EFFECTIVE MODELS FOR LOW-ENERGY HADRON PHYSICS

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Two effective models are considered: the Nambu-Jona-Lasinio model (NJL) and the chiral σ model (Cσ). It is shown that in the tree-level approximation for boson fields, the Cσ model represents an approximation to the NJL model. The results are in agreement up to 15%.

1. Introduction

Perturbative QCD continues to be entirely consistent with experiments in hadron physics, but at low energies perturbation theory becomes inapplicable. To obtain meaningful results in this low-energy region (up to $\gtrsim$1 GeV), it is necessary to develop another way of performing calculations. It seems that the best choice are effective models, especially three of them:

- the Nambu-Jona-Lasinio model (NJL),
- the chiral σ model (Cσ),
- the Skyrme model.

The aim of this work is to show a connection between the NJL and Cσ models. This is an interesting question because the Cσ model is solved, and from the papers by 't Hooft and Witten[38] it may be concluded that the NJL model can be closely connected with QCD. In fact, it seems that under the assumption of confinement,the chiral-symmetry breakdown and the $1/N_c$ expansion, the NJL model can be "derived" from QCD[4,36,38].

These models are formulated in such a way as to include the essential properties of the QCD Lagrangian (symmetries, spontaneous breaking of chiral symmetry). An especially important approach is based on the concept of different "phases" in QCD: this means that phenomena of chiral-symmetry breaking exist in a definite energy range (below 1 GeV). This provides us with an important argument for definite cut-offs needed in the calculation (to obtain definite results).

2. The chiral σ model (Cσ) with valence quarks

The chiral σ model was formulated by Gell-Mann and Levi[15].From this model we can get the mass of the nucleon without explicit breaking of chiral SU(2) symmetry. Also, the famous concept of PCAC[12,25] can be incorporated into it, which is especially important because the PCAC has been experimentally proved very well.*

The Lagrangian of the model is

$$L = i\bar{q}\gamma^{\mu}\partial_{\mu}q - g\bar{q}(\sigma + i\underline{\tau}\underline{\pi}\gamma_5)q + \frac{1}{2}\partial_{\mu}\sigma\partial^{\mu}\sigma + \frac{1}{2}\partial_{\mu}\underline{\pi}\partial^{\mu}\underline{\pi} - u(\underline{\pi}^2 + \sigma^2) + c\sigma, \qquad (2.1)$$

where $q,\bar{q}$ are fermion (quark) fields, σ and $\underline{\pi}$ are the boson scalar isoscalar and the pseudoscalar isovector fields, respectively, g is the nucleon-meson coupling constant, the last factor is connected with the PCAC (non-zero mass of the pion field) and $u(\underline{\pi}^2+\sigma^2)$ is the interaction term between boson fields, commonly used as

$$u = \frac{\lambda^2}{4}(\sigma^2 + \underline{\pi}^2 - \nu^2)^2 . \qquad (2.2)$$

The assumption $\nu^2>0$ leads to the spontaneous breaking of chiral symmetry (χSB) and to the definite mass of the nucleon.

*Owing to all that, this model has been widely and successfully used through years[7,12,13,16-19,26].

Analysis of (2.1) gives

$$\sigma_V = f_\pi , \qquad m_q = g f_\pi , \tag{2.3}$$

where σ_V is the vacuum value of the σ field (the analogous value of the pion field is zero), m_q is the constituent (dynamic) mass of quark fields (we work in the u-d sector)* and f_π is the pion decay constant.

A natural condition in the sense of χSB leading to the soliton hedgehog solution of this model is (the linear σ model)

$$\langle \sigma^2(\underline{r}) + \underline{\pi}^2(\underline{r}) \rangle = f_\pi^2 , \tag{2.4}$$

where we have assumed that boson fields are static.

The baryon wave function is

$$|\psi_H\rangle = |q_H^3\rangle |P\rangle |Z\rangle , \tag{2.5}$$

where $|q_H^3\rangle$ is the product of wave functions of three constituent (valent) quarks (antisymmetrized in colour space); $|P\rangle$ and $|Z\rangle$ are coherent states for boson fields.

