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DIAGNOSING POSSIBLE INCONSISTENCIES OF FEYNMAN RULES FOR THEORIES OF GAUGE FIELDS QUANTIZED IN THE AXIAL GAUGE

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Abstract

An attempt to define precisely the Feynman rules for Yang-Mills theories in the axial gauge is made by requiring consistency with the theories in covariant gauges. Line integrals of gauge potential are used to construct a fermion correlation function which is invariant under a restricted class of gauge transformations comprising the ones that relate axial to covariant gauges. It is found that the Cauchy principal value prescription for spurious singularities of the gauge field propagator violates the compatibility condition in the one-loop order of perturbation theory.

The axial gauge choice in quantizing Yang-Mills fields is sometimes useful for proving general theorems. One example from QCD deals with the proof of factorization of long- and short-distance effects in deep inelastic scattering. This proof has first been completed in the axial gauge[1]. Also various calculations may be shorter in axial than in covariant gauges, as is the case in QCD at finite temperature[2]. The main reason for usefulness of the axial gauge resides in the following features of Yang-Mills theories in this gauge: (i) no ghost fields are required to calculate the S-matrix elements[3], (ii) the Ward identities retain their classical form[4], (iii) there is no need to renormalize gauge transformations[5], (iv) the leading collinear divergences reside only in ladder diagrams[6]. These features were derived by formal manipulations.

Over the last few years, it has become clear, however, that there are problems in defining axial gauges precisely. It has been a long-standing source of worry that the bare propagator of the Yang-Mills field in the axial gauge is plagued with singularities which apparently cannot be made precise by the usual definition of the propagator as the inverse of the quadratic part of the Lagrangian. The correct prescription how to integrate over these spurious axial gauge singularities, and the rest of consistent Feynman rules, remain still in dispute. The most commonly adopted recipe, which prescribes that the non-covariant poles of the axial gauge propagator are to be treated as Cauchy principal values, was caught, in the case of perturbative evaluation of rectangular Wilson loops[7], yielding results in disagreement with results of calculation in the Feynman gauge. This is in contradiction with gauge invariance of Wilson loops.

There were attempts to rectify the situation by putting forward amendments to the Feynman rules in the axial gauge. One of these amendments sacrifices translational invariance of the gauge field propagator[8], and another invokes ghost fields[9]. In either case, most of the appeal of the axial gauge choice is lost, primarily because these amendments are incompatible with at least one of the desirable properties of formal theory.

In the search for consistent, yet practicable axial gauge Feynman rules, an obstacle appears to be verification of compatibility with the firmly established Feynman rules in covariant gauges; the Wilson loops reveal difficulties with the Cauchy principal value prescription only at the two-loop order of perturbation theory.

In principle, any gauge invariant quantum object, constructed out of fields of the theory, may be used as a necessary condition to verify compatibility of Feynman rules in various gauges. It is difficult to imagine a simpler gauge invariant object than a rectangular Wilson loop; calculation of Wilson loops, defined on more complicated closed contours, or of scattering amplitudes for physical proceses, as the next most obvious gauge invariant quantities, proved to be a much more taxing demand.

But unrestricted gauge invariance is more than what we need in order to be able to question compatibility of a proposed set of axial gauge Feynman rules with the conclusively-established covariant gauge Feynman rules. A completely gauge-invariant object allows us to disprove or to give evidence of compatibility of a given set of Feynman rules in the axial gauge with Feynman rules in all other gauges, not only the covariant ones. For our purpose, therefore, it would be sufficient to have an object which is invariant not under all gauge transformations, but under those ones, and only those ones, that relate any gauge potential satisfying the axial gauge-fixing condition

$$n.A = 0 \tag{1}$$

to a gauge potential satisfying the Lorentz gauge-fixing condition

$$\partial.A = 0. \tag{2}$$

In a classical gauge field theory every part of a Wilson loop (i.e. trace of the path-ordered exponential of the contour integral of the gauge potential, defined on any curve, not just the closed ones) fulfils the requirement in the sense that allows for the fact that the Lorentz condition (2) only constrains A_μ, but does not fix it uniquely. In the abelian case, for example, $A_\mu(x)$ can change by $\partial_\mu\Lambda(x)$, say, and still satisfy (2), provided

