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Variable Structure Correlation Analysis Based on a New Asymmetric Copula

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Abstract: Traditional dependence modeling has predominantly relied on symmetric copulas to capture variable relationships. However, empirical evidence reveals the prevalence of asymmetric dependence structures in real-world data. Addressing this limitation, contemporary research has progressively shifted toward asymmetric copulas for more precise dependence characterization. Building on this development, this paper develops an innovative construction method for asymmetric copulas by extending Archimedean copula theory. Employing a piecewise modeling framework, we empirically validate the proposed method using daily returns of the Shanghai Composite Index (SHCI) and Shenzhen Component Index (SZCI). The findings demonstrate that the mixed variable-structure copula model constructed using the novel methodology provides superior goodness-of-fit relative to a single-structure asymmetric copula model, which itself outperforms traditional symmetric Archimedean copulas. Furthermore, the paper explores the mutual transformation of tail copula functions through survival copulas, broadening the practical scope of asymmetric copula applications.

Keywords: asymmetric copula; goodness-of-fit; tail distribution characteristic; the square Euclidean distance; variable- structure copula;

1 INTRODUCTION

As a powerful tool for modeling the dependence between joint distributions and their marginals, copulas have been extensively applied in diverse domains, particularly in financial risk management, climate science, and engineering systems reliability assessment [1-9]. Traditional dependence analysis predominantly employs symmetric copulas to characterize inter-variable dependence structures. Moreover, conventional dependence metrics - including Pearson's linear correlation coefficient, Spearman's ρ , Kendall's τ , and Gini's γ - primarily quantify symmetric dependence relationships. While these measures are sufficient for elliptical distributions, they fail to capture complex, non-linear dependence structures prevalent in empirical data. Empirical evidence demonstrates that dependence structures frequently exhibit significant asymmetry, particularly in tail dependence. In recent years, copula-based regression dependence measures have attracted growing scholarly attention, particularly for their ability to capture asymmetric dependence patterns.

Dabrowska [10] pioneered a regression-based correlation measure utilizing the correlation ratio. This approach quantifies the relationship between variables by assessing the proportion of variance in one variable explainable by another, thus establishing a regression-oriented framework for dependence measurement. Building upon prior research, Sungur [11] pioneered the copula dependence coefficient as a novel dependence measure. Shih and Emura [12] refined its definition by reformulating it as the Spearman's ρ of the convolved copula. The copula dependence coefficient serves as a regression-based asymmetric dependence measure, capable of capturing directional dependence structures only for asymmetric copulas. To address the limitations of symmetric dependence modeling, scholars have developed various methodological frameworks for constructing asymmetric copula functions [13-19], significantly advancing the analysis of asymmetric dependence structures. Wei and Kim [13] proposed a new method for exploring the asymmetric correlation of multidimensional variables through the multidimensional regression correlation function and substantiated the feasibility of the method via empirical analysis. Wei et al. [14] put forward an innovative stochastic frontier model (SFM) incorporating skew-normal copula functions and studied the asymmetric dependence structures between technical inefficiency disturbances and noise disturbances. Wu [15] proposed a novel construction method for asymmetric copula models, investigated their theoretical properties, and empirically analyzed the impact of asymmetric dependence characteristics on component lifetime using failure time data of mechanical parts. Zhang et al. [16] employed asymmetric copula models to capture the asymmetric dependence structures among natural soil parameters, thereby enhancing the reliability of geotechnical engineering design through improved soil condition assessment. Alfonsi and Brigo [17] developed an innovative asymmetric copula construction framework utilizing periodic functions, accompanied by rigorous theoretical analysis of the model's statistical properties - specifically tail dependence coefficients and rank correlation measures. Liebscher [18] developed two asymmetric copula construction methods: one derived from product copulas and another based on generalized Archimedean copulas. The study also proposed corresponding Monte Carlo simulation algorithms, which collectively established a methodological foundation for dependence structure analysis. Lee et al. [19] developed a novel asymmetric copula construction framework using the Khoudraji transformation (k-transformation), systematically analyzing its impact on copula-based asymmetric dependence measures. Asymmetric copula functions exhibit enhanced flexibility in characterizing complex dependence structures among random variables, particularly when incorporating additional shape parameters.

Existing asymmetric copula models predominantly exhibit upper-tail dependence and tail asymptotic independence characteristics, which inadequately capture the diverse dependence patterns observed in empirical data. To address this methodological limitation, this study proposes a novel asymmetric copula construction framework based on generalized Archimedean copulas. This advancement establishes a robust foundation for more accurate characterization of complex dependence structures among variables.

The paper is structured into five main sections as follows: Section 1. Introduction. This section provides an overview of the research background, a review of relevant literature, and the main contributions of this paper. Section 2. Construction of asymmetric copula models. Building upon the asymmetric copula construction method proposed by Liebscher, this section proposes a novel constructing asymmetric copulas approach and verifies their lower-tail dependence distribution characteristics which fills the deficiency of the previous asymmetric copula model in the description of dependent structure. Section 3. Correlation analysis based on single-structure copula models. This section performs a comparative dependence analysis between financial variables using Archimedean copula and the asymmetric copula model. The empirical results systematically demonstrate the superiority of asymmetric copulas in capturing nonlinear tail dependence. These findings establish the theoretical and empirical foundations for the comprehensive comparison in Section 5. Section 4. The construction of variable-structure copula models. Building upon the dependence analysis of financial variables, this section develops a variable-structure copula framework through piecewise modeling, with distinct functional specifications for each dependence regime: (i) upper-tail, (ii) lower-tail, and (iii) intermediate dependence regions. Section5. Comparative dependence analysis based on variable-structure copula models. This section conducts a comparative dependence analysis between the Shanghai Composite Index (SHCI) and Shenzhen Component Index (SZCI) returns using the two constructed variable-structure copula models (Model 1 and Model 2). The results show that variable-structure copulas exhibit better fit than the single-structure asymmetric copula, which in turn outperforms the symmetric Archimedean copula. These findings simultaneously validate the effectiveness of the newly proposed asymmetric copula construction method and demonstrate the practical application of tail dependence transformation through survival copulas.

2 CONSTRUCTION OF ASYMMETRIC COPULA MODELS

This section firstly reviews Liebscher's constructive approach for asymmetric copulas, verifying their upper-tail dependence distribution properties. Given the inherent complexity of the dependence structures observed in empirical data, this section develops a novel asymmetric copulas construction method and discusses the distribution characteristics of the proposed copula models that significantly expands the application scope of asymmetric copulas. Additionally, this section also highlights that asymmetric copulas can achieve transformation of tail dependence structures through survival copula operations. This research addresses critical limitations in existing asymmetric copulas - particularly their restrictive tail dependence structures - while significantly improving practical applicability in empirical analyses.

We first introduce the asymmetric copula construction framework proposed by Liebscher [18], which generalizes the Archimedean copula structure.

