THE VALENCE APPROXIMATION FOR FERMION-ANTIFERMION BOUND STATES AND DISCRETIZED LIGHT-CONE QUANTIZATION IN (3+1) DIMENSIONS*

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We study the feasibility of extending the method of discretized light-cone quantization (DLCQ) from (1+1)-dimensional to (3+1)-dimensional theories. We restrict ourselves to the maximally truncated Fock space of only one valence fermion and one valence antifermion of an $SU(N_c)$ gauge theory. We obtain the spectrum and wave functions of bound states.

*Presented at the Meeting "Snopovi i čestice I", Haludovo, Malinska, Yugoslavia, 27-29 April 1989.

1. Introduction

Discretized light-cone quantization (DLCQ) is a method based on the diagonalization of light-cone Hamiltonians, i.e., the Hamiltonians of quantum field theories formulated in light-cone coordinates. This is a nonperturbative method which has been successfully applied to bound states in (1+1)-dimensional theories, namely, to $\bar{\psi}\phi\psi$ theory[1)], ϕ^4 and ϕ^3 scalar theories[2)], QED_{1+1} encompassing the Schwinger model[3)] and QCD_{1+1}[4)], as well as to scattering in QED_{1+1}[5)].

We want to explore the feasibility of extending DLCQ to (3+1)-dimensional gauge theories[6)]. Consider a fermion and an antifermion interacting through an $SU(N_c)$ gauge theory in (3+1) dimensions (an electron and a positron for QED, or a quark and an antiquark for QCD). For simplicity, we restrict the Fock space to the valence $f\bar{f}$ space[6)] of only one fermion and one antifermion. Therefore, we again get a toy model in which realistic, phenomenology-related calculations are not possible, but they are not the goal of this paper. The goal is to explore the feasibility of formulating DLCQ in (3+1) dimensions and to develop appropriate techniques for studying the spectra and wave functions of $f\bar{f}$ bound states resulting from specific interactions within this model on the light cone.

2. Essentials of $SU(N_c)$ in (3+1) dimensions in the light-cone formulation

We use the light-cone coordinates $x^\mu = (x^+,\underline{x}) \equiv (x^+,x^-,x^1,x^2) \equiv (x^0+x^3,x^0-x^3,x^1,x^2)$. Similarly, $V^\pm \equiv V^0 \pm V^3$ for any 4-vector V^μ. The nonvanishing components of the metric tensor are $g_{+-} = g_{-+} = \frac{1}{2}$ and $g_{11} = g_{22} = -1$.

$$\mathcal{L} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m_F)\psi \tag{1}$$

is the usual $SU(N_c)$ Lagrangian density, where $iD_\mu = i\partial_\mu - gA^a$ and $G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - gf^{abc}A_b^\mu A_c^\nu$. The fermion field ψ has N_c color components ψ_c. It is convenient to decompose it by use of the projectors $\Lambda^{(\pm)} \equiv \frac{1}{4}\gamma^\mp\gamma^\pm$ like this:

$$\psi = \psi_+ + \psi_- \equiv \Lambda^{(+)}\psi + \Lambda^{(-)}\psi \ . \tag{2}$$

We shall work in the axial gauge $A_a^+ \equiv A_a^0 + A_a^3 = 0$, whereby it turns out that only A_a^i (i=1,2) and ψ_+ satisfy the dynamical equations of motion (involving the LC time derivative $\partial_+ = \frac{1}{2}\partial^-$) and are therefore independent degrees of freedom. In contrast to that, ψ_- and A_a^- are dependent fields, being given at every LC time x^+ by ψ_+ and A_μ^i through constraint equations which do not involve time derivatives. (For details, see Ref. 6.) Only the independent fields ψ_+ and A_μ^i are subject to quantization.

3. Discretization

We expand the independent fields ψ_+ and A^i_μ in the creation and annihilation operators in the plane-wave basis. We discretize the single-particle momenta k^μ by imposing on the plane waves the periodic boundary conditions with box lengths $(2L,\ 2L_\perp)$. (For a formulation with antiperiodic boundary conditions, see Ref. 4.) Then

$$k_- = \frac{1}{2}\, k^+ = \frac{\pi}{L}\, n\ , \qquad n=1,2,\ldots,N_{||}, \tag{3}$$

$$k_i = -k^i = \frac{\pi}{L_\perp}\, n_i, \qquad n_i=0,\pm 1,\pm 2,\ldots,\pm N_\perp. \tag{4}$$

Note that $k^+= 2k_- > 0$ to ensure the positive light-cone single-particle energy $k^- = (m_p^2+\vec{k}_\perp^2)/k^+$. Obviously, for the total longitudinal momentum, $P^+ \equiv \sum_p k_p^+$, it is the same: $P^+ > 0$.

