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Moderate deviation principles for the weighted sums of sums of independent random variables

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Abstract. In the present paper, we consider the weighted sums of sums of independent and identically distributed random variables and study their moderate deviation principles, which can be applied to study the asymptotic behavior of time series and branching processes. These results can be seen as the complements of the results in Imomov and Nazarov [6].

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1. Introduction

In many practical situations, we have to deal with statistical data that arise in some time series. Examples of this process are the number of patients in a hospital at a specific point in time, or the number of persons in a queue waiting for service at a certain moment. In order to study the statistical properties of these time series, we often need to consider some weighted sums of random variables. For example, let $\{X_n, n \geq 1\}$ be a stochastic process based on a linear autoregressive model,

$$X_{n+1} = \theta X_n + \xi_{n+1}, \quad X_0 = 0, \tag{1}$$

where $\theta \in \Theta \subset \mathbb{R}$ (the space of parameters) is unknown, and $\{\xi_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables representing the noises. For the observation samples $\{X_n, n \geq 1\}$, we have the following weighted form:

$$X_n = \sum_{k=1}^{n} \theta^{n-k} \xi_k. \tag{2}$$

Several researchers have discussed the asymptotic behavior of model (1). Cumberland and Sykes [2] stated that the distribution of the nearly unstable first-order autoregressive process can be approximated by an Ornstein-Uhlenbeck process. Miao [10, 11] and Miao et al. [13] established an asymptotic normality and large deviation principle for autoregressive processes. Miao [12] and Miao et al. [15] studied the discounted large deviation principle, while Berry-Esséen explored the analogue for autoregressive processes. Miao and Shen [14] and Miao and Yin [16] proved the moderate deviation principle and Cramér moderate deviations for the LS estimator of the autoregressive processes in the neighborhood of the unit root. When the model parameters are in $(0, 1)$, Minkosch and Samorodnitsky [17] investigated the asymptotic with regularly varying innovations. Olvera-Cravioto [18] obtained a new uniform approximation of a nearly unstable first-order autoregressive process with regularly varying innovations.

Let us recall another counting processes. Assume that $\{X_n, n \geq 1\}$ is a stochastic process based on a branching process with immigration,

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j} + \varepsilon_n, \quad X_0 = 0, \tag{3}$$

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where $\{\xi_{n,j}, \varepsilon_n, n \geq 1, j \geq 1\}$ are independent nonnegative integer-valued random variables such that $\{\xi_{n,j}, n \geq 1, j \geq 1\}$ and $\{\varepsilon_n, n \geq 1\}$ are mutually independent. When $\{\xi_{n,j}, n \geq 1, j \geq 1\}$ are Bernoulli random variables, process (3) is the so-called INAR(1) model, which was proposed by Al-Osh and Alzaid [1]. Ispány et al. [8] investigated the asymptotic normality and convergence rate of the least squares estimator for the coefficient INAR(1) model, where the autoregressive type coefficient converges to 1. Ispány et al. [9] studied a sequence of Galton-Watson type branching processes with immigration (3), where the offspring mean tends to its critical value 1 and the offspring variance tends to 0. Imomov [5] considered the discrete-time branching process allowing immigration and investigated the limit properties of transition functions and their convergence to invariant measures. Imomov and Tukhtaev [7] studied the Galton-Watson branching process allowing immigration, in which the immigration law has infinite mean and the offspring law has an infinite variance. For model (3), assume that

$$m := \mathbb{E}\xi_{1,1}, \quad \lambda := \mathbb{E}\varepsilon_1, \quad \sigma^2 := Var(\xi_{1,1}), \quad b^2 = Var(\varepsilon_1)$$

are finite. For $n \geq 1$, let $\mathcal{F}_n$ denote the σ-algebra generated by $\{X_0, X_1, \cdots, X_n\}$; then we have that

$$M_n := X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_n - mX_{n-1} - \lambda$$

defines a martingale difference sequence $\{M_n, n \geq 1\}$ with respect to the filtration $\{\mathcal{F}_n, n \geq 1\}$. Moreover, we obtain the recursion

$$X_n - \mathbb{E}X_n = \sum_{k=1}^{n} m^{n-k} M_k, \tag{4}$$

which is also a form of weighted sums as (2).