Let $C(u)=\omega^{-1}\left(\omega(u_1)+\cdots+\omega(u_d)\right)$, where $u_i\in[0,1]$, ω is the strictly descending function (called the generator) from $[0,1]$ to $[0,\infty]$, if ω^{-1} is d-monotonic, namely $(-1)^k\frac{d^k}{dt^k}\omega^{-1}(t)\geq 0$, $k=0,1,2\cdots d$, then $C(u)$ is an Archimedean copula [20]. It has another form $C(u)=\omega_0^{-1}\left(\omega_0(u_1)\times\cdots\times\omega_0(u_d)\right)$, where $u_i\in[0,1]$, ω_0 is a function on $[0,1]$ to $[0,1]$, and $\omega_0(t)=e^{-\omega(t)}$.

Archimedean copulas are widely employed in investment risk management owing to their symmetric structure and flexible tail dependence properties. The most prevalent copulas include Gumbel, Clayton, Frank copulas - exhibit distinct tail dependence patterns: upper-tail dependence, lower-tail dependence and tail independence, respectively. Archimedean copulas exhibit structural symmetry because their generator function ω (or ω_0) applies equally to every marginal variable u_i. We can construct asymmetric copulas through distinct functional composition of generator functions. Based on this idea, Liebscher [18] developed a construction framework for asymmetric copulas by generalizing the Archimedean copula structure.

Theorem 1:

Let $C(u)=\omega\left(\frac{1}{m}\sum_{j=1}^{m}h_{j1}\left(\omega^{-1}(u_1)\right)\cdots h_{jd}\left(\omega^{-1}(u_n)\right)\right)$,

where $\omega^{(n)}$ exists, $\omega'(u)>0$, $\omega^{(i)}(u)\geq 0$, $i=2,\cdots,n$, $u\in[0,1]$, let $\omega(0)=0$, $\omega(1)=1$, and for each $j\in\{1,\cdots,m\}$ and $k\in\{1,\cdots,n\}$, h_{jk} is derivable and strictly increased from [0,1] to [0,1], $h_{jk}(0)=0$, $h_{jk}(1)=1$, $\frac{1}{m}\sum_{j=1}^{m}h_{jk}(x)=x$, $k=1,\cdots,n$, $x\in[0,1]$, so $C(u)$ is an absolutely continuous copula.

In Theorem 1, if $h_{jk}(x)$ is equal to x for every $j\in\{1,\cdots,m\}$ and $k\in\{1,\cdots,d\}$, $C(u)=\omega\left(\omega^{-1}(u_1)\cdots\left(\omega^{-1}(u_d)\right)\right)$ is an Archimedean copula, where $\psi(t)=\omega^{-1}(t)$ is the generator of copula $C(u)$. Therefore, the function $\omega(t)$ of Theorem 1 can be got through generators of Archimedean copulas.

Example 1:

Let $\omega(t)=\dfrac{e^{-\alpha(1-t)^{\frac{1}{r}}}-e^{-\alpha}}{1-e^{-\alpha}}$, so $\omega^{-1}(t)=1-\left(-\dfrac{ln\left(e^{-\alpha}+x-xe^{-\alpha}\right)}{\alpha}\right)^r$, where $r\in(1,+\infty)$, $\alpha\in(0,\infty)$. Let $k=2$, $m=2$, $h_{1k}(x)=x^{\xi_k}$, $h_{2k}(x)=2x-x^{\xi_k}$, $\xi_k\in[1,2]$, according to Theorem 1, an asymmetric copula $C^*(u,v)=\omega\left(\frac{1}{2}\sum_{j=1}^{2}h_{j1}\left(\omega^{-1}(u)\right)h_{j2}\left(\omega^{-1}(v)\right)\right)$ can be constructed by composing the generator function $\omega(t)$ with the function $h_{ij}(x)$. By calculation, its upper-tail dependence coefficient $\lambda_U=2-2^{\frac{1}{r}}$, its lower-tail dependence coefficient $\lambda_L = 0$ .Then the copula C^* is an asymmetric copula with upper-tail dependence distribution characteristic.

In fact, we can rigorously verify that the bivariate asymmetric copula derived in Theorem 1 possesses lower-tail asymptotic independence.

Theorem 2:

Let copula $C(u,v) = \omega\left(\frac{1}{m}\sum_{j=1}^{m} h_{j1}\left(\omega^{-1}(u)\right) h_{j2}\left(\omega^{-1}(v)\right)\right)$, ω'' exists, $\omega'(u) > 0$, $\omega'' \ge 0$, $u \in [0,1]$. Let $\omega(0) = 0$, $\omega(1) = 1$ and h_{jk} is differentiable and strictly increased from [0,1] to [0,1] for every $j \in \{1, \cdots, m\}$ and $k \in \{1,2\}$, $h_{jk}(0) = 0$, $h_{jk}(1) = 1$, $\frac{1}{m}\sum_{j=1}^{m} h_{jk}(x) = x$, $k = 1,2$, $x \in [0,1]$. Then the copula C is lower-tail asymptotic independent.

Proof:

$$\lambda_L = \lim_{u\to 0}\frac{\partial C(u,v)}{\partial u} = \frac{1}{m}\lim_{u\to 0}\omega'$$

$$\left(\frac{1}{m}\sum_{k=1}^{2} h_{k1}(\omega(u)) h_{k2}(\omega(u))\right)\left(\omega^{-1}(u)\right)'$$

$$\cdot\sum_{k=1}^{m} h'_{j1}(\omega(u)) h_{j2}(\omega(u)) + h_{j1}(\omega(u)) h'_{j2}(\omega(u)))$$

because $\lim_{u\to 0} h_{j1}(\omega(u)) = \lim_{u\to 0} h_{j2}(\omega(u)) = 0$, if $\omega'\left(\frac{1}{m}\sum_{k=1}^{m} h_{k1}(\omega(u)) h_{k2}(\omega(u))\right) < \infty$, then $\lambda_L = 0$.

Theorem 1 essentially extends Archimedean copulas from an arithmetic mean perspective, while this paper proposes a further generalization through geometric mean, thereby developing a novel asymmetric copula construction method.

Theorem 3:

Let

$$C(u_1,u_2) = \omega\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}(u_1)\right) + h_{j2}\left(\omega^{-1}(u_2)\right)\right)\right)^{\frac{1}{2}},$$

where $\omega^{-1}(t)$ is the decreasing function from [0,1] to $[0,\infty)$, $\omega' > 0$, h_{jk} is the increasing function from $[0,\infty)$ to $[0,\infty)$, then the function C is a copula.

Refer to Appendix for detailed proof.

In Theorem 3, if $h_{jk}(t) = t$, that is, for every j and k, $h_{jk}(t)$ is equal, then $C(u) = \omega\left(\omega^{-1}(u_1)\right) + \left(\omega^{-1}(u_2)\right)$ is an Archimedean copula.

The Archimedean copula has two forms, one form is $C(u) = \omega\left(\left(\omega^{-1}(u_1)\right) + \cdots + \left(\omega^{-1}(u_d)\right)\right)$, another is $C(u) = \bar{\omega}\left(\left(\bar{\omega}^{-1}(u_1)\right)\cdots\left(\bar{\omega}^{-1}(u_d)\right)\right)$, $\bar{\omega}(t) = exp(-\omega(t))$.

Therefore, the construction methods proposed by Liebscher [21] and developed in this paper can be formally characterized as dual extensions of Archimedean copulas, each employing distinct functional formulations. Furthermore, the generator functions ω in Theorem 3 may be adapted from conventional Archimedean generators.

