

CONFORMAL FIELD THEORIES ON ALGEBRAIC SURFACES †

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ABSTRACT: subject of this talk is a summary of recently developed techniques for string theories and two dimensional conformal models on general algebraic Riemann surfaces of higher genus. The problem of spin structures on Z_n symmetric algebraic surfaces is pointed out.

In this talk we consider conformal field theories on algebraic Riemann surfaces Σ associated to a polynomial equation of the kind:

$$F(z, w) = w^n + P_{n-1}(z)w^{n-1} + \dots + P_1(z)w + P_0(z) = 0 \quad (1),$$

$w, z \in \mathbb{C}$ and $P_i(z)$ are rational functions of z . An important subcase of eq. (1) is:

$$w^n = \prod_i (z - c_i) \prod_j \frac{(z - a_j)}{(z - b_j)} \quad (2)$$

corresponding to Z_n symmetric surfaces [1,2]. Hyperelliptic curves occur when $n = 2$. If we solve eq. (1) in w , considering z as the independent variable, $w(z)$ turns out to

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be a multivalued function of z with n branches denoted by $w^{(l)}(z)$, ($l = 0, \dots, n-1$). Σ is determined, though not uniquely, by the condition of single-valuedness of $w(z)$ on it [3]. The actual construction of Σ in terms of sheets and branch points is matter of classical books on complex analysis such as [3,4] to which we refer the interested reader. The motivations behind investigation of conformal field theories on these surfaces are twofold:

- a) the partition functions of free scalar fields $X(z)$ and of b-c systems on Z_n symmetric surfaces can be interpreted as multipoint blocks of conformal models. Examples are given by Z_n symmetric orbifolds [5] and critical Ashkin-Teller model [6].
- b) For each abstract closed and orientable Riemann surface there exists a conformally equivalent algebraic surface Σ which provides an explicit representation of it.

Point b) makes algebraic surfaces convenient for construction of conformal field theories on higher genus. As a matter of fact the moduli space of Σ is parametrized by branch points without Shottky problem, factorization properties of string amplitudes can be checked just contracting to a single point two or more branch points, differentials and prime form are simply expressed in terms of the functions $w(z)$ and $F(z, w)$ and modular invariance, on hyperelliptic curves, consists in permutations of pairs of branch points. The computation of correlation functions and partition function is possible until now only on Z_n symmetric surfaces for the following free conformal field theories:

- i) usual b-c systems with conformal weight j (j integer) and scalar fields X [1,2];
- ii) b-c systems (j integer) with Z_n symmetric twisted boundary conditions along the homology cycles A_i, B_i ($i = 1, \dots, g$) [7,8]. These fields are coupled with an external vector field such that they pick up a phase of the kind $e^{2\pi i \frac{k}{n}}$ ($k = 0, \dots, n-1$) when transported around non-trivial homology cycles [9];
- iii) free fermions. Only on hyperelliptic curves the whole set of 2^{2g} spin structures can be realized.

In what follows we will restrict ourselves to b-c systems. Scalar fields can be treated in a similar way [2,10]. Z_n symmetric surfaces have found an application in string theory [11,12] and minimal models on hyperelliptic surfaces [13]. Due to the Z_n symmetry of these surfaces the multivaluedness of fields at the branch points α_i can be simulated with the introduction of some primary fields $V(\alpha_i)$ called twist fields [1]. In this formalism correlation functions and chiral determinants become correlation

functions on the complex plane with insertion of twist fields:

$$\langle \prod_{l=1}^N b(z_l) \prod_{k=1}^M c(z_k) \rangle = \frac{\prod_{l=1}^N b(z_l) \prod_{k=1}^M c(z_k) (\text{zero modes}) \prod_i V(c_i) \prod_j V(a_j) V(b_j)}{(\text{zero modes}) \prod_i V(c_i) \prod_j V(a_j) V(b_j)} \quad (3).$$

Unfortunately spin structures and twisted Z_m boundary conditions, m integer, entail non-trivial transition functions of fields which destroy the original Z_n symmetry between the sheets. Z_n symmetric twisted boundary conditions represent just a few exceptions [8]. Moreover a general surface of genus $g \geq 3$ covering all moduli space cannot be Z_n symmetric [14]. In both these cases the method of twist fields is no more applicable. To overcome these difficulties we provide a different recipe to compute the correlation functions for b-c systems and scalar fields on the general algebraic surface of eq. (1) [10]. Following refs. [15,16] we construct the two point functions in terms of abelian differentials and of third kind differentials. The former are of the form:

$$\omega_i(z) = \frac{\phi_i(z, w)}{\partial_w F(z, w)} dz \quad (4)$$

where $\phi_i(z, w)$ are polynomials of degree $n-3$ in z and w . The latter can be expressed using the Weierstrass kernel [17]:

$$G_{(l)}(z)_a = \frac{F(a, w(z))}{(w(z) - w^{(l)}(a)) F_w(z, w(z))} \frac{dz}{z - a} \quad (5).$$

This kernel has a pole at $z = a$ on the l -th sheet and other spurious poles which occur symmetrically on all the sheets. Therefore it is easy to find counterterms to subtract these extra singularities in the expression of the non-renormalized third kind differential with poles in a and b :

$$\omega_{ab} = G_{(l)}(z)_a - G_{(l')}(z)_b - (\text{spurious pole terms}) \quad (6).$$

With the aid of the kernel $G_{(l)}(z)_a$ it is also possible to construct a tensor $T dz^j da^{1-j}$ with a single pole in a on the l -th sheet:

$$\Omega_{a,(j)}(z) = \frac{F(a, w(z))}{(w(z) - w^{(l)}(a))(z - a)} \left[\frac{dz}{F_w(z, w(z))} \right]^j \left[\frac{da}{F_w(a, w^{(l)}(a))} \right]^{1-j} - (\text{spurious pole terms}) \quad (7).$$

Eqs. (5),(6) and (7) are sufficient to yield all correlation functions for b-c systems. For scalar fields we can express also the prime form in terms of third kind differentials [18].

Explicit examples for surfaces of genus three and four are given in ref. [10]. Here we quote just the formula for the Green function of b-c systems with $j \neq 1$:

$$G_{j,1-j}(z, a) = \frac{\langle b(z)c(a) \prod_{i=1}^{(2j-1)(g-1)} b(z_i) \rangle}{\langle \prod_{i=1}^N b(z_i) \rangle} = \frac{\det \begin{vmatrix} \Omega_a(z) & \Omega_1(z) & \dots & \Omega_N(z) \\ \Omega_a(z_1) & \Omega_1(z_1) & \dots & \Omega_N(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_a(z_N) & \Omega_1(z_N) & \dots & \Omega_N(z_N) \end{vmatrix}}{\det |\Omega_j(z_i)|} \quad (8)$$

where $N = (2j - 1)(g - 1)$ and $\Omega_i(z)$ ($i = 1, \dots, (2j - 1)(g - 1)$) are the zero modes which can be easily derived using the methods of ref. [10].

In this way it is possible to compute the n-points functions for bosonic string theories and, by means of Coulomb gas representation, to generalize the results of [13] on minimal models. An open problem left is to derive a measure of integration over the moduli space which, in our case, is parametrized by branch points. Also it would be nice to express the spin structures of fermions on the Z_n symmetric surfaces of reference [12] in order to obtain high energy superstring amplitudes.

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