$N_{||}$ and $N_\perp$ are the parallel and transverse cutoffs, respectively. The discretization destroys Lorentz invariance, which must be recovered in the continuum limit $N_{||}, L, N_\perp, L_\perp \rightarrow \infty$.

4. Formulating the model

We truncate the Fock space to only one valence fermion and one antifermion, i.e., we have no dynamical gluons A^i_a (but just A^-_a from the constraint equations) and no $f\bar{f}$ pairs from the sea. The single fermion state is $b^+_q|0\rangle$, where $q = \{c,s,n,\vec{n}_\perp\}$ are the quantum numbers of color, spin, and (dimensionless and discretized) longitudinal and transverse momenta, respectively [see formulas (3) and (4)]. What are the quantum numbers of composite fermion-antifermion states? The total light-cone energy P^- is a very complicated operator. However, the space components of the total momentum, P^+, P^1, and P^2, are simple, and since they do not contain interaction, they are purely kinematical and are simply the sum of the single-particle momenta k_p^+, k_p^1, k_p^2 of all the constituents p. In fact, in the plane-wave basis, P^+, P^1, and P^2 are diagonal. We choose the total momentum eigenvalues $P^1 = P^2 = 0$ (i.e., parton transverse momenta are equal and opposite) and $P^+ = \frac{2\pi}{L} K > 0$, where the integer K is the so-called "resolution" or the total dimensionless longitudinal momentum.

Remember that single-particle (parton) longitudinal momenta are positive. This means that there are only K-1 ways how K can be partitioned among <u>massive</u> f and $\bar{f}$: $n = 1,2,\ldots,(K-1)$, i.e., $(K-1)$ becomes $N_{||}$, the parallel cutoff. Therefore, there are $N_D = (K-1)(2N_\perp+1)^2$ eigenstates with eigenvalues $P^+ = \frac{2\pi}{L} K$, $P^1 = 0$, $P^2 = 0$, and the total helicity Σ. The eigenstates can be written as

$$|n,\vec{n}_\perp\rangle \equiv \frac{1}{\sqrt{N_c}} \sum_{s,c=1}^{N_c} \sum_{q'} b_q^\dagger d_q^\dagger |0\rangle G(n',\vec{n}_\perp') \times$$
$$\times \langle \tfrac{1}{2}\, s, \tfrac{1}{2}\, s' | \Sigma M\rangle \delta_{c'}^{c} \delta_{n'}^{K-n'} \delta_{-\vec{n}_\perp'}^{\vec{n}_\perp} , \qquad (5)$$

where $G(n',\vec{n}_\perp')$ is the regulating function. For the time being, we can take the naive, noncovariant choice ("square cutoff")

$$G(n,\vec{n}_\perp) = \Theta(N_{||}-n)\Theta(N_\perp-|n_1|)\Theta(N_\perp-|n_2|), \quad (N_{||}=K-1), \tag{6}$$

where the cutoffs in the parallel and perpendicular directions are not related to each other. However, we will later introduce a covariant cutoff, when we address the continuum limit where Lorentz invariance must be restored.

Our goal is to find the eigenfunctions and the eigenvalues of the invariant mass operator $M^2 = P_\mu P^\mu$ of our $f\bar{f}$ system. Since $P^1 = P^2 = 0$, $M^2 = P^+P^- = \frac{2\pi}{L} KP^- \equiv H$, i.e., since P^+ is already diagonal, the diagonalization of the invariant mass operator amounts to the diagonalization of the light-cone energy P^-. We therefore identify M^2 with the light-cone Hamiltonian H, which is to be diagonalized. P^-, and consequently H, is very complicated, and also becomes very long when we insert plane-wave expansions for the fields. However, working in the truncated $f\bar{f}$ valence space simplifies it considerably, because the only parts that contribute are[6)]

$$H = \sum_q (b_q^\dagger b_q + d_q^\dagger d_q)\left(\frac{m_F^2+\vec{k}_\perp^2}{n/K} + \tilde{g}X_n\right) +$$

$$+ \frac{g^2}{4\pi}\frac{K}{L_\perp^2} \sum_{q_1\ldots q_4} b_{q_1}^\dagger d_{q_2}^\dagger b_{q_3} d_{q_4} \delta_{n_3+n_4}^{n_1+n_2} \delta_{\vec{n}_{\perp 3}+\vec{n}_{\perp 4}}^{\vec{n}_{\perp 1}+\vec{n}_{\perp 2}} \times$$

$$\times \; (T^a_{c_1c_3}T^a_{c_4c_2}[n_1-n_3|]\delta^{s_1}_{s_3}\delta^{s_2}_{s_4} - T^a_{c_1c_2}T^a_{c_4c_3}[n_1+n_2|]\delta^{s_1}_{-s_2}\delta^{s_3}_{-s_4}) \, . \tag{7}$$