From the condition for minimum energy, using (2.1) we can get (after a rather lengthy calculation) the following shapes (profiles) for boson fields:

$$\sigma = \sigma(r), \qquad \underline{\pi} = \underline{r}/r \; h(r) \; . \tag{2.6}$$

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*This means that the model gives a prediction for g. Since f_π = 93 MeV and $m_q \simeq$ 300-500 MeV, g is 4∿7.

With these results, we are in a position to connect this model with the NJL model. This is the subject of the next section.

3. The Nambu-Jona-Lasinio (NJL) model

The NJL model[29,30] was formulated in analogy with the BCS theory of superconductivity. This is the simplest model with chiral symmetry and with fourth-order self-interaction between fermion fields:

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + \frac{G}{2}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\underline{\tau}\psi)^2\right]. \qquad (3.1)$$

In addition to U(1) symmetry, this Lagrangian has $SU(2)_L \times SU(2)_R$ symmetry (adequate transformations are $\psi \to \exp(i\underline{\alpha}P_{L,R}\underline{\tau})$, with $P_{L,R} = \frac{1}{2}(1 \pm \gamma_5)$)*.

It is easy to show that the generating functional for (3.1) is

$$Z = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\sigma\,\mathcal{D}\underline{\pi}\,\exp\left[i\int d^4x\,\bar{\psi}(i\gamma^{\mu}\partial_{\mu} - gf_{\pi}u)\psi\right], \qquad (3.2)$$

with

$$\begin{aligned} \sigma &= -\frac{g}{\mu^2}\bar{\psi}\psi\,, \\ \underline{\pi} &= -\frac{g}{\mu^2}\bar{\psi}i\gamma_5\underline{\tau}\psi\,, \\ G &= \frac{g^2}{\mu^2}\,, \\ u &= \frac{1}{f_{\pi}}(\sigma + i\underline{\tau}\,\underline{\pi}\gamma_5)\,. \end{aligned} \qquad (3.3)$$

After performing a formal integration of quark fields and using classical boson fields (the tree-level

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*With use of a simple dimensional analysis it is possible to show that G has the dimension of M^{-2}, so the model is not renormalizable. We should remember that this is only a model that represents a possible way to a real physical theory.

approximation), we get

$$Z = \exp(-S_E^{eff}), \tag{3.4}$$

where S_E^{eff} is the effective action in Euclidean space:

$$S_E^{eff} = -S_p \ln(-i\gamma^\mu \partial_\mu^E + gf_\pi u) . \tag{3.5}$$

Here S_p denotes the functional trace

$$S_p = N_c \sum_\tau \sum_\gamma \int d_E^4 x .$$

Eq. (3.5) can be solved in two different ways. One is the method of proper time regularization (PTR), which is exact at the numerical level. The other is the procedure of gradient expansion. The first method means that we have to solve (3.5) directly. This equation can bè written in a slightly different way as

$$S_E^{eff} = -S_p \ln(\frac{\partial}{\partial x_o} + h), \tag{3.6}$$

where x_o = it and the operator h is given by

$$h = \frac{\alpha\nabla}{i} + gf_\pi \gamma^o u . \tag{3.7}$$

Because of the condition (2.4), the fields σ and $\underline{\pi}$ have to be chosen as $\sigma = f_\pi \cos\theta(r)$, $\underline{\pi} = f_\pi\, \underline{r}/r \sin\theta(r)$, where the profile $\theta(r)$ can be chosen as exponential ($\theta(r) = -\pi \exp(-r/R)$ or linear ($\theta(r) = -\pi(1 - \frac{r}{R})$); R is the variational parameter.

After performing calculations, we obtain

$$E_o(R) = N_c [\sum_\lambda R_1(e_\lambda,\Lambda) - \sum_k R_1(e_k,\Lambda)] , \tag{3.8}$$

where $e_k^2 = g^2 f_\pi^2 + k^2$ and e_λ is the eigenvalue of the operator h ((3.7)), Λ is a cut-off to be incorporated in the result, because the integral in (3.9) is divergent.*

The value of this cut-off has to be found from the connection between the second way of solving the NJL model (gradient expansion) and the chiral σ model.

The gradient expansion means that we have to expand the logarithm in (3.6) and to take only the first few terms. The reasons for this are the following. We assume that boson fields do not change rapidly and, on the other hand, we can get physically meaningful results and the connection with the chiral σ model.