Recently, Imomov and Nazarov [6] considered the weighted sums of partial sums of i.i.d. random variables, which have forms similar to (2) and (4). Let $\{\xi_n, n \geq 1\}$ be a sequence of i.i.d. random variables with

$$\mathbb{E}\xi_1 = 0 \quad \text{and} \quad Var(\xi_1) = \sigma^2 > 0.$$

Denote $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$ for each $n \geq 1$. For any $n \geq 1$, let $\{\xi_k^{(n)}, k \geq 1\}$ be a sequence of i.i.d. random variables with

$$\mathbb{E}\xi_1^{(n)} = 0 \quad \text{and} \quad Var(\xi_1^{(n)}) = \sigma_n^2 > 0.$$

Denote $S_0^{(n)} = 0$ and $S_n^{(n)} = \xi_1^{(n)} + \cdots + \xi_n^{(n)}$ for each $n \geq 1$. Define the following sums of random variables:

$$\begin{aligned} X_n &= \sum_{k=1}^{n} m^k S_k, \quad Y_n = \sum_{k=1}^{n} m^{n-k} S_k, \\ Z_n &= \sum_{k=1}^{n} m_n^k S_k, \quad T_n = \sum_{k=1}^{n} m_n^{n-k} S_k, \\ J_n^{(n)} &= \sum_{k=1}^{n} m_n^k S_k^{(n)}, \quad K_n^{(n)} = \sum_{k=1}^{n} m_n^{n-k} S_k^{(n)}, \end{aligned} \tag{5}$$

where m is a positive constant and for some $\alpha \neq 0$,

$$m_n = 1 + \frac{\alpha}{n} + o\left(\frac{1}{n}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

The law of large numbers (LLN) and the central limit theorem (CLT) for those special forms in (5) have been investigated by Imomov and Nazarov [6].

Motivated by the above results of Imomov and Nazarov [6], in the present paper, we continue to study these weighted sums of the forms in (5) and prove their moderate deviation principles (MDP). For the theory of large deviation, one can refer to the books, Dembo and Zeitouni [3] and Deuschel and Stroock [4]. The rest of this paper is organized as follows. In Section 2, the main results are stated and their proofs are given in Section 3. Throughout this paper, C always stands for a positive constant which may differ from one place to another.

2. Main results

In the section, we state the MDP of the weighted sums $X_n, Y_n, Z_n, T_n, J_n^{(n)}, K_n^{(n)}$. Assume that $\{b_n, n \geq 1\}$ is a sequence of positive constants satisfying

$$b_n \to \infty \quad \text{and} \quad \frac{\sqrt{n}}{b_n} \to \infty \quad \text{as} \quad n \to \infty. \tag{6}$$

Our main results are stated as follows.

Theorem 1. *For any sequence $\{b_n, n \geq 1\}$ of positive constants satisfying (6), assume that there exists a positive constant t, such that*

$$\mathbb{E}\exp\{t|\xi_1|\} < \infty. \tag{7}$$

(i) *Let $m \geq 1$; then for any $\lambda > 0$,*

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{|X_n|}{b_n\sqrt{Var(X_n)}} > \lambda\right) = -\frac{\lambda^2}{2}. \tag{8}$$

(ii) *Let $m \leq 1$; then for any $\lambda > 0$,*

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{|Y_n|}{b_n\sqrt{Var(Y_n)}} > \lambda\right) = -\frac{\lambda^2}{2}.$$

Remark 1. From the proof of Theorem 1, the MDP of X_n (for the case $m < 1$) and Y_n (for the case $m > 1$) do not hold.