The symbols $X_n \equiv \xi_n + \lambda_n$ are called "self-induced inertias":

$$\xi_n \equiv \frac{K}{2}\frac{\pi}{L_\perp^2}\sum_m \sum_{m_1m_2} G(m,\vec{m}_\perp)(\{n-m|\}+\{n+m|\}), \tag{8}$$

$$\lambda_n \equiv \frac{K}{2}\frac{\pi}{L_\perp^2}\sum_m \sum_{m_1m_2} G(m,\vec{m}_\perp)([n-m|]-[n+m|]), \tag{9}$$

where

$$\{n|\} = \begin{cases} 1/n & n\neq 0 \\ 0 & n=0 \end{cases} , \tag{10}$$

$$[n|] = \begin{cases} 1/n^2 & n\neq 0 \\ 0 & n=0 \end{cases} , \tag{11}$$

and

$$\tilde{g}^2 = C_F \frac{g^2}{4\pi} , \quad C_F = \frac{(N_c^2 - 1)}{2N_c} \, . \tag{12}$$

We note that in spite of the absence of dynamical, transverse gauge bosons, we still have nontrivial interactions[6)] in our truncated Hamiltonian, because of the constraint equations for the dependent A_a^- and ψ_- fields, and because of the contractions of the boson fields in the pieces of the Hamiltonian where they appear bilinearly. Therefore, what we consider here is the valence model of quarkonium with instantaneous, contact interactions specific for the light-cone formulation.

5. Diagonalization

We want to solve the eigenvalue problem

$$H|\Psi_i\rangle = M_i^2|\Psi_i\rangle, \tag{13}$$

where M_i^2 is the i-th eigenvalue and $|\Psi_i\rangle$ the i-th eigenstate of the Hamiltonian H given by (7). In the basis of states (5), the eigenvalue problem becomes

$$M_i^2\Psi_i(n,\vec{n}_\perp) = \sum_{n',\vec{n}'_\perp} \langle n,\vec{n}_\perp|H|n',\vec{n}'_\perp\rangle G(n',\vec{n}'_\perp)\Psi_i(n',\vec{n}'_\perp), \tag{14}$$

where

$$\Psi_i(n,\vec{n}'_\perp) \equiv \langle n,\vec{n}_\perp|\Psi_i\rangle . \tag{15}$$

It is convenient to divide the matrix element of H in the diagonal part and the off-diagonal part:

$$\langle n,\vec{n}_\perp|H_{DIA}|n,\vec{n}_\perp\rangle = \tilde{g}^2(X_n+X_{K-n}) + K^2\frac{(m_F^2+\vec{k}_\perp^2)}{n(K-n)} \equiv a_{\underline{n}} \equiv a_n(\vec{n}_\perp^2), \tag{16}$$

$$\langle n,\vec{n}_\perp|H_{OFF}|n',\vec{n}'_\perp\rangle = -[|n-n'|]\frac{g^2}{\pi}\frac{K}{(2L_\perp)^2}C_F \equiv c_{\underline{nn}'} \equiv c_{nn'}, \tag{17}$$

i.e., the off-diagonal elements $c_{nn'}$ do not depend on the transverse quantum numbers. Then

$$M_i^2\Psi_i(n,\vec{n}_\perp) = a_{\underline{n}}\Psi_i(n,\vec{n}_\perp) + \sum_{n'} c_{nn'}\sum_{\vec{n}'_\perp} G(n',\vec{n}'_\perp)\Psi_i(n',\vec{n}'_\perp), \tag{18}$$

so that the matrix to be diagonalized consists of the elements $a_n(\vec{n}_\perp)$ on the diagonal, and of $(2N_\perp+1)^2$-dimensional <u>blocks</u> $c_{nn'}P$. Here P is the so-called <u>pairing matrix</u> whose elements are all unity:

$$P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot\ \cdot \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} . \qquad (19)$$

We further simplify the matrix by working in the strong-coupling limit, $\tilde{g}^2 >> \frac{3}{4\pi} \sim \frac{1}{4}$, where the term with $\vec{k}_\perp^2$ can be neglected, so that the dependence on $\vec{n}_\perp$ is eliminated:

$$a_{\underline{n}} = a_n(\vec{n}_\perp) \rightarrow a_n(0) \equiv a_n \, . \qquad (20)$$

