^s/(a\sqrt{5})$ and (11):

$$\begin{aligned} D :=& \frac{1}{\sqrt{5}}\phi^{n-u-s} + \sum_{i \in A'_u(t_k)} \alpha\phi^{n-u} + \sum_{i \in B_u(t_k)} \alpha\phi^{n-u-1} \\ =& \frac{1}{\sqrt{5}}\phi^{n-u-s} + (F_u - a)\frac{\omega^s}{a\sqrt{5}}\phi^{n-u} + F_{u-1}\,\alpha\phi^{n-u-1}. \end{aligned}$$

By (17), we have

$$D = \alpha\left(F_u\phi^{n-u} + F_{u-1}\phi^{n-u-1}\right) = \alpha\phi^n.$$

Let us calculate the error terms of the RHS of (18). By (15) and $C_{n-u} \le C_{n-1}$, we have

$$\begin{aligned} &O(\omega^{n-u-s})_1 + (F_u - a)O(\beta^{n-u})_{C_{n-1}} + F_{u-1}O(\beta^{n-u-1})_{C_{n-1}} \\ &\quad = O(\omega^{n-u-s})_1 + \left((F_u - a)\beta + F_{u-1}\right) O\left(\beta^{n-u-1}\right)_{C_{n-1}} \\ &\quad < O(\beta^n + \omega^{n-u-s})_{C_{n-1}} = \beta^n O(1 + \phi^{u+s}(\omega/\beta)^n)_{C_{n-1}} \\ &\quad = \beta^n O(1)_{C_n} = O(\beta^n)_{C_n}. \end{aligned}$$

Notice that $C_n = C_{s+u} \prod_{k=s+u+1}^{n} (1 + \phi^{u+s}(\omega/\beta)^k)$. The fact that $\omega/\beta < 1$ implies that $\ln(C_n)$ is a bounded increasing sequence. Let $C := \lim_{n\to\infty} C_n$. Then, $x_n(j) = \alpha\phi^n + O(\beta^n)_C$.

7. The probability

Theorem 2 can be interpreted as a local probability. Notice that there are F_{n-1} integers in the interval $[F_n, F_{n+1})$. By Theorem 2, $\#X_n(j) = \frac{\omega^s}{a} F_{n-1} + O(\beta^n)$, which implies

$$\operatorname{Prob}\left\{ m \in [F_n, F_{n+1}) : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a} \right\} = \frac{\omega^s}{a} + O((\beta/\phi)^n).$$

Let us show that this local estimate is sufficient to calculate the global probability.

Theorem 4. *Let $F_n \le U < F_{n+1}$. Then, for $j \in \{0, 1, \dots, a-1\}$,*

$$\#\{m < U : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} = \frac{\omega^s}{a} U + O(\beta^n). \tag{19}$$

Proof. We shall prove that there is a positive integer C such that

$$\#\{m < U : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} = \frac{\omega^s}{a} U + O(\textstyle\sum_{k=1}^{n} \beta^k + \omega^{k+2})_C. \tag{20}$$

Then, the asserted result (19) follows from (20). We use induction on n. The first few cases $n \in N_0 := \{1, 2, \dots, n_0\}$ are obviously true with some $C \in \mathbb{N}$ and $n_0 \in \mathbb{N}$. Use Theorem 2 to choose a larger value of C, if necessary, so that $\sum_{k=1}^{n} x_k(j) = \alpha\phi^{n+2} + O(\beta^n)_C$ for all $n \in \mathbb{N}$.

Assume that there is $n \ge n_0$ such that (20) is true for all $n' \le n$ and $F_{n'} \le U' < F_{n'+1}$. Let $U = F_{n+1} + \sum_{k\in A} F_k$, where $\ell := \max(A) \le n-1$. Then,

$$\begin{aligned} &\#\{m < U : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} \\ &\quad = \#\{m < F_{n+1} : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} \\ &\quad\quad + \#\{F_{n+1} \le m < U : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\}, \end{aligned}$$

$$\#\{m < F_{n+1} : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} = \sum_{k=1}^{n} x_k(j) = \alpha\phi^{n+2} + O(\beta^n)_C.$$

If $F_{n+1} \le m < U$, $\mathrm{TB}_s(m) = \mathbf{d}$, and $m \equiv j \pmod{a}$, then $m - F_{n+1} < \sum_{k\in A} F_k$, and hence,

$$\begin{aligned} &\#\{F_{n+1} \le m < U : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} \\ &\quad = \#\{0 \le m < U - F_{n+1} : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j_n \pmod{a}\}, \end{aligned}$$