Theorem 2. *For any sequence $\{b_n, n \geq 1\}$ of positive constants satisfying (6), assume that there exists a positive constant t, such that*

$$\mathbb{E}\exp\{t|\xi_1|\} < \infty. \tag{9}$$

Then, for any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{|Z_n|}{b_n\sqrt{Var(Z_n)}} > \lambda\right) = -\frac{\lambda^2}{2} \tag{10}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{|T_n|}{b_n\sqrt{Var(T_n)}} > \lambda\right) = -\frac{\lambda^2}{2}.$$

Theorem 3. *Assume that there exists a positive constant t, such that*

$$\sup_{n\geq 1}\mathbb{E}\exp\{t|\xi_1^{(n)}|\} < \infty. \tag{11}$$

Furthermore, suppose that the sequences $\{\sigma_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ satisfy $b_n \to \infty$ and

$$\begin{cases} \frac{\sqrt{n}\sigma_n}{b_n} \to \infty & \text{if } \sigma_n \to \infty \\ \frac{\sqrt{n}\sigma_n^3}{b_n} \to \infty & \text{if } \sigma_n \to 0 \end{cases} \quad \text{as } n \to \infty. \tag{12}$$

Then, for any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{\left|J_n^{(n)}\right|}{b_n\sqrt{Var(J_n^{(n)})}} > \lambda\right) = -\frac{\lambda^2}{2}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\frac{\left|K_n^{(n)}\right|}{b_n\sqrt{Var(K_n^{(n)})}} > \lambda\right) = -\frac{\lambda^2}{2}.$$

Remark 2. When condition (12) does not hold, there are positive constants c, c_1 and c_2, such that $c_1 \leq \sigma_n \leq c_2$ and $\sigma_n \to c$ as $n \to \infty$. In this case, we can return to Theorem 2.

3. Proofs of main results

Firstly, we rewrite the weighted sums $X_n, Y_n, Z_n, T_n, J_n^{(n)}, K_n^{(n)}$ as follows:

$$X_n = \begin{cases} \sum_{i=1}^n \dfrac{m^{n+1} - m^i}{m-1}\xi_i, & \text{if } m \neq 1 \\ \sum_{i=1}^n (n-i+1)\xi_i, & \text{if } m = 1 \end{cases},$$

$$Y_n = \begin{cases} \sum_{i=1}^n \dfrac{m^{n-i+1} - 1}{m-1}\xi_i, & \text{if } m \neq 1 \\ \sum_{i=1}^n (n-i+1)\xi_i, & \text{if } m = 1 \end{cases},$$

$$Z_n = \sum_{i=1}^n \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i, \qquad T_n = \sum_{i=1}^n \frac{m_n^{n-i+1} - 1}{m_n - 1}\xi_i,$$

$$J_n^{(n)} = \sum_{i=1}^n \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i^{(n)}, \qquad K_n^{(n)} = \sum_{i=1}^n \frac{m_n^{n-i+1} - 1}{m_n - 1}\xi_i^{(n)}.$$

Before giving the proofs of the main theorems, we recall the following results in Imomov and Nazarov [6], which are important for the proof processes of our main results.

Lemma 1 ([6, Section 2]). *For any $n \geq 1$, we have*

(1) $\mathbb{E}X_n = \mathbb{E}Y_n = \mathbb{E}Z_n = \mathbb{E}T_n = \mathbb{E}J_n^{(n)} = \mathbb{E}K_n^{(n)} = 0.$

(2) *As $n \to \infty$, we have*

$$Var(X_n) = \begin{cases} \dfrac{\sigma^2}{(m-1)^2}\left(nm^{2n+2} - 2\dfrac{m^{n+2}(m^n-1)}{m-1} + \dfrac{m^2(m^{2n}-1)}{m^2-1}\right), & \textit{if } m \neq 1, \\ \dfrac{n(n+1)(2n+1)}{6}\sigma^2, & \textit{if } m = 1. \end{cases}$$
$$\sim \begin{cases} \dfrac{\sigma^2}{(m-1)^2}\dfrac{m^2}{1-m^2}, & \textit{if } m < 1, \\ \dfrac{n^3}{3}\sigma^2, & \textit{if } m = 1, \\ \dfrac{\sigma^2}{(m-1)^2}nm^{2n+2}, & \textit{if } m > 1. \end{cases}$$