The Hamiltonian matrix element is then

$$\langle n,\vec{n}_\perp | H | n',\vec{n}_\perp' \rangle = c_{nn'} + a_n \delta^n_{n'} \, \delta^{n_1}_{n_1'} \, \delta^{n_2}_{n_2'} \, . \qquad (21)$$

This is a simple matrix, but also very huge. For instance, even for quite low cutoffs, $K = 32$, $N_\perp = 10$, the dimension $N_D = (K-1)(2N_\perp+1)^2$ is already 13671, so that it is obvious that the direct numerical diagonalization would be hopeless. Fortunately, it is possible to prediagonalize the problem analytically to a large extent by recoupling each $d_P = \sum_{\vec{n}_\perp} G(n,\vec{n}_\perp) = (2N_\perp+1)^2$-dimensional n-subspace of the initial basis (5) to the basis which diagonalizes the pairing matrix P, namely, the so-called pairing basis:

$$|n;t\rangle = \sum_{\vec{n}_\perp} C_t(\vec{n}_\perp) G(n,\vec{n}_\perp) |n,\vec{n}_\perp\rangle , \quad t = 1,\ldots,d_P . \qquad (22)$$

It is well known that P has only one nonvanishing eigenvalue for $|n;1\rangle$:

$$c_{nn'} P|n;t\rangle = c_{nn'} d_P \delta_{t1} |n;1\rangle \ , \qquad (23)$$

and that the recoupling coefficients for t = 1 are all equal:

$$C_1(\vec{n}_\perp) = 1/[\sum_{\vec{n}_\perp} G(n,\vec{n}_\perp)]^{1/2} = 1/\sqrt{d_P} \ , \quad \forall \vec{n}_\perp . \qquad (24)$$

Other C_t, $t = 2,\dots,d_P$, are not even needed. It is enough to know that the eigenvalues of $|n;t\neq 1\rangle$ are 0. From the orthogonality we find that

$$\sum_{\vec{n}_\perp} G(n,\vec{n}_\perp) C_t(\vec{n}_\perp) C_{t'}(\vec{n}_\perp) = \delta_{tt'} \ , \qquad (25)$$

$$\sum_{\vec{n}_\perp} G(n,\vec{n}_\perp) C_t(\vec{n}_\perp) = \delta_{1t'} \sqrt{d_P} \ . \qquad (26)$$

We then easily compute the Hamiltonian matrix in the new basis:

$$\langle n;t|H|n';t'\rangle = \delta_{tt'}(a_n \delta_{nn'} + \delta_{1t} d_P c_{nn'}) . \qquad (27)$$

If we now regroup $|n;t\rangle$ in subspaces of the same t, the prediagonalization is completed, because the eigenvalue problem in the pairing basis

$$M_i^2 \langle n;t|\Psi_i\rangle = \sum_{n't'} \langle n;t|H|n';t'\rangle\langle n';t'|\Psi_i\rangle \qquad (28)$$

falls apart in two distinct classes:

i) $\forall t \neq 1$, the trivial eigenvalue problem

$$M_i^2 \langle n;t|\Psi_i\rangle = a_n \langle n;t|\Psi_i\rangle, \quad i = K,\dots,N_D, \qquad (29)$$

$$M_i^2 = a_n = \frac{K^2 m_F^2}{n(K-n)} + \tilde{g}^2 (X_n + X_{K-n}) . \qquad (30)$$

In the continuum limit, this spectrum would become continuous. It can be shown that the minimal such eigenvalue, the so-called "band head" $\bar{M}^2_{f\bar{f}}$, appears for $n = K/2$ and that for $K>>1$

$$\bar{M}^2_{f\bar{f}} \equiv \mathrm{MIN}(a_n) = 4m_F^2 + \tilde{g}^2 \frac{4\pi}{3} Kq_\perp^2 , \tag{31}$$

where $q_\perp \equiv \pi N_\perp / L_\perp$. The minimal trivial eigenvalue is thus comparable with the invariant mass of an isolated quark, which can be shown to be (again in the limit $K>>1$)

$$M_q^2 = m_F^2 + \tilde{g}^2 \frac{2\pi}{3} Kq_\perp^2 . \tag{32}$$

So, they both diverge linearly for $K \to \infty$.