The real part of (3.5) can be written as

$$\mathrm{Re}\, S_E^{eff} = -\frac{1}{2} S_p \ln(1 + GV) =$$

$$= -\frac{1}{2} [S_p GV - \frac{1}{2} S_p (GV)^2 + \ldots] , \qquad (3.10)$$

with

$$G = \frac{1}{-\partial_E^2 + (gf_\pi)^2} , \qquad (3.11)$$

$$V = -igf_\pi \gamma^\mu \partial_\mu^E u^+ .$$

If we use a second-order expansion, we can get the kinetic part of the Lagrangian in the form

$$L_k = 2N_c g^2 I_2(\Lambda) [(\partial_\mu \sigma)^2 + (\partial_\mu \underline{\pi})^2] , \qquad (3.12)$$

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*This cut-off is discussed in the Introduction.

where $I_2(\Lambda)$ is defined by

$$I_2(\Lambda) = (16\,\pi^2)^{-1} \int_{(gf_\pi/\Lambda)^2}^{\infty} \frac{du}{u}\, e^{-u}. \qquad (3.13)$$

The cut-off Λ is chosen in such a way that L_k is identical with the kinetic part in (2.1)*.

We have obtained two results: one is the "exact" equation (3.8), with Λ determined (we assume that Λ from PTR is the same as Λ from the gradient expansion)** and the other result is approximate:

$$E_0^G(R) = I_2(\Lambda)\int d^3x\, [\tfrac{1}{2}(\nabla\sigma)^2 + \tfrac{1}{2}(\nabla\underline{\pi})^2] \quad . \qquad (3.14)$$

The difference between these two results is about 15%. This shows us that our approximation in the gradient expansion is really satisfactory.

A solution with the baryon number B=1 (nucleon) can be obtained if quark fields are expanded not about zero, but about its classical value (which minimizes the action). Formally, the calculations are performed with the new variable μ (the chemical potential), which is used to specify the baryon number. We have to calculate the expression

$$S_E^{eff}(u,\mu) = S_p \ln(-i\gamma^\mu \partial_\mu^E + gf_\pi u + i\mu\gamma_E^0). \qquad (3.15)$$

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*It is clear that with the above choice of Λ, $E_0^G(R)$ in (3.14) is equal to the kinetic energy of meson fields, i.e. this kinetic energy term represents the polarization of the vacuum caused by static meson fields.

** f_π can also be reproduced with this choice of the cut-off Λ[20].

The calculations are, in principle, the same as before. (A method similar to PTR is used.) The results are

$$E_1(R) = N_c e_{val}(R) + E_o(R) \; ,$$
$$E_1^G(R) = N_c e_{val}(R) + E_o^G(R) \; , \qquad (3.16)$$

with B=1. e_{val} has been obtained as an eigenvalue of the operator h (Eq. (3.7)), with the classical quark wave function as an eigenfunction. Numerical results for the energy (mass) and for the minimal radius of the nucleon are shown in Table 1[20). The exponential profile has been taken into account in the calculation.

Table 1

	g	Λ (MeV)	E_{min} (GeV)	R_{min} (fm)
E_1	4	637	1 130	0.6
E_1^G	4	-	1 330	0.7
Ref. 26	4	∞	1 070	0.4
E_1	4.8	638	1 072	0.7
E_1^G	5.5	-	1 079	0.8
Ref. 26	4	∞	1 070	0.4

We have used the zero-pion-mass approximation throughout this section. It is easy to show that the finite pion mass will not affect our results very much. Also, if the condition (2.4) is not used, after some more complicated calculations we find that the results are similar to those obtained under this condition.

Now we can calculate the quark condensate

$\langle 0|\bar{\psi}\psi|0\rangle$. The result is $\langle\bar{\psi}\psi\rangle = f_\pi$, and this means that like the Cσ model, the NJL model allows for XSB. This is one more step in the process of connecting the NJL model with the fundamental theory - QCD.

4. Conclusion

In this paper we have shown the connection between the NJL model and the Cσ model, and also to what extent this connection is satisfactory. Numerical results show that the approximation of the Cσ model to the NJL model is quite reasonable. As we have already shown, the difference is about 15%.