(3) *As $n \to \infty$, we have*

$$Var(Y_n) = \frac{\sigma^2}{(m-1)^2}\left(\frac{m^2(m^{2n}-1)}{m^2-1} - 2\frac{m(m^n-1)}{m-1} + n\right)$$
$$\sim \begin{cases} \dfrac{\sigma^2}{(m-1)^2}n, & \textit{if } m < 1, \\ \dfrac{\sigma^2}{(m-1)^2}\dfrac{m^{2n+2}}{m^2-1}, & \textit{if } m > 1. \end{cases}$$

(4) *As $n \to \infty$, we have*

$$Var(Z_n) = \frac{\sigma^2}{(m_n-1)^2}\left(nm_n^{2n+2} - 2\frac{m_n^{n+2}(m_n^n-1)}{m_n-1} + \frac{m_n^2(m_n^{2n}-1)}{m_n^2-1}\right)$$
$$\sim \frac{n^3\sigma^2}{2\alpha^3}(e^{2\alpha}(2\alpha-3) + 4e^\alpha - 1).$$

(5) *As $n \to \infty$, we have*

$$\begin{aligned} Var(T_n) =& \frac{\sigma^2}{(m_n-1)^2}\left(\frac{m_n^2(m_n^{2n}-1)}{m_n^2-1} - 2\frac{m_n(m_n^n-1)}{m_n-1} + n\right) \\ \sim& \frac{n^3\sigma^2}{2\alpha^3}(e^{2\alpha} - 4e^{\alpha} + 2\alpha + 3). \end{aligned}$$

(6) *As $n \to \infty$, we have*

$$\begin{aligned} Var(J_n^{(n)}) =& \frac{\sigma_n^2}{(m_n-1)^2}\left(nm_n^{2n+2} - 2\frac{m_n^{n+2}(m_n^n-1)}{m_n-1} + \frac{m_n^2(m_n^{2n}-1)}{m_n^2-1}\right) \\ \sim& \frac{n^3\sigma_n^2}{2\alpha^3}(e^{2\alpha}(2\alpha-3) + 4e^{\alpha} - 1). \end{aligned}$$

(7) *As $n \to \infty$, we have*

$$\begin{aligned} Var(K_n^{(n)}) =& \frac{\sigma_n^2}{(m_n-1)^2}\left(\frac{m_n^2(m_n^{2n}-1)}{m_n^2-1} - 2\frac{m_n(m_n^n-1)}{m_n-1} + n\right) \\ \sim& \frac{n^3\sigma_n^2}{2\alpha^3}(e^{2\alpha} - 4e^{\alpha} + 2\alpha + 3). \end{aligned}$$

The proofs of our main results are given below.

Proof of Theorem 1. (i) The MDP is an application of the Gärtner-Ellis Theorem [3, Theorem 2.3.6], where b_n^2 replaces n. In order to prove (8), by using the Gärtner-Ellis Theorem, it is sufficient to show that for any $\gamma \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{E} \exp\left\{\frac{\gamma b_n X_n}{\sqrt{Var(X_n)}}\right\} = \frac{\gamma^2}{2}. \tag{13}$$

For the case $m = 1$, we have

$$\begin{aligned} &\left|\frac{1}{b_n^2}\log\mathbb{E}\exp\left\{\frac{\gamma b_n X_n}{\sqrt{Var(X_n)}}\right\} - \frac{\gamma^2}{2}\right| \\ =& \left|\frac{1}{b_n^2}\log\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(X_n)}}\sum_{i=1}^n (n-i+1)\xi_i\right\} - \frac{\gamma^2}{2}\right| \\ \le& \sum_{i=1}^n \frac{1}{b_n^2}\left|\log\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\} - \left(\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\} - 1\right)\right| \\ &+ \sum_{i=1}^n \frac{1}{b_n^2}\left|\left(\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\} - 1\right) - \frac{\gamma^2 b_n^2}{2n}\right| \\ =:& \Delta_1 + \Delta_2. \end{aligned}$$