ii) For $t = 1$, we have the nontrivial eigenvalue problem

$$M_i^2 \psi_i(n) = a_n \psi_i(n) + \sum_{n'=1}^{K-1} d_P c_{nn'} \psi_i(n), \; i = 1,2,\ldots,K-1, \tag{33}$$

where $\psi_i(n) \equiv \langle n;t=1|\Psi_i\rangle$ are the "transverse coherent states" for $t = 1$. We have to diagonalize (33) numerically, but note that $i = 1,2,\ldots,K-1$, that is, we have to diagonalize numerically just a $(K-1)$-dimensional matrix which corresponds to the $t = 1$ subspace. Thus, for $K = 32$ and $N_\perp = 10$, the numerical effort is reduced from diagonalizing the $N_D = 13671$-dimensional matrix to diagonalizing the $n_K=K-1=31$-dimensional matrix. Thus, the amount of numerical effort is comparable to the one encountered in (1+1) dimensions.

By use of the definitions for X_n and by simplification of various parts of (33) as much as possible, the $(K-1)$-

dimensional problem becomes

$$\left[M^2 - \frac{K^2 m_F^2}{n(K-n)} - 4\tilde{g}^2 \frac{q_\perp^2}{\pi} \left(\frac{K}{2n} \sum_{m=1}^{n} \frac{1}{m} - \frac{K}{2(K-n)} \sum_{m=1}^{K-n} \frac{1}{m}\right)\right] \psi(n) =$$

$$= 4\tilde{g}^2 \frac{q_\perp^2}{\pi} K \sum_{\substack{m=1 \\ m \neq n}}^{K-1} \frac{\psi(n)-\psi(m)}{(n-m)^2} . \tag{34}$$

By diagonalization for increasing K, we get the logarithmic behavior for the lowest level $M_{f\bar{f}}^2$:

$$M_{f\bar{f}}^2 \sim \tilde{g}^2 q_\perp^2 \ln K, \tag{35}$$

and similarly for higher coherent levels. Therefore, we find the favorable result that $\bar{M}_{f\bar{f}}^2$ and M_q^2 diverge as $K/\ln K$ at the scale of $M_{f\bar{f}}^2$, the scale of coherent states. However, it is also obvious that we cannot simply take the continuum limit by letting $K \to \infty$, as we would also induce the divergent dependence of the coherent states on the unphysical, auxiliary parameter K. This is the consequence of the noncovariant regularization (6) we have used up to now. For the continuum limit, we must use covariant regularization.

6. The model in the covariant continuum limit

The regulating function G must be a Lorentz-invariant function of momenta and of some invariant mass Λ. We adopt the Brodsky-Lepage regularization which admits only the Fock states with invariant masses smaller than some sufficiently large cutoff scale Λ^2:

$$M^2_{Fock} = P_\mu P^\mu = -P_\perp^2 + \sum_{p=1}^{A} \frac{m_p^2 + (\vec{k}_\perp)_p^2}{x_p} < \Lambda^2 \quad . \tag{36}$$

The sum is over partons p. For two partons of mass m_p, we get

$$G(n,\vec{n}_\perp) \to G(x,\vec{k}_\perp) = \Theta(\Lambda^2 - \frac{m_p^2 + \vec{k}_\perp^2}{x(1-x)}) \quad . \tag{37}$$

Therefore, the allowed values of x and $\vec{k}_\perp^2/\Lambda^2$ are within the sphere centered at $x = 1/2$, $k_\perp = 0$:

$$(x - \tfrac{1}{2})^2 + \frac{\vec{k}_\perp^2}{\Lambda^2} < \frac{1}{4} - \frac{m_p^2}{\Lambda^2} = (\tfrac{1}{2} - \varepsilon)^2 \quad . \tag{38}$$

Here ε says how much the radius of the permitted momenta is smaller than 1/2, and it is determined by the ratio m_p^2/Λ^2 through

$$\varepsilon(1 - \varepsilon) \equiv m_p^2/\Lambda^2, \quad 0 < \varepsilon < \tfrac{1}{2} \ . \tag{39}$$

In the continuum limit, where the regulating function (37) has been used, the self-induced inertias $X_n = \xi_n + \lambda_n$ are given by

$$\xi(x) = \frac{\Lambda^2}{4} \left[\ln(\frac{1+x}{1-x}) + x \ln(\frac{1-x^2}{x^2})\right] , \tag{40}$$

$$\lambda(x) = \Lambda^2 \left[\frac{1}{2} \int_{\varepsilon}^{1-\varepsilon} dy \, \frac{y(1-y) - \varepsilon(1-\varepsilon)}{(x-y)^2} + (1-2\varepsilon) - \right.$$

$$\left. - (\tfrac{1}{2} + x) \ln \frac{1+x-\varepsilon}{x+\varepsilon}\right] \ . \tag{41}$$