However, there are many open problems concerning this subject. First of all, the Cσ model gives rather poor results for some of the physical observables[12]. On the other hand, it is clear that the NJL model can be solved in a more satisfactory way if we take into account the one-loop contribution to boson fields, or, eventually, if we incorporate the vector meson in the model.

Anyway, should the present results be confirmed and should the NJL model turn out to be indeed a good approximation to low-energy QCD, then one would have a direct link between QCD and the chiral σ model with valence quarks.

Acknowledgments

The major part of this work was done at the Nuclear Institute KFA, Jülich, F.R. Germany, in the Soliton Group under the guidance of Professor K. Goeke (presently at the Institute for Theoretical Physics II, Bochum, F.R. Germany). I gratefully acknowledge the excellent facilities which enabled me to obtain most of the results presented in this paper.

References

1) Aitchison, Fraser, Miron, Phys. Rev. D33 (1986) 1994;

2) Aitchison, Krakow Workshop (1987), World Scientific, Singapore (1987);

3) Aitchison, preprint (1988) (to be published in Surveys in High Energy Physics);

4) Ball, Krakow Workshop (1987), World Scientific, Singapore (1987);

5) Barducci, Cassalbuoni, Curtis, Dominici, Gato, Phys. Rev. D38 (1988) 238;

6) Bernstein, Rev. Mod. Phys. 46 (1974) 7;

7) Birse, Phys. Rev. D33 (1986) 1934;

8) Dashen, Hasslacer, Neveu, Phys. Rev. D10 (1974) 4114;

9) Dyakonov, Krakow Workshop (1987), World Scientific, Singapore (1987);

10) Dyakonov, Petrov, Nucl. Phys. B272 (1986) 457;

11) Dyakonov, Petrov, Pobylitsa, Nucl. Phys. B303 (1988) 809;

12) N. Fiolhais, Ph.D. thesis, Coimbra, Portugal (1988);

13) Friedberg, T.D. Lee, Phys. Rev. D15 (1977) 1694;

14) Gasser, Leutwyler, Phys. Rep. 87 (1982) 78;

15) Gell Mann, Levy, Il Nuovo Cimento 16 (1960) 705;

16) K. Goeke, Urbano, Fiolhais, Harvey, Phys. Lett. 164B (1985) 249;

17) K. Goeke, Harvey, Grümmer, Urbano, Phys. Rev. D37 (1988) 754;

18) Fiolhais, Nippe, Goeke, Grümmer, Urbano, Phys. Lett. B194 (1987) 187;

19) Fiolhais, Goeke, Grümmer, Urbano, preprint (published in Nucl. Phys. A481 (1988) 727;

20) Meissner, Arriola, Grümmer, Goeke, preprint (1988);

21) A. Gorski, lectures at the KFA, Jülich, F.R. Germany, September-October 1988;

22) Halzen, Martin: Quarks and Leptons, ed. John Wiley & Sons, New York (1984);

23) Holzwarth, Schwesinger, Rep. Prog. Phys. 49 (1986) 825;

24) K. Huang: Quarks, Leptons & Gauge Fields, ed. World Scientific, New York (1982);

25) Itzykson, Zuber: Quantum Field Theory, ed. McGraw-Hill (1980);

26) Kahana, Ripka, Soni, Nucl. Phys. A415 (1984) 351;

27) B.W. Lee: Chiral Dynamics, ed. Gordon and Breach, Science Publ. New York (1970);

28) A. Messiah: Quantum Mechanics, ed. North Holland Publ. Co., Amsterdam (1967)

29) Nambu, Jona-Lasinio, Phys. Rev. 122 (1961) 345;

30) Nambu, Jona-Lasinio, Phys. Rev. 124 (1961) 246;

31) Pagels, Phys. Rep. 16 (1974) 221;

32) Rajamaran: Solitons and Instantons, ed. North Holland Publ.Co., New York (1982);

33) Reinhart, preprint (1986);

34) Ryder: Quantum Field Theory, ed. Cambridge University Press (1984);

35) Shuryak, Phys. Rep. 115 (1984) 153;

36) P. Simić, Phys. Rev. D34 (1986) 1903;

37) Williams, Cahill, Phys. Rev. D28 (1983) 1966;

38) Witten, Nucl. Phys. B160 (1979) 57.