From conditions (6)-(7) and Lemma 1, for n large enough, we have

$$\sup_{1\le i\le n} \mathbb{E}\exp\left\{\frac{2\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}|\xi_i|\right\} \le \mathbb{E}\exp\left\{C\frac{b_n}{\sqrt{n}}|\xi_1|\right\} \le C. \tag{14}$$

For the term Δ_1, by using (14), Taylor's formula and Cauchy-Schwarz's inequality, for n large enough, we have

$$\begin{aligned}
&\sup_{1\le i\le n}\left|\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\}-1\right| \\
&\le \sup_{1\le i\le n}\mathbb{E}\left(\frac{\gamma^2 b_n^2(n-i+1)^2}{2Var(X_n)}\xi_i^2\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\,|\xi_i|\right\}\right) \\
&\le C\frac{b_n^2}{n},
\end{aligned}$$

which implies that

$$\sup_{1\le i\le n}\left|\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\}-1\right|\longrightarrow 0,$$

as $n\longrightarrow\infty$. Hence, for n large enough, we have

$$\sup_{1\le i\le n}\left|\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\}-1\right|\le\frac{1}{2}. \tag{15}$$

Using (15) and the elementary inequality

$$|\log(1+x)-x|\le 2x^2,\quad\text{for }\ |x|\le\frac{1}{2}, \tag{16}$$

for n large enough, we have

$$\begin{aligned}
\Delta_1 &= \sum_{i=1}^n\frac{1}{b_n^2}\left|\log\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\}-\left(\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\}-1\right)\right| \\
&\le \sum_{i=1}^n\frac{2}{b_n^2}\left(\mathbb{E}\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\xi_i\right\}-1\right)^2 \\
&\le \sum_{i=1}^n\frac{2}{b_n^2}\left(\mathbb{E}\left(\frac{\gamma^2 b_n^2(n-i+1)^2}{2Var(X_n)}\xi_i^2\exp\left\{\frac{\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\,|\xi_i|\right\}\right)\right)^2 \\
&\le \sum_{i=1}^n\frac{\gamma^4 b_n^2(n-i+1)^4}{2(Var(X_n))^2}\mathbb{E}(\xi_i^4)\mathbb{E}\exp\left\{\frac{2\gamma b_n(n-i+1)}{\sqrt{Var(X_n)}}\,|\xi_i|\right\} \\
&\le C\sum_{i=1}^n\frac{b_n^2(n-i+1)^4}{(Var(X_n))^2} \\
&\le C\frac{b_n^2}{n},
\end{aligned}$$

which yields that $\Delta_1\longrightarrow 0$ as $n\longrightarrow\infty$. Furthermore, by using Taylor's formula, Cauchy-Schwarz's inequality and inequality (14), we obtain

$$\begin{aligned}
\Delta_2 &= \sum_{i=1}^{n} \frac{1}{b_n^2} \left| \left(\mathbb{E}\exp\left\{ \frac{\gamma b_n (n-i+1)}{\sqrt{Var(X_n)}} \xi_i \right\} - 1 \right) - \frac{\gamma^2 b_n^2}{2n} \right| \\
&\le \frac{1}{6} \sum_{i=1}^{n} \frac{1}{b_n^2} \mathbb{E}\left(\frac{\gamma^3 b_n^3 (n-i+1)^3}{(Var(X_n))^{3/2}} |\xi_i|^3 \exp\left\{ \frac{\gamma b_n (n-i+1)}{\sqrt{Var(X_n)}} |\xi_i| \right\} \right) \\
&\le \frac{1}{6} \sum_{i=1}^{n} \frac{\gamma^3 b_n (n-i+1)^3}{(Var(X_n))^{3/2}} (\mathbb{E}(\xi_i^6))^{1/2} \left(\mathbb{E}\exp\left\{ \frac{2\gamma b_n (n-i+1)}{\sqrt{Var(X_n)}} |\xi_i| \right\} \right)^{1/2} \\
&\le C \sum_{i=1}^{n} \frac{b_n (n-i+1)^3}{(Var(X_n))^{3/2}} \\
&\le C \frac{b_n}{\sqrt{n}},
\end{aligned}$$

which implies that $\Delta_2 \to 0$ as $n \to \infty$. Hence, claim (13) holds for the case $m = 1$.