where $j_n := j - F_{n+1}$. By the induction hypothesis,

$$\begin{aligned} &\#\{0 \le m < U - F_{n+1} : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j_n \pmod{a}\} \\ &\qquad = \frac{\omega}{a}(U - F_{n+1}) + O(\textstyle\sum_{k=1}^{\ell} \beta^k + \omega^{k+2})_C, \quad \ell \le n-1 \\ \Rightarrow\quad &\#\{m < U : \mathrm{TB}_s(m) = \mathbf{d},\ m \equiv j \pmod{a}\} \\ &\qquad = \alpha\phi^{n+2} + O(\beta^n)_C + \frac{\omega^s}{a}(U - F_{n+1}) + O(\textstyle\sum_{k=1}^{\ell} \beta^k + \omega^{k+2})_C \\ &\qquad = \frac{\omega^s}{a} U + O(\textstyle\sum_{k=1}^{n} \beta^k)_C + O(\omega^{n+2})_1 + O(\textstyle\sum_{k=1}^{\ell} \omega^{k+2})_C \\ &\qquad = \frac{\omega^s}{a} U + O(\textstyle\sum_{k=1}^{n} \beta^k)_C + O(\textstyle\sum_{k=1}^{n} \omega^{k+2})_C. \end{aligned}$$

Thus, (20) is true for all n and $U \in [F_n, F_{n+1})$. □

Let us prove Theorem 1. Notice that the subset described in (19) is bijective to $\{n < (U-j)/a : \mathrm{TB}_s(an+j) = \mathbf{d}\}$ under $m \mapsto (m-j)/a$. Thus, if we write $U' := (U-j)/a$, where $F_n \le U < F_{n+1}$, then $U' \ge (F_n - j)/a$ and

$$\begin{aligned} \#\{n < U' : \mathrm{TB}_s(an+j) = \mathbf{d}\} &= \tfrac{\omega^s}{a} U + O(\beta^n) = \omega^s U' + O(\beta^n) + j\omega^s \\ \Rightarrow \quad \mathrm{Prob}\left\{\, n \in \mathbb{N} : \mathrm{TB}_s(an+j) = \mathbf{d} \,\right\} &= \omega^s. \end{aligned}$$

8. Other sequences and questions

For sequences $K_n = f(n)$, where $f(x)$ is an integer-valued polynomial of degree ≥ 2 or an exponential function such as 2^x, computer calculations suggest that the probability of having $\mathrm{TB}_s(K_n) = \mathbf{d}$ remains to be ω^s. An effort to apply the method introduced in this paper to these sequences may begin with the following sets:

$$X_n(j) := \{m \in [F_n, F_{n+1}) : \mathrm{TB}_s(m) = \mathbf{d},\ m \in f(\mathbb{N}) + j\}.$$

Then, we have a recursion (7) and for the next step, we need to find u (depending on n) such that $A_u(F_n)$ covers Z_{n-u}, i.e.,

$$\bigcup_{i \in A_u(F_n)} X_{n-u}(i) = Z_{n-u}.$$

Also, there needs to be a subtuple J_u of $A_u(F_n)$ such that

$$\sum_{i \in J_u} x_{n-u}(i) = \tfrac{1}{\sqrt{5}} \phi^{n-u-s} + O(\omega^{n-u-s}).$$

Furthermore, to afford an error estimate $O(\beta^n)$, we need to find $1 < \beta < \phi - \epsilon$ such that

$$\beta^{u+1} > (F_u - \#J_u)\beta + F_{u-1}.$$

We shall explore this path in a subsequent paper.

It is natural to ask how this applies to the prime number sequence ρ_n. Based on computer calculations, the following conjecture appears to be true, and we will investigate in the subsequent paper the conjecture incorporating the approach introduced in [6]:

Conjecture 1 (Prime Number Theorem for the Zeckendorf expansion)**.** *Let* **d** *be a standard Zeckendorf digit block of length s. Then,*

$$\mathrm{Prob}\left\{\, n \in \mathbb{N} : \mathrm{TB}_s(\rho_n) = \mathbf{d} \,\right\} = \omega^s.$$

There are other interesting questions regarding digit distribution under the Zeckendorf expansion. As mentioned in the introduction, what is the average number of summands of $an + b \in [F_m, F_{m+1})$ under the Zeckendorf expansion? The normality is one of the toughest problems in the area of digit distribution. An infinite tuple of integers in $\{0, 1, \dots, q\}$ is called *normal* if all blocks of the same size occur with equal probability. For example, let K_n be the positive integer obtained by concatenating the decimal expansion of n^2, e.g., $K_5 = 1491625$. It is proven in [12] that the infinite tuple $\lim_{n\to\infty} K_n$ is normal, i.e., decimal digit blocks of all lengths s will occur with probability $1/10^s$. What about the normality of concatenating the Zeckendorf digits of $K_n = n^2$ after prepending 0? For example, $K_5 = \mathbf{0}1\mathbf{0}101\mathbf{0}10001\mathbf{0}100100\mathbf{0}1000101$, where the prepended zeros are in bold face to visualize the concatenation positions. Does the probability of a standard Zeckendorf digit block of length s occurring in K_n approach ω^s as $n \to \infty$?

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