The wave functions $\langle n,\vec{n}_\perp|\Psi\rangle \equiv \Psi(n,\vec{n}_\perp)$ become $\langle x,\vec{k}_\perp|\Psi\rangle \equiv$ $\equiv \Psi(x,\vec{k}_\perp)$. The eigenvalue problem now becomes

$$M^2\Psi_i(x,\vec{k}_\perp) = a(x,\vec{k}_\perp)\Psi(x,\vec{k}_\perp) +$$

$$+ \text{PV}\int dx' \int d^2k_\perp c(x,x')G(x',k_\perp')\Psi_i(x',\vec{k}_\perp'), \tag{42}$$

where, just as in the discrete case, we have decomposed the Hamiltonian matrix element into the diagonal part

$$a(x,\vec{k}_\perp) = \langle x,\vec{k}_\perp|H_{DIA}|x,\vec{k}_\perp\rangle = \frac{m_F^2+\vec{k}_\perp^2}{x(1-x)} + \tilde{g}^2[X(x)+X(1-x)] \tag{43}$$

and the off-diagonal part

$$c(x,x') = \langle x,\vec{k}_\perp|H_{OFF}|x',\vec{k}_\perp'\rangle = -\frac{\tilde{g}^2}{\pi}\frac{1}{(x-x')} . \tag{44}$$

Now we can step by step follow the procedure developed in the discrete case, and, after taking the strong-coupling limit in order to be able to neglect the $\vec{k}_\perp^2$ term, we can again prediagonalize the eigenvalue problem.

We go over to the continuous pairing basis

$$|x;t\rangle \equiv \int d^2k_\perp G(x,\vec{k}_\perp)\mathcal{C}_t(x,\vec{k}_\perp)|x,\vec{k}_\perp\rangle . \tag{45}$$

The only essential difference with respect to the discrete case is that the dependence of the range of the index t now depends nontrivially on the subspace of the longitudinal momentum x to which this index t is attached:

$$\forall x,\ t = t_x \in [0,d(x)],$$
$$d(x) = \int d^2k_\perp G(x,k_\perp) = \pi\Lambda^2[x(1-x)-\varepsilon(1-\varepsilon)]. \tag{46}$$

Analogously to the discrete pairing basis, the minimal value of t, now t = 0, labels the transverse coherent states for each x subspace:

$$\text{for } t=0, \quad \mathcal{C}_0(x,\vec{k}_\perp) = \frac{1}{\sqrt{d(x)}}, \quad \forall x, \tag{47}$$

$$\int d^2k_\perp G(x,\vec{k}_\perp)\, \mathcal{C}_t(x,\vec{k}_\perp) = \int dk_\perp k_\perp G(x,\vec{k}_\perp) \int d\phi\, \mathcal{C}_t(x,\vec{k}_\perp) = $$
$$= \delta(t)\sqrt{d(x)}\,. \tag{48}$$

The matrix element in the new basis is then

$$\langle x;t|H|x';t'\rangle = \delta(t-t')\,[a(x)\,\delta(x-x') +$$
$$+ c(x,x')\sqrt{d(x)}\sqrt{d(x')}\,\delta(t)]\,. \tag{49}$$

Therefore, we again have the prediagonalization, because the problem again falls apart in two distinct classes:

i) $t \neq 0$ gives the trivial case ("incoherent states"):

$$M^2\langle x;t|\Psi\rangle = a(x)\langle x;t|\Psi\rangle, \quad t \neq 0. \tag{50}$$

ii) t = 0 gives the eigenvalue problem for the transverse coherent states:

$$M^2\psi(x) = \left[\frac{\varepsilon(1-\varepsilon)}{x(1-x)}\Lambda^2 + \tilde{g}^2\Lambda^2 F(x,\varepsilon)\right]\psi(x) +$$
$$+ \frac{\tilde{g}^2}{\pi}\,PV\int_\varepsilon^{1-\varepsilon} dx'\,\frac{\psi(x)d(x')-\psi(x')\sqrt{d(x')}\sqrt{d(x)}}{(x-x')^2}\,, \tag{51}$$

where $\psi(x) \equiv \langle x;t=0|\Psi\rangle$. Note the cancellation of singularity at $x = x'$. $F(x,\varepsilon)$ is a slowly varying function of x and ε:

$$F(x,\varepsilon) = \alpha(x,\varepsilon) + \alpha(1-x,\varepsilon) ,$$

$$\alpha(x,\varepsilon) = 1 - 2\varepsilon + \frac{1}{4} \ln \left[\frac{1+x}{1-x} \left(\frac{x+\varepsilon}{1+x-\varepsilon}\right)^2\right] + \qquad (52)$$

$$+ \frac{x}{4} \ln \left[\frac{1-x^2}{x^2} \left(\frac{x+\varepsilon}{1+x-\varepsilon}\right)^4\right] .$$

7. Results on the spectra and the eigenfunctions

Details of solving (51) are given in Ref. 6, as well as considerations on approximate solutions and eigenvalues. Here we just give the results.