For the case $m > 1$, we have

$$\begin{aligned}
&\left| \frac{1}{b_n^2} \log \mathbb{E}\exp\left\{ \frac{\gamma b_n X_n}{\sqrt{Var(X_n)}} \right\} - \frac{\gamma^2}{2} \right| \\
&\qquad = \left| \frac{1}{b_n^2} \log \mathbb{E}\exp\left\{ \frac{\gamma b_n}{\sqrt{Var(X_n)}} \sum_{i=1}^{n} \frac{m^{n+1} - m^i}{m-1} \xi_i \right\} - \frac{\gamma^2}{2} \right|.
\end{aligned}$$

From Lemma 1, it is easy to check that for n large enough, we obtain

$$\sup_{1 \le i \le n} \mathbb{E}\exp\left\{ \frac{2\gamma b_n}{\sqrt{Var(X_n)}} \frac{m^{n+1} - m^i}{m-1} |\xi_i| \right\} \le \mathbb{E}\exp\left\{ C \frac{b_n}{\sqrt{n}} |\xi_1| \right\} \le C$$

and

$$\begin{aligned}
&\sup_{1 \le i \le n} \left| \mathbb{E}\exp\left\{ \frac{\gamma b_n}{\sqrt{Var(X_n)}} \frac{m^{n+1} - m^i}{m-1} \xi_i \right\} - 1 \right| \\
&\le \mathbb{E}\left(\frac{\gamma^2 b_n^2}{2Var(X_n)} \frac{(m^{n+1} - m^i)^2}{(m-1)^2} \xi_i^2 \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(X_n)}} \frac{m^{n+1} - m^i}{m-1} |\xi_i| \right\} \right) \\
&\le \frac{\gamma^2 b_n^2}{2Var(X_n)} \frac{(m^{n+1} - m^i)^2}{(m-1)^2} \left(\mathbb{E}(\xi_i^4) \right)^{1/2} \left(\mathbb{E}\exp\left\{ \frac{2\gamma b_n}{\sqrt{Var(X_n)}} \frac{m^{n+1} - m^i}{m-1} |\xi_i| \right\} \right)^{1/2} \\
&\le C \frac{b_n^2}{Var(X_n)} \frac{(m^{n+1} - m^i)^2}{(m-1)^2} \\
&\le C \frac{b_n^2}{n}.
\end{aligned}$$

Therefore, in the same method as the case $m = 1$, we can obtain claim (13).

(ii) For the case $m = 1$, we have $X_n = Y_n$, so it is enough to prove the case $m < 1$. Similarly to the proof of (13), for any $\gamma \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \frac{1}{b_n^2} \log \mathbb{E}\exp\left\{ \frac{\gamma b_n Y_n}{\sqrt{Var(Y_n)}} \right\} = \frac{\gamma^2}{2}.$$

□

Proof of Theorem 2. We only claim that (10) holds. Applying the Gärtner-Ellis Theorem [3, Theorem 2.3.6], we only need to show that for any $\gamma \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{E} \exp\left\{\frac{\gamma b_n Z_n}{\sqrt{Var(Z_n)}}\right\} = \frac{\gamma^2}{2}.$$

Indeed, it can be easily seen that

$$\begin{aligned}
&\left|\frac{1}{b_n^2} \log \mathbb{E} \exp\left\{\frac{\gamma b_n Z_n}{\sqrt{Var(Z_n)}}\right\} - \frac{\gamma^2}{2}\right| \\
&= \left|\frac{1}{b_n^2} \log \mathbb{E} \exp\left\{\frac{\gamma b_n}{\sqrt{Var(Z_n)}} \sum_{i=1}^{n} \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i\right\} - \frac{\gamma^2}{2}\right| \\
&\le \sum_{i=1}^{n} \frac{1}{b_n^2}\left|\log \mathbb{E} \exp\left\{\frac{\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i\right\}\right. \\
&\quad \left. - \left(\mathbb{E} \exp\left\{\frac{\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i\right\} - 1\right)\right| \\
&\quad + \sum_{i=1}^{n} \frac{1}{b_n^2}\left|\left(\mathbb{E} \exp\left\{\frac{\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i\right\} - 1\right) - \frac{\gamma^2 b_n^2}{2n}\right|.
\end{aligned}$$