Example 2:

Let $\omega^{-1}(t) = exp(t^{-\theta}) - e$, then $\omega(t) = \left(\frac{1}{ln(t+e)}\right)^{\theta}$, $\theta \in (1,+\infty)$. Assume $h_{1k}(t) = t^{\alpha_k}$, $h_{2k}(t) = t^{2-\alpha_k}$, where $\alpha_1, \alpha_2 \in [0,2]$, for simplicity, let $\alpha_1 > \alpha_2$. Obviously, $h_{jk}(t)$ and $\omega(t)$ satisfy the conditions of the Theorem 3, so $C^{\#}(u,v) = \omega\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}(u)\right) + h_{j2}\left(\omega^{-1}(v)\right)\right)\right)^{\frac{1}{2}}$ is a copula. By calculation, its lower-tail dependence coefficient

$$\lambda_L = \left(\frac{2+\alpha_1-\alpha_2}{2}\right)\left(\frac{2}{2+\alpha_1-\alpha_2}\right)^{\frac{1}{\theta}+1} = \left(\frac{2}{2+\alpha_1-\alpha_2}\right)^{\frac{1}{\theta}},$$

its upper-tail dependence coefficient λ_U does not exist.

As demonstrated in Example 2, the proposed asymmetric copula exhibits pronounced lower-tail dependence distribution characteristic. The construction method proposed in Theorem 3 enhances the capability of asymmetric copulas to characterize diverse tail dependence structures. In addition, the modeling flexibility of asymmetric copulas can be further enhanced through functional transformations of tail copulas.

Definition 1 (Tail Copula):

Let C be bivariate copula C with joint distribution function H. Given any arbitrary point $(x,y) \in \overline{\mathbb{R}}_t^2 = \frac{[0,\infty]^2}{\{[\infty,\infty]\}}$, if the limit $\Lambda_L(x,y) = \lim_{t\to\infty} t\cdot C\left(\frac{x}{t},\frac{y}{t}\right)$ exists, $\Lambda_L(x,y): \overline{\mathbb{R}}^2 \to \mathbb{R}$ is called the lower-tail copula associated with C.

Similarly, if the limit $\Lambda_U(x,y) = \lim_{t\to\infty} t\cdot \hat{C}\left(\frac{x}{t},\frac{y}{t}\right)$ exists, $\Lambda_U(x,y)$ is called the upper-tail copula that associated with C, where $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$, the copula $\hat{C}$ is the survival copula of copula C.

Theorem 4 (Tail copula transformation) Let C be a bivariate copula with survival $\hat{C}(u,v) = u + v - 1 + C(1-u,1-v)$, then

(i) the lower-tail copula of C is given by

$$\Lambda_U(x,y) = \lim_{t\to\infty} t\cdot C\left(\frac{x}{t},\frac{y}{t}\right)$$

(ii) the upper-tail copula of C is given by

$$\Lambda_L(x,y) = \lim_{t\to\infty} t\cdot \hat{C}\left(\frac{x}{t},\frac{y}{t}\right)$$

Crucially, since $\hat{C}$ is itself a copula, Λ_U equivalently represents the lower-tail copula of $\hat{C}$. This established an exact tail dependence inversion $\Lambda_L^{C}(x,y) = \Lambda_U^{\hat{C}}(x,y)$.

Example 3:

According to Exmple1, the asymmetric copula $C^*(u,v)=\omega\left(\frac{1}{2}\sum_{j=1}^{2}h_{j1}\left(\omega^{-1}(u)\right)h_{j2}\left(\omega^{-1}(v)\right)\right)$ is upper-tail dependent and lower-tail asymptotically independent, where $\omega(t)=\frac{exp\left(-(1-t)^{\frac{1}{r}}\right)-e^{-1}}{1-e^{-1}}$, $r\in(1,+\infty)$, $h_{11}(t)=t^{\frac{3}{2}}$, $h_{21}(t)=2t-t^{\frac{3}{2}}$, $h_{12}(t)=t^2$, $h_{22}(t)=2t-t^2$.

According to Definition 1, its upper-tail copula $\Lambda_U^{C^*}(x,y)=\lim_{t\to\infty}t\cdot\hat{C}^*\left(\frac{x}{t},\frac{y}{t}\right)$, by calculation $\Lambda_U^{C^*}(x,y)=x+y-x\cdot 2^{\frac{1}{r}-1}-y\cdot 2^{\frac{1}{r}-1}$. Let $\tilde{C}(u,v)=\hat{C}^*(u,v)=u+v-1+C^*(1-u,1-v)$, then

$$\begin{aligned}\Lambda_L^{\tilde{C}}(x,y)&=\lim_{t\to\infty}t\cdot\tilde{C}\left(\frac{x}{t},\frac{y}{t}\right)\\&=\lim_{t\to\infty}t\cdot\tilde{C}^*\left(\frac{x}{t},\frac{y}{t}\right)\\&=x+y-x\cdot 2^{\frac{1}{r}-1}-y\cdot 2^{\frac{1}{r}-1}\end{aligned}$$

Given that $\lambda_L^{\tilde{C}}(x,y)=\Lambda_L^{\tilde{C}}(1,1)=2-2^{\frac{1}{r}}$, then copula $\tilde{C}$ is an asymmetric copula with lower-tail dependence structure.

The preceding analysis confirms tail dependence transformation through survival copula operations. This methodological advance significantly extends the applicability of asymmetric copulas, establishing a theoretical foundation for their enhanced utilization in dependence modeling. To validate the construction method proposed in Theorem 3 and demonstrate the practical utility of tail copula transformations, the study conducts a comparative dependence analysis using two variable-structure copulas - respectively constructed from copula $\tilde{C}$ in Example 3 and copula $C^{\#}$ in Example 2. The empirical results demonstrate that the two variable-structure copula models exhibit superior goodness-of-fit in modeling the dependence structure between the SHCI and SZCI compared to the single-structure asymmetric and the conventional Archimedean copula. This finding not only validates the practical feasibility of the novel construction methodology but also reveals its enhanced capability in capturing complex market linkages. The detailed comparative analyses are provided in section 4.

3 CORRELATION ANALYSIS BASED ON SINGLE-STRUCURE COPULA MODELS

3.1 Basic Descriptive Statistics of the Data

This paper takes Shanghai Composite Index (SHCI) and Shenzhen Component Index (SZCI) as the research object, denoted by X_1, X_2 respectively. The daily closing price P_t serves as the primary price measure. The sample period spans from 31 July 2015 to 24 September 2021, yielding 1499 valid daily observations sourced from Sina Finance. The logarithmic return R_{it} for day t is computed as $R_{it}=100\left(ln\,P_{it}-ln\,P_{it-1}\right)$, $i=1,2$, $t=2,\cdots,1499$.

We conduct a comparative analysis of the dependence structure between the logarithmic returns R_{1t} (SHCI) and R_{2t} (SZCI) using three distinct copula frameworks: Archimedean, the single-structure asymmetric and variable- structure copulas in this paper.