The spectrum of the eigenvalues of $\tilde{M} \equiv M/\tilde{g}\Lambda$, plotted against the parameter ε, is shown in Fig. 1. Obviously, the regime $\varepsilon \sim 0$ is the most interesting one because $\varepsilon \sim 0$ is equivalent to $\Lambda >> m_F$ and the region of x and $k_\perp$ is not very restricted, while $\varepsilon \sim 1/2$ is obviously an unphysical regime, not only because of $\Lambda \sim m_F$, but even more because $\varepsilon \sim 1/2$ constrains x and $k_\perp$ to the small neighborhood of the point $x = 1/2$ and $k_\perp = 0$. For $\varepsilon = 1/2$, only this single point is allowed and all levels therefore collapse to a single level. However, for ε not close to 1/2, we have discrete spectra, which indicates the existence of $f\bar{f}$ bound states. Being similar to QED_{1+1} spectra[3)] in the $f\bar{f}$ approximation, the discrete spectra show no clear sign of an ionization threshold, indicating that the model may be confining, although we have

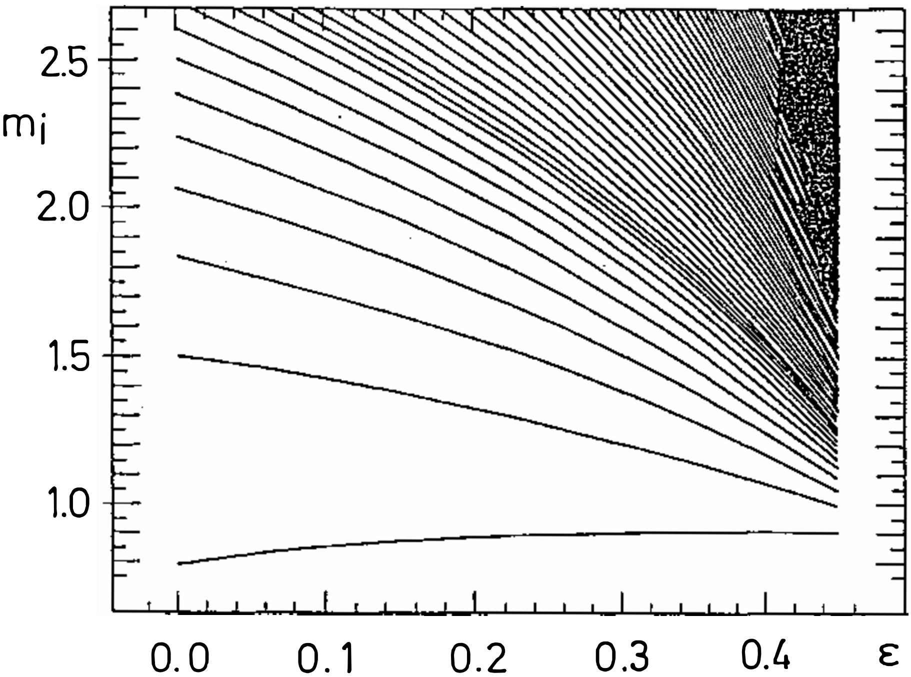

Fig. 1. The spectrum m_i (in units of $\tilde{g}\Lambda$) of the covariant integral equation (51) is plotted versus ε over the range from $\varepsilon = 0$ to $\varepsilon = 0.45$. At $\varepsilon = 1/2$ <u>all</u> levels collapse to a single eigenvalue $\sqrt{(3/4)\ \ln 3}$.

just the residual, contact interactions from the constraints on the dependent fields.

It can be shown[6)] that the transverse coherent states in the color neutral sector are in fact the only states whose masses are finite at the scale of Λ. The others (trivial, prediagonalized eigenvalues, masses of free fermions and masses of colored fermion-antifermion combinations) all diverge because they contain a divergent integral [the same one as appears in formula (41)], whose singularity can be cancelled by the off-diagonal part in the integral equation only for the coherent states, while in all other cases it remains uncompensated.