Note that for any $\alpha \neq 0$,

$$\lim_{n\to\infty} m_n^n = e^{\alpha}.$$

Hence, from condition (9) and Lemma 1, we can assert that for n large enough,

$$\sup_{1\le i\le n} \mathbb{E} \exp\left\{\frac{2\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}|\xi_i|\right\} \le \mathbb{E} \exp\left\{C\frac{b_n}{\sqrt{n}\sigma_n}|\xi_1|\right\} \le C$$

and

$$\begin{aligned}
&\sup_{1\le i\le n}\left|\mathbb{E} \exp\left\{\frac{\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}\xi_i\right\} - 1\right| \\
&\le \mathbb{E}\left(\frac{\gamma^2 b_n^2}{2Var(Z_n)} \frac{(m_n^{n+1} - m_n^i)^2}{(m_n - 1)^2}\xi_i^2 \exp\left\{\frac{\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}|\xi_i|\right\}\right) \\
&\le \frac{\gamma^2 b_n^2}{2Var(Z_n)} \frac{(m_n^{n+1} - m_n^i)^2}{(m_n - 1)^2}\left(\mathbb{E}(\xi_i^4)\right)^{1/2}\left(\mathbb{E} \exp\left\{\frac{2\gamma b_n}{\sqrt{Var(Z_n)}} \frac{m_n^{n+1} - m_n^i}{m_n - 1}|\xi_i|\right\}\right)^{1/2} \\
&\le C\frac{b_n^2}{Var(Z_n)} \frac{(m_n^{n+1} - m_n^i)^2}{(m_n - 1)^2} \\
&\le C\frac{b_n^2}{n}.
\end{aligned}$$

Therefore, by using the same method of the proof of Theorem 1, we can establish (10). □

Proof of Theorem 3. We only give the proof of (17). By using the Gärtner-Ellis Theorem [3, Theorem 2.3.6], it is sufficient to show that for any $\gamma \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{E} \exp\left\{\frac{\gamma b_n J_n^{(n)}}{\sqrt{Var(J_n^{(n)})}}\right\} = \frac{\gamma^2}{2}. \tag{17}$$

Note that

$$\begin{aligned}
&\left|\frac{1}{b_n^2}\log\mathbb{E}\exp\left\{\frac{\gamma b_n J_n^{(n)}}{\sqrt{Var(J_n^{(n)})}}\right\}-\frac{\gamma^2}{2}\right|\\
&=\left|\frac{1}{b_n^2}\log\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\sum_{i=1}^{n}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}-\frac{\gamma^2}{2}\right|\\
&\le\sum_{i=1}^{n}\frac{1}{b_n^2}\left|\log\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}\right.\\
&\qquad\left.-\left(\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}-1\right)\right|\\
&\quad+\sum_{i=1}^{n}\frac{1}{b_n^2}\left|\left(\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}-1\right)-\frac{\gamma^2b_n^2}{2n}\right|\\
&=:\Delta_3+\Delta_4.
\end{aligned}$$

As a consequence, in order to obtain (17), it is enough to show that $\Delta_3\longrightarrow 0$ and $\Delta_4\longrightarrow 0$ as $n\longrightarrow\infty$.