Table 1 Statistical characteristics analysis of yield rate

	Mean	Standard deviation	Skewness	Kurtosis	Range	ADF test
SHCI	–0.009855	1.611986	0.914784	4.51023	15.079	–11.041**
SZCI	0.001489	1.274330	1.114244	7.92333	14.427	–11.757**
Note: ** means significant at 0.01 level						

Tab. 1 presents the descriptive statistics of the logarithmic returns for both SHCI and SZCI. The results reveal two key characteristics: both return series exhibit positive skewness (SHCI: 0.914784; SZCI: 1.114244) and leptokurtic distributions, with excess kurtosis values of 1.51023 and 4.92333, respectively. The Augmented Dickey-Fuller (ADF) tests reject the null hypothesis of non-stationarity at the 1% significance level for both series.

3.2 The Selection of Edge Distribution

Given the well-established stylized facts of financial time series - particularly time-varying volatility, volatility clustering, skewness, and leptokurtosis with heavy tails - we employ GARCH models to capture these conditional distributional characteristics. Specifically, the marginal distributions of the return series $\{R_{it}\}$, $i=1,2$, are modeled using the GARCH (1,1)-t model:

$$R_{nt}=\mu_{nt}+\varepsilon_{nt},\quad n=1,2,\ t=1,2,\cdots,T,$$

$$\varepsilon_{nt}=h_{nt}^{\frac{1}{2}}\xi_{nt}$$

$$h_{nt}=W_n+\alpha_n\varepsilon_{n,t-1}^2+\beta_n h_{n,t-1},$$

$$\left(\xi_{1t},\xi_{2t}\right)\sim C\left(T_{v_1}\left(\xi_{1t}\right),T_{v_2}\left(\xi_{2t}\right)\right)$$

where ξ_{1t} and ξ_{2t} follow separately standard Student's t-distributions with zero mean, unit variance, and degrees of freedom v_1 and v_2, respectively, namely $\sqrt{\frac{v_1}{v_1-2}}\xi_{1t}\sim t(v_1)$, $\sqrt{\frac{v_2}{v_2-2}}\xi_{2t}\sim t(v_2)$.

The parameter estimations of the two GARCH models are presented in Tab. 2.

Table 2 Parameter simulations of GARCH model

	SHCI		SZCI	
u	–0.014895	(0.4709)	–0.06555	(0.0273)*
w	0.013206	(0.05863) ·	0.02178	(0.19804)
α	0.084388	(5.26e^{-06})***	0.06580	(0.00166)**
β	0.918962	(2e^{-16})***	0.93298	(2^{e-16})***
V	3.91628	(2e^{-16})***	4.36013	(4.44e^{-16})***
Log- likelihood	–2157.952		–2610.869	
K-S Statistic	0.026299		0.024336	
K-S Probability	0.2510		0.3372	
Note: * means significant at 0.05 level, ** means significant at 0.01 level, ***means significant at 0.001 level, · means significant at 0.1 level.				

The Kolmogorov-Smirnov (K-S) test results confirm that probability integral transforms $\{u_t\}$ and $\{v_t\}$ obtained from the GARCH $(1,1)$ - t models follow uniform distributions on the interval $[0,1]$: K-S statistic(SHCI) $= 0.026299$, p-value $= 0.2510$, K-S statistic(SZCI) $= 0.024336$, p-value $= 0.3372$. This provides strong evidence that the GARCH-t model accurately captures the marginal distributions of the original return series R_{1t} (SHCI) and R_{2t} (SZCI).

3.3 The Selection of Copula Models

We evaluate the dependence structure between SHCI and SZCI returns employing three canonical Archimedean copulas: Gumbel, Clayton, and Frank. The maximum likelihood estimation results are shown in Tab. 3:

Table 3 Parameter estimates of copula models

Copula	Parameter	Log-likelihood	AIC	BIC
Gumbel	3.5	1321.95	-2641.89	-2636.58
Frank	12.7	1221.16	-2440.31	-2435
Clayton	3.22	1042.52	-2083.03	-2077.72

Based on the Akaike Information Criterion (AIC), the Gumbel copula demonstrates superior goodness-of-fit compared to the Clayton and Frank copulas, indicating its appropriateness for modeling the dependence structure between SHCI and SZCI returns. Because the Gumbel copula exhibits upper-tail dependence characteristics with an estimated upper-tail dependence coefficient $\lambda_U = 0.7809863$ this indicates there is an upper- tail dependence relationship between SHCI and SZCI.

According to Example 1, the copula C^* also exhibits upper-tail dependence characteristics, its upper-tail dependence coefficient $\lambda_U^{C^*} = 2 - 2^{\frac{1}{r}}$. Let $2 - 2^{\frac{1}{r}} = 0.78$, we can derive its parameter estimate $\hat{r} = 3.5$.

Table 4 d^2 between copula and empirical copula functions

Copula	d^2
Gumbel	0.2889
C_5^*	0.2909
C_6^*	0.2866
C_7^*	0.2859
C_8^*	0.2863
C_9^*	0.2869

To compare the goodness-fit of the asymmetric copula C^* and the Gumbel copula, we compute the squared Euclidean distances d^2 between each parametric copula and the empirical copula. As summarized in Tab. 4 (where the copula C_α^* represents the asymmetric copula with parameter $\alpha = i$): when $\alpha = 7$, the asymmetric copula C^* achieves the minimum squared Euclidean distance ($d^2 = 0.2859$) smaller than the Gumbel copula ($d^2 = 0.2889$). This demonstrates that the copula C^* exhibits superior performance in capturing the dependence structure between SHCI and SZCI returns. Moreover, the asymmetric copula C^*, with its two estimable parameters, offers greater modeling flexibility compared to the Gumbel copula.

4 THE CONSTRUCTION OF VARIABLE-STRUCTURE COPULA MODELS

In Section 3, we employ both the Gumbel copula and the asymmetric copula framework to comparatively analyze the dependence structure between SHCI and SZCI returns. Both models exhibit upper-tail dependence characteristics. However empirical evidence reveals that although lower-tail dependence between SHCI and SZCI returns is less pronounced than upper-tail dependence, they are not asymptotically independent either.

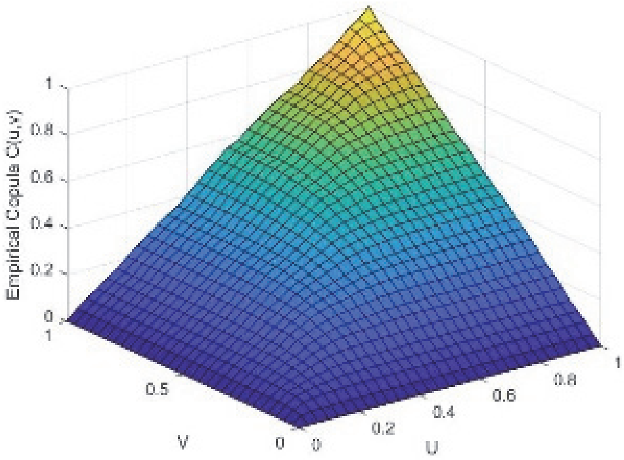

Figure 1 Empirical distribution function graph of variable sequence $\{u_t\},\{v_t\}$