The wave functions, or, more precisely, the wave functions squared, $f_i(x) = |\psi_i(x)|^2$, are shown in Fig. 2 for several lowest states. The ground-state coherent wave function is in a good approximation proportional to $\sqrt{d(x)}$, i.e.,

$$\psi(x) \approx \mathcal{N} \sqrt{x(1-x) - \varepsilon(1-\varepsilon)} \tag{53}$$

where $\mathcal{N}$ is the normalization constant. In fact, the eigenfunctions look qualitatively like vibration modes of the string with free ends (except that the amplitudes at the edges are much larger) multiplied by the factor $\sqrt{x(1-x)-\varepsilon(1-\varepsilon)}$. This means that as we go up in the mass spectrum, the excited wave functions are more and more depleted for $x \sim 1/2$, and

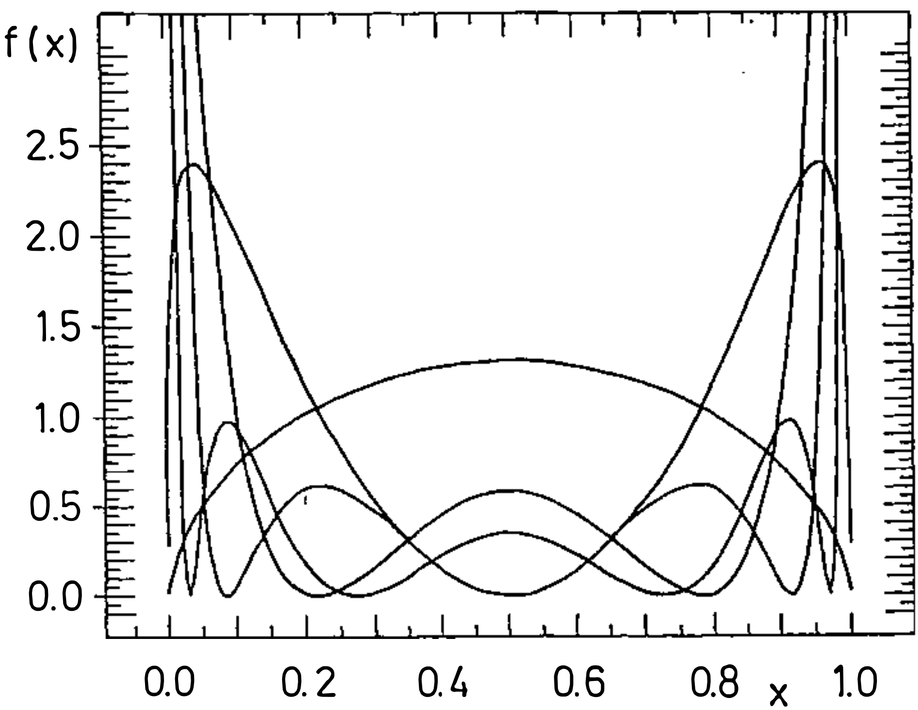

Fig. 2. The structure functions $f(x) \equiv |\psi(x)|^2$ of the lowest five eigenstates of the covariant integral equation (51) are plotted versus the scaled momentum x for $\varepsilon = 0$.

enhanced close to the edges, $x = 0$ and $x = 1$. At infinitely high energies, we would ultimately get delta functions at $x = 0$ and $x = 1$, which would correspond to plane waves in coordinate space, i.e., the fermion and the antifermion would be delocalized, deconfined at infinitely high energies. In contrast, the ground-state wave function, of the shape $x(1-x)-\varepsilon(1-\varepsilon)$, has the maximum at $x = 1/2$, and the momenta x are quite widely spread around this maximal value. This means that the fermion and the antifermion tend to stay relatively close together and form a well-localized but extended configuration in the coordinate space, as one would expect from a bound ground state.

References

1) H.C. Pauli and S.J. Brodsky, Phys. Rev. D32 (1985) 1993, ibid. 2001;

2) A. Harindranath and J.P. Vary, Phys. Rev. D36 (1987) 1141 and Phys. Rev. D37 (1988) 1064;

3) Th. Eller, H.C. Pauli and S.J. Brodsky, Phys. Rev. D35 (1987) 1493;

4) K.J. Hornbostel, Ph.D. thesis, SLAC report 333, December 1988, to be published;

5) J.F. Briere and H. Kröger, Phys. Lett. B217 (1989) 215;

6) D. Klabučar and H.C. Pauli, "Discretized Light-Cone Quantization in a (3+1)-Dimensional Valence Model for Quarkonium", Max-Planck-Institut für Kernphysik, Heidelberg preprint 1989.