From conditions (11)-(12) and Lemma 1, for n large enough, we have

$$\sup_{1\le i\le n}\mathbb{E}\exp\left\{\frac{2\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\left|\xi_i^{(n)}\right|\right\}\le\mathbb{E}\exp\left\{C\frac{b_n}{\sqrt{n}\sigma_n}\left|\xi_1^{(n)}\right|\right\}\le C.$$

Hence, we can deduce that for n large enough,

$$\begin{aligned}
&\sup_{1\le i\le n}\left|\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}-1\right|\\
&\le\sup_{1\le i\le n}\mathbb{E}\left(\frac{\gamma^2b_n^2}{2Var(J_n^{(n)})}\frac{(m_n^{n+1}-m_n^i)^2}{(m_n-1)^2}\left|\xi_i^{(n)}\right|^2\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\left|\xi_i^{(n)}\right|\right\}\right)\\
&\le C\frac{b_n^2}{n\sigma_n^2},
\end{aligned}$$

which means that

$$\sup_{1\le i\le n}\left|\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}-1\right|\longrightarrow 0,$$

as $n\longrightarrow\infty$. Hence, for n large enough, we have

$$\sup_{1\le i\le n}\left|\mathbb{E}\exp\left\{\frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}}\frac{m_n^{n+1}-m_n^i}{m_n-1}\xi_i^{(n)}\right\}-1\right|\le\frac{1}{2}. \tag{18}$$

Using the elementary inequality (16) and (18), for n large enough, we obtain

$$
\begin{aligned}
\Delta_3 &= \sum_{i=1}^{n} \frac{1}{b_n^2} \Bigg| \log \mathbb{E} \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}} \frac{m_n^{n+1} - m_n^i}{m_n - 1} \xi_i^{(n)} \right\} \\
&\quad - \left(\mathbb{E} \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}} \frac{m_n^{n+1} - m_n^i}{m_n - 1} \xi_i^{(n)} \right\} - 1 \right) \Bigg| \\
&\le \sum_{i=1}^{n} \frac{2}{b_n^2} \left(\mathbb{E} \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}} \frac{m_n^{n+1} - m_n^i}{m_n - 1} \xi_i^{(n)} \right\} - 1 \right)^2 \\
&\le \sum_{i=1}^{n} \frac{2}{b_n^2} \left(\mathbb{E} \left(\frac{\gamma^2 b_n^2}{2Var(J_n^{(n)})} \frac{(m_n^{n+1} - m_n^i)^2}{(m_n - 1)^2} |\xi_i^{(n)}| \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}} \frac{m_n^{n+1} - m_n^i}{m_n - 1} |\xi_i^{(n)}| \right\} \right) \right)^2 \\
&\le C \sum_{i=1}^{n} \frac{b_n^2}{(Var(J_n^{(n)}))^2} \frac{(m_n^{n+1} - m_n^i)^4}{(m_n - 1)^4} \\
&\le C \frac{b_n^2}{n\sigma_n^4},
\end{aligned}
$$

which yields $\Delta_3 \longrightarrow 0$.

Furthermore, we can deduce that

$$
\begin{aligned}
\Delta_4 &= \sum_{i=1}^{n} \frac{1}{b_n^2} \left| \left(\mathbb{E} \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}} \frac{m_n^{n+1} - m_n^i}{m_n - 1} \xi_i^{(n)} \right\} - 1 \right) - \frac{\gamma^2 b_n^2}{2n} \right| \\
&\le \frac{1}{6} \sum_{i=1}^{n} \frac{1}{b_n^2} \mathbb{E} \Bigg(\frac{\gamma^3 b_n^3}{(Var(J_n^{(n)}))^{3/2}} \frac{(m_n^{n+1} - m_n^i)^3}{(m_n - 1)^3} |\xi_i^{(n)}|^3 \\
&\quad \times \exp\left\{ \frac{\gamma b_n}{\sqrt{Var(J_n^{(n)})}} \frac{m_n^{n+1} - m_n^i}{m_n - 1} |\xi_i^{(n)}| \right\} \Bigg) \\
&\le C \sum_{i=1}^{n} \frac{b_n}{(Var(J_n^{(n)}))^{3/2}} \frac{(m_n^{n+1} - m_n^i)^3}{(m_n - 1)^3} \\
&\le C \frac{b_n}{\sqrt{n}\sigma_n^3},
\end{aligned}
$$

which implies that $\Delta_4 \longrightarrow 0$ as $n \longrightarrow \infty$. □

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