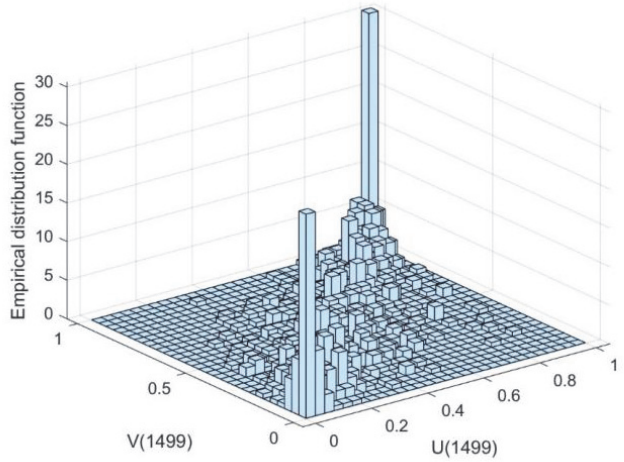

Figure 2 The frequency histogram of all sample data

Therefore, in order to more accurately depict the dependent structure between SHCI and SZCI returns, this paper constructs a mixed variable-structure copula with the following specification:

$$C^T(u_t, v_t) = D_t^L C^L(u_t, v_t) + D_t^U C^U(u_t, v_t) + \\ + \left(1 - D_t^L - D_t^U\right) C^M(u_t, v_t)$$

where C^T denotes the asymmetric copula characterizing the dependence structure between u_t and v_t over the entire sample period, the segmented-specific asymmetric copulas are defined as C^L, C^U and C^M, respectively capturing dependence patterns in the lower-tail, upper-tail and the intermediate region. D_t^L and D_t^U represent respectively indicator function in $\{u_t \le \varepsilon, v_t \le \varepsilon\}$ and $\{u_t \ge 1-\varepsilon, v_t \ge 1-\varepsilon\}$,their definitions are as follows:

$$D_t^L = \begin{cases} 1, & u_t \le \varepsilon, v_t \le \varepsilon \\ 0, & \text{else} \end{cases},$$

$$D_t^U = \begin{cases} 1, & u_t \ge 1-\varepsilon, v_t \ge 1-\varepsilon \\ 0, & \text{else} \end{cases}$$

where, ε denotes the given threshold, take $\varepsilon = 0.25$. Namely if $u_t \le 0.25$, $v_t \le 0.25$, the sample data is considered to be in the lower-tail region; if $u_t \ge 0.75$, $v_t \ge 0.75$, the sample data is considered to be in the upper-tail region; if $0.25 < u_t < 0.75$ or $0.25 < v_t < 0.75$, the sample data is considered to be in the intermediate region. The analytical procedure consists of the following steps:

Step 1. Marginal distribution modeling. The analysis commences by modeling the marginal distributions of the logarithmic returns $\{R_{it}\}$, $i = 1,2$ for both the SHCI and SZCI series.

Step 2. Probability integral transformation. We perform probability integral transformation on the logarithmic returns $\{R_{it}\}, i = 1,2$ using the marginal distributions obtained in Step 1, the transformed variables are denoted as $\{u_t\}, \{v_t\}, t = 1,2,\cdots,T$.

Step 3. The construction of variable-structure copulas. Building upon transformed variables from step 2, we construct a mixed variable-structure copula model by strategically combining the upper-tail asymmetric copula C^U, the lower-tail asymmetric copula C^L and the intermediate-region asymmetric copula C^M.

The variable-structure copula is formulated as:

$$C^T(u_t, v_t) = D_t^L C^L(u_t, v_t) + D_t^U C^U(u_t, v_t) + \\ + \left(1 - D_t^L - D_t^U\right) C^M(u_t, v_t)$$

Based on Example 1, the asymmetric copula C^* exhibits upper-tail dependence distribution structure. Therefore, we select copula C^* as the upper-tail asymmetric copula model C^U.

$$C^*(u,v) = \omega\left(\frac{1}{2}\sum_{j=1}^{2} h_{j1}\left(\omega^{-1}(u)\right) h_{j2}\left(\omega^{-1}(v)\right)\right)$$

where $\omega(t) = \dfrac{exp\left(-\alpha(1-t)^{\frac{1}{r}}\right) - e^{-\alpha}}{1-e^{-\alpha}}$, $r \in (1,+\infty)$, $\alpha \in (0,\infty)$, $h_{11}(x) = x^{\frac{3}{2}}$, $h_{12}(x) = x^2$, $h_{21}(x) = 2x - x^{\frac{3}{2}}$, $h_{22}(x) = 2x - x^2$.

For financial time series, we are most concerned with the characteristics of their tail distributions. Moreover, as analyzed in Section 3.3, the copula C^* can effectively capture the dependence relationship between SHCI and SZCI returns. Therefore, we employ the asymmetric copula C^* to model the dependence structure in both the upper tail and the middle region. The variable-structure copula model can be simplified to the following form:

$$C^T(u_t, v_t) = \left(1 - D_t^L\right) C^U(u_t, v_t) + D_t^L C^L(u_t, v_t)$$

The selection of the variable-structure copula model can be further supported by the frequency histogram of the sample data, as shown in Figs. 3 and 4.

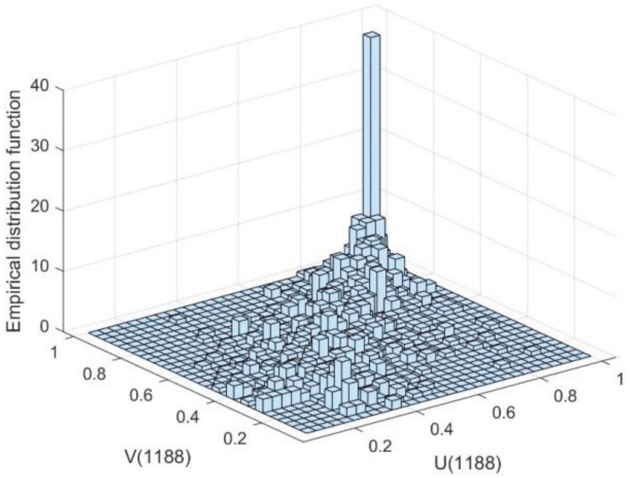

Figure 3 The frequency histogram of upper tail data

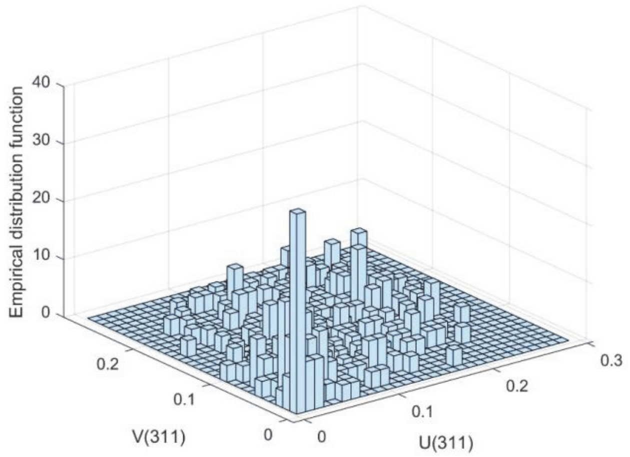

Figure 4 The frequency histogram of the lower tail data

Based on our predetermined threshold of $\varepsilon = 0.25$, the sample data contains 311 observations that satisfy the joint condition $\{u_t \le 0.25, v_t \le 0.25\}$, with the remaining 1188 observations falling outside this range. The paper presents frequency histograms for the complete sample data as well as these two subsets, as shown in Figs. 3 and 4, respectively. The frequency histograms reveal distinct tail dependence patterns: the 311 subsample points exhibit lower-tail dependence, while the 1,188 subsample points demonstrate upper-tail dependence. When integrated, these complementary histograms provide a complete characterization of the distributional properties across the entire sample. The frequency histogram analysis validates the appropriateness of the selected threshold, which is consequently employed for all subsequent piecewise modeling in our empirical analyses.

For the lower-tail asymmetric copula C^L, this paper selects two copula models: (i) the copula $\hat{C}^*$, employed to demonstrate the practical implementation of tail dependence transformation via survival copulas; (ii) the copula $C^\#$ introduced in Example 2, utilized to illustrate the effectiveness of the novel asymmetric copula construction method proposed in Theorem 3 .

The variable-structure copula models are constructed based on the selected asymmetric copulas for upper and lower tails.

Step 4. Parameter estimation and comparative dependence analysis. This paper performs parameter estimation for the asymmetric copula models in each segmented region using the corresponding subsample data. Subsequently, we conduct a comparative dependence analysis between SHCI and SZCI returns through variable-structure copula models constructed in step 3.

5 COMPARATIVE DEPENDENCE ANALYSIS BASED ON VARIABLE-STRUCTURE COPULA MODELS

In the preceding dependence analysis, the paper employs both the Gumbel copula and the asymmetric copula C^*; both exhibit upper-tail dependence and lower-tail independence - to model the dependence structure between SHCI and SZCI returns. However, as evidenced by the frequency histograms of the sample data, these indices exhibit non-independent behavior in the lower-tail region. Consequently, this section implements a piecewise modeling framework, constructing a variable-structure copula to conduct a systematic comparative dependence analysis between the two indices.

According to the foregoing analysis, the variable-structure copula is as follows:

$$C^T\left(u_t,v_t\right)=\left(1-D_t^L\right)C^U\left(u_t,v_t\right)+D_t^LC^L\left(u_t,v_t\right)$$

In the above equation, the function C^U denotes the upper-tail asymmetric copula. We select the copula C^* as C^U. The function C^L denotes the lower -tail asymmetric copula. We respectively select two model forms ($C^\#$ and $\hat{C}^*$) as copula C^L for comparative analysis. The specific analysis process is as follows:

Step 1. Construct variable-structure copula Model 1 by strategically combining the copula C^* for upper-tail dependence and copula $C^\#$ for lower-tail dependence modeling. The complete mathematical formulation of Model 1 is expressed as:

$$C^T\left(u_t,v_t\right)=\left(1-D_t^L\right)C^*\left(u_t,v_t\right)+D_t^LC^\#\left(u_t,v_t\right)$$

Step 2. Construct variable-structure copula Model 2 by combining the copula C^* for upper-tail dependence and copula $\hat{C}^*$ for lower-tail dependence modeling. The complete mathematical specification of Model 2 is defined as:

$$C^T\left(u_t,v_t\right)=\left(1-D_t^L\right)C^*\left(u_t,v_t\right)+D_t^L\hat{C}^*\left(u_t,v_t\right)$$

Step 3. The parameters of copulas C^U, C^L and C^M in Model 1 and Model 2 are estimated using partitioned sample data.

We estimate model parameters by systematically examining the relationship between tail dependence coefficients and copula parameters, with key results summarized in Tab. 5.

Table 5 Tail dependence coefficients and generators of copulas

Copula	Generator $\omega(t)$	Dependence Coefficient
Gumbel	$\omega(t)=(-ln\,t)^{\frac{1}{\alpha}}$, $\alpha\in(0,1]$	$\lambda_U=2-2^{\alpha}$, $\lambda_L=0$
Clayton	$\omega(t)=\frac{1}{r}\left(t^{-r}-1\right)$ $r\in[-1,0)\cup(0,+\infty)$	$\lambda_U=0$ $\lambda_L=2^{-\frac{1}{r}}$
Asymmetric Copula	$\omega(t)=\frac{e^{-\alpha(1-t)^{\frac{1}{r}}}-e^{-\alpha}}{1-e^{-\alpha}}$ $r\in(1,+\infty),\alpha\in(0,+\infty)$	$\lambda_U=2-2^{\frac{1}{r}}$ $\lambda_L=0$
Asymmetric Copula	$\omega(t)=\left(\frac{1}{ln(t+e)}\right)^{\theta}$ $\theta\in(0,+\infty)$	λ_U non-existent, $\lambda_L=\left(\frac{2}{2+\alpha_1-\alpha_2}\right)^{\frac{1}{\theta}}$ $\alpha_1,\alpha_2\in[0,2]$

According to the analysis in Section 3, when $a=7$ and $r=3.5$, the asymmetric copula C^* effectively captures the dependence structure between SHCI and SZCI returns. Therefore, we retain the asymmetric copula C^* with $a=7$ and $r=3.5$ to model the dependence in the upper tail and intermediate regions. For the lower tail, we estimate the parameters of the lower-tail copula C^L using the 311 sample observations from the lower-tail region. The specific parameter estimation procedure is as follows:

Step 1. First, the Kendall's τ of the sample data is computed.

Step 2. Estimation of the Clayton copula parameter. Given the one-to-one relationship between Kendall's τ and the parameter a of the Clayton copula, the parameter a can be directly derived from Kendall's τ. The estimated value $a=1.7154$.

Step 3. Estimation of the lower-tail dependence coefficient. Using the known relationship between the Clayton copula's parameter and its lower-tail dependence coefficient $\lambda_L=2^{-\frac{1}{a}}$, compute the estimated value $\lambda_L=0.6676$.

Step 4. Parameter estimation of the asymmetric copula $\hat{C}^*$. The lower-tail dependence coefficient of copula $\hat{C}^*$ coincides with the upper-tail dependence coefficient of copula C^* in functional form, both yielding $2-2^{\frac{1}{r}}$. Given that $\hat{C}^*$ and the Clayton copula exhibit identical tail dependence properties, we can estimate the parameters of copula $\hat{C}^*$ using the lower-tail dependence coefficient from the Clayton copula. Specifically, letting $2-2^{\frac{1}{r}}=2^{-\frac{1}{a}}$, we obtain its parameter estimation $r=2.4150$.

Following the same methodology, we obtain the parameter estimate for copula $C^\#$. As established in the preceding analysis, the lower-tail dependence coefficient of copula $C^{\#}$ satisfies $\lambda_L = \left(\frac{2}{2+\alpha_1-\alpha_2}\right)^{\frac{1}{\theta}}$. By setting $\left(\frac{2}{2+\alpha_1-\alpha_2}\right)^{\frac{1}{\theta}} = 0.6676$, we derive $\theta = \frac{ln\frac{2}{2+\alpha_1-\alpha_2}}{ln\,0.6676}$. This demonstrates that the parameter θ is functionally dependent on both α_1 and α_2.

Step 5. Comparative analysis of the goodness-of-fit for Model 1 and Model 2. We compute the squared Euclidean distances between the empirical copula and each variable-structure copula model (Model 1 and Model 2), followed by a comparative goodness-of-fit analysis between the two models.

The calculation results are shown in Tab. 6. According to Tab. 6, the squared Euclidean distance between copula C^* and the empirical copula is $d_1^2 = 0.2359$ in both the upper-tail and intermediate regions. For the lower-tail region, when selecting the asymmetric copula $C^{\#}$ with parameters $\alpha_1 = 1.4$ and $\alpha_2 = 0.5$, the squared Euclidean distance between copula $C^{\#}$ and the empirical copula is $d_2^2 = 0.04383$, resulting in a total squared Euclidean distance of $d_{12}^2 = d_1^2 + d_2^2 = 0.27973$ for Model 1 across the entire sample region. Alternatively, when choosing the asymmetric copula $\hat{C}^*$ with parameters $a = 8$ and $r = 2.415$ for the lower-tail region, the Euclidean distance between copula $\hat{C}^*$ and the empirical copula is $d_3^2 = 0.02526$,yielding a total squared Euclidean distance of $d_{13}^2 = d_1^2 + d_3^2 = 0.26116$ for Model 2 over the complete sample range.

Table 6 Parameter estimates and Euclidean distances

Copula	Parameter Estimates	Euclidean distance
Gumbel	$\alpha = 0.2857$	$d_G^2 = 0.2889$
Asymmetric Copula	$\alpha = 7, r = 3.5$	$d_{C^*}^2 = 0.2859$
The Variable-Structure Copula ModelI	Copula $C^*: \alpha = 7, r = 3.5$ Copula $C^{\#}$: $\alpha_1 = 1.4, \alpha_2 = 0.5$ $\theta = 0.9196$	$d_1^2 = 0.2359$ $d_2^2 = 0.04383$ $d_{12}^2 = 0.27973$
The Variable-Structure Copula Model 2	Copula $C^*: \alpha = 7, r = 3.5$ Copula $\hat{C}^*: \alpha = 8, r = 2.415$	$d_1^2 = 0.2359$ $d_3^2 = 0.02526$ $d_{13}^2 = 0.26116$

It should be noted that in calculating the Euclidean distance between copula $\hat{C}^*$ and the empirical copula using lower-tail region data, two outliers are excluded, resulting in 309 valid observations. To ensure consistent sample sizes ($N = 1499$) across both Model 1 and Model 2 and the complete dataset, we selected 1190 sample data points for calculating the Euclidean distances corresponding to the upper-tail and intermediate regions. Thus, in the piecewise modeling of both Model 1 and Model 2, a consistent total of 1499 sample data points is employed-identical to the data volume used in the prior correlation analyses with the Gumbel copula and the single-structure asymmetric copula C^*. This methodological treatment facilitates comparative analysis of dependence structures across different models.

As demonstrated in Tab. 6, when employing the single-structure asymmetric copula C^* with parameters $\alpha = 7$ and $r = 3.5$ to characterize the dependence structure between SHCI and SZCI returns, the minimum squared Euclidean distance between copula C^* and the empirical copula is $d_G^2 = 0.2889$, which is greater than the corresponding distances for both variable-structure copula Models 1 and 2, indicating that the goodness-of-fit of the variable-structure copula models is superior to that of the single-structure asymmetric copula.

Comprehensive analysis demonstrates that the variable-structure copula models provide better goodness-of-fit ($d_{12}^2 = 0.27973$, $d_{13}^2 = 0.26116$) than the single-structure asymmetric copula ($d_{C^*}^2 = 0.2859$), which itself outperforms the symmetric Gumbel copula ($d_G^2 = 0.2889$). These results validate both the effectiveness of the novel asymmetric copula construction method and the superiority of the variable-structure copula models derived from asymmetric copulas in dependence analysis.

6 CONCLUSIONS

This study proposes a novel asymmetric copula construction method by extending Archimedean copulas from a geometric mean perspective. To validate the practical applicability of this innovative approach, we conduct empirical research using the SHCI and SZCI as study subjects. Through comparative analysis with conventional copula models, our findings demonstrate that the variable-structure copulas exhibit superior goodness-of-fit compared to the single-structure asymmetric copulas and the symmetric Gumbel copula. This conclusion validates the effectiveness of the novel construction method proposed in Theorem 3. Furthermore, this paper presents a theoretical exploration of tail copula function transformations based on survival copulas, thereby substantially expanding the potential applications of asymmetric copulas in empirical modeling.

With the gradual advances in asymmetric copula theory, copula correlation coefficients - as a regression-based measure for asymmetric dependence between variables - have been increasingly employed in dependence analysis. The development of asymmetric copulas provides a theoretical foundation for characterizing asymmetric dependence structures using copula correlation coefficients. Nevertheless, their complex functional forms pose substantial computational challenges in deriving copula correlation coefficients. Therefore, developing more effective methodologies to analyze asymmetric dependence structures using copula correlation coefficients constitutes a key focus of our future research.

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Appendix

Theorem 2:
Let

$$C(u_1,u_2)=\omega(\prod_{j=1}^{2}(h_{j1}\left(\omega^{-1}(u_1)\right)+h_{j2}\left(\omega^{-1}(u_2)\right)))^{\frac{1}{2}},$$

where $\omega^{-1}(t)$ is the decreasing function from $[0,1]$ to $[0,\infty)$, $\omega'>0$, h_{jk} is the increasing function from $[0,\infty)$ to $[0,\infty)$, then the function C is a copula.

Proof: First, $C(0,u_2)=C(u_1,0)=0$, and $C(u_1,1)=\omega(\prod_{j=1}^{2}h_{j2}\left(\omega^{-1}(u_1)\right))^{\frac{1}{2}}$, according to known conditions $\prod_{j=1}^{2}h_{jk}\left(\omega^{-1}(u_k)\right)=\left(\omega^{-1}(u_k)\right)^2$, so, $C(u_1,1)=u_1$, similarly, $C(1,u_2)=u_2$;

Second, verify $\frac{\partial C(u_1,u_2)}{\partial u_1\partial u_2}\ge 0$. By calculation,

$$\frac{\partial C(u_1,u_2)}{\partial u_1\partial u_2}=\frac{1}{4}\omega''\left(\prod_{\mathrm{j}=1}^{2}\left(h_{\mathrm{j}1}\left(\omega^{-1}(u_1)\right)+h_{\mathrm{j}2}\left(\omega^{-1}(u_2)\right)\right)\right)^{\frac{1}{2}}.$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}(u_1)\right)+h_{j2}\left(\omega^{-1}(u_2)\right)\right)\right)^{-1}$$

$$\left(h'_{11}(h_{21}+h_{22})+h'_{21}(h_{11}+h_{12})\right)$$

$$\cdot\left(h'_{12}(h_{21}+h_{22})+h'_{22}(h_{11}+h_{12})\right)\cdot$$

$$\frac{1}{\omega'\left(\omega^{-1}(u_1)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}(u_2)\right)}$$

$$-\frac{1}{4}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{3}{2}}$$

$$\cdot\left(h'_{11}\left(h_{21}+h_{22}\right)+h'_{21}\left(h_{11}+h_{12}\right)\right)$$

$$(h'_{12}\left(h_{21}+h_{22}\right)+h'_{22}\left(h_{11}+h_{12}\right)$$

$$\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_1\right)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_2\right)\right)}$$

$$+\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h'_{11}h'_{22}+h'_{21}h'_{12}\right)\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_1\right)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_2\right)\right)}$$

where

$$-\frac{1}{4}\omega'(\prod_{j=1}^{2}(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)))^{\frac{1}{2}}$$

$$(\prod_{j=1}^{2}(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)))^{-\frac{3}{2}}$$

$$\cdot\left(h'_{11}\left(h_{21}+h_{22}\right)+h'_{21}\left(h_{11}+h_{12}\right)\right)$$

$$(h'_{12}\left(h_{21}+h_{22}\right)+h'_{22}\left(h_{11}+h_{12}\right)$$

$$\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_1\right)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_2\right)\right)}$$

$$=-\frac{1}{4}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h'_{11}h'_{12}\cdot\frac{h_{21}+h_{22}}{h_{11}+h_{12}}+h'_{21}h'_{12}+h'_{11}h'_{22}+h'_{21}h'_{22}\cdot\frac{h_{11}+h_{12}}{h_{21}+h_{22}}\right)$$

In order to verify $\frac{\partial C\left(u_1,u_2\right)}{\partial u_1\partial u_2}\geq 0$, we just have to verify $h'_{11}h'_{12}\cdot\frac{h_{21}+h_{22}}{h_{11}+h_{12}}+h'_{21}h'_{22}\cdot\frac{h_{11}+h_{12}}{h_{21}+h_{22}}>h'_{21}h'_{12}+h'_{11}h'_{22}$

Namely

$$h'_{11}h'_{12}\cdot\frac{\left(h_{21}+h_{22}\right)^2}{\left(h_{11}+h_{12}\right)\left(h_{21}+h_{22}\right)}+h'_{21}h'_{22}\cdot\frac{\left(h_{11}+h_{12}\right)^2}{\left(h_{11}+h_{12}\right)\left(h_{21}+h_{22}\right)}>h'_{21}h'_{12}+h'_{11}h'_{22}$$

That is just to verify

$$h'_{11}h'_{12}\left(h_{21}+h_{22}\right)^2+h'_{21}h'_{22}\left(h_{11}+h_{12}\right)^2$$

$$>\left(h'_{11}h'_{12}\left(h_{21}+h_{22}\right)^{2'}{}_{21}h'_{12}+h'_{11}h'_{22}\right)\left(h_{11}+h_{12}\right)\left(h_{21}+h_{22}\right)$$

Namely

$$h'_{11}h'_{12}\cdot\left(h_{21}+h_{22}\right)^2-\left(h'_{11}h'_{22}\right)\left(h_{11}+h_{12}\right)\left(h_{21}+h_{22}\right)$$

$$+h'_{21}h'_{22}\cdot\left(h_{11}+h_{12}\right)^2-h'_{21}h'_{12}\left(h_{11}+h_{12}\right)\left(h_{21}+h_{22}\right)>0$$

$$\mathrm{h}'_{11}\left(\mathrm{h}_{21}+\mathrm{h}_{22}\right)\left(\mathrm{h}'_{12}\left(h_{21}+h_{22}\right)-h'_{22}\left(h_{11}+h_{12}\right)\right)+\left(h_{21}+h_{22}\right)$$

$$+h'_{21}\left(h_{11}+h_{12}\right)\left(h'_{22}\left(h_{11}+h_{12}\right)-h'_{12}\left(h_{21}+h_{22}\right)\right)>0$$

In order to get the above result, need to verify,

$$\left(h'_{12}\left(h_{21}+h_{22}\right)-h'_{22}\left(h_{11}+h_{12}\right)\right)$$

$$\left(h'_{11}\left(h_{21}+h_{22}\right)-h'_{21}\left(h_{11}+h_{12}\right)\right)>0$$

Because $\frac{h'_{12}}{h'_{22}}$ and $\frac{h_{11}+h_{12}}{h_{21}+h_{22}}$, $\frac{h'_{11}}{h'_{21}}$ and $\frac{h_{11}+h_{12}}{h_{21}+h_{21}}$ have the same size relationship. So, the above equation is greater than 0.

It can be seen that,

$$h'_{11}h'_{12}\cdot\frac{h_{21}+h_{22}}{h_{11}+h_{12}}+h'_{21}h'_{12}+h'_{11}h'_{22}+h'_{21}h'_{22}\cdot\frac{h_{11}+h_{12}}{h_{21}+h_{22}}>$$

$$2h'_{21}h'_{12}+2h'_{11}h'_{22}$$

So

$$-\frac{1}{4}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{3}{2}}$$

$$\cdot\left(h'_{11}h'_{12}\cdot\frac{h_{21}+h_{22}}{h_{11}+h_{12}}+h'_{21}h'_{12}+h'_{11}h'_{22}+h'_{21}h'_{22}\cdot\frac{h_{11}+h_{12}}{h_{21}+h_{22}}\right)$$

$$\geq-\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h'_{21}h'_{12}+h'_{11}h'_{22}\right)$$

and

$$-\frac{1}{4}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{3}{2}}$$

$$\cdot\left(h'_{11}\left(h_{21}+h_{22}\right)+h'_{21}\left(h_{11}+h_{12}\right)\right)$$

$$\left(h'_{12}\left(h_{21}+h_{22}\right)+h'_{22}\left(h_{11}+h_{12}\right)\right)$$

$$\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_1\right)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_2\right)\right)}$$

$$=-\frac{1}{4}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{11}'h_{12}'\cdot\frac{h_{21}+h_{22}}{h_{11}+h_{12}}+h_{21}'h_{12}'+h_{11}'h_{22}'+h_{21}'h_{22}'\cdot\frac{h_{11}+h_{12}}{h_{21}+h_{22}}\right)$$

$$>-\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{21}'h_{12}'+h_{11}'h_{22}'\right)$$

$$>-\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{11}'h_{22}'+h_{21}'h_{12}'\right)\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_1\right)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_2\right)\right)}$$

So, the sum of the above two formulas is that

$$-\frac{1}{4}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{11}'h_{12}'\cdot\frac{h_{21}+h_{22}}{h_{11}+h_{12}}+h_{21}'h_{12}'+h_{11}'h_{22}'+h_{21}'h_{22}'\cdot\frac{h_{11}+h_{12}}{h_{21}+h_{22}}\right)$$

$$+\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{11}'h_{22}'+h_{21}'h_{12}'\right)\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_1\right)\right)}\cdot\frac{1}{\omega'\left(\omega^{-1}\left(u_2\right)\right)}$$

$$>-\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{21}'h_{12}'+h_{11}'h_{22}'\right)$$

$$+\frac{1}{2}\omega'\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

$$\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{-\frac{1}{2}}$$

$$\cdot\left(h_{21}'h_{12}'+h_{11}'h_{22}'\right)=0$$

Then

$$C\left(u_1,u_2\right)=\omega\left(\prod_{j=1}^{2}\left(h_{j1}\left(\omega^{-1}\left(u_1\right)\right)+h_{j2}\left(\omega^{-1}\left(u_2\right)\right)\right)\right)^{\frac{1}{2}}$$

satisfies the condition of copula, it is a copula function.