$$\Box \Lambda(x) = 0. \tag{3}$$

This implies that, given a gauge potential A_μ which satisfies the Lorentz condition (2), there always exists a gauge-equivalent potential A'_μ in the same gauge (2), that is related to a gauge potential A''_μ satisfying the axial gauge-fixing condition (1), by a gauge transformation that maps a hyperplane, perpendicular to the gauge-fixing vector n, to the identity of the group.

I provide here a proof of this statement separately for the space-like case $n^2 < 0$ and the time-like case $n^2 > 0$. In the temporal case, we can, without loss of generality, choose $n = (1,0,0,0)$ and the hyperplane $(T,\underline{x})$, T fixed. Since it is always possible to find a solution to Eq.(3), which on the hyperplane satisfies the initial condition

$$\partial_0\Lambda(T,\underline{x}) = - A_0(T,\underline{x}), \tag{4}$$

where A_0 is given by the starting gauge potential in the Lorentz gauge, it follows that

$$A'_0(T,\underline{x}) = A_0(T,\underline{x}) + \partial_0\Lambda(T,\underline{x}) = A_0(T,\underline{x}) - A_0(T,\underline{x}) = 0 \tag{5}$$

and so it is, trivially, also in the axial gauge on the hyperplane.

For the space-like case, we can, without loss of generality, choose $n = (0,0,0,1)$ and the hyperplane (x^0,x^1,x^2,R), R fixed. It is always possible to make a gauge transformation such that $A''_3(x) = 0$. Namely, given $A_\mu(x)$, let us construct

$$U(P_x) := \exp [-ig \int_0^{x^3} dx^3 A_3(x)] \tag{6}$$

so that

$$\partial_3 U(P_x) = U(P_x) - ig\, A_3(x). \tag{7}$$

If we transform $A_\mu(x)$ by this $U(P_x)$, we get

$$ig\, A_3(x) + ig\, A''_3(x) = ig\, U(P_x)\, A_3(x)\, U(P_x)^{-1} + [\partial_3 U(P_x)]U(P_x)^{-1} = 0$$

by Eq.(7).

Instead of constructing $U(P_x)$ out of the original covariant $A_\mu(x)$, we now construct it out of $A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$, where Λ satisfies Eq.(3) and the boundary condition

$$\Lambda(x^0,x^1,x^2,R) = \Lambda(x^0,x^1,x^2,0) - \int_0^R dx^3 A_3(x^0,x^1,x^2,x^3). \tag{8}$$

Since there always exists such a Λ, we have

$$\exp[-ig \int_0^R dx^3 A'_3(x)] = \exp [-ig \int_0^R dx^3(A_3 + \partial_3\Lambda)] =$$
$$\exp[-ig \int_0^R dx^3 A_3(x) - ig\Lambda(x^0,x^1,x^2,R) + ig\Lambda(x^0,x^1,x^2,0)] = 1 ,$$

i.e. the transformation to the axial gauge is trivial on the hyperplane.

In the non-abelian case, at least in perturbation theory, i.e. for small A_μ and $U(x) = 1 + \Lambda(x) = 1 + \Lambda^a(x)\, T^a$, Eq. (3) gets replaced by

$$\Box \Lambda - [A_\mu , \partial_\mu\Lambda] + \ldots = 0, \tag{9}$$

where the dots, denoting higher orders in Λ, are negligible in perturbation theory. Since Eq.(9) is soluble when suitable boundary conditions on the hyperplane are supplied, we can get the conclusions from the abelian case across to the non-abelian case.

Therefore, if both ends R_x and R_y of a curve C, by which the Wilson line

$$P \exp [-ig \int_{R_x}^{R_y} A_\mu(z)\, dz^\mu] \tag{10}$$

is defined, lie on the hyperplane, the Wilson line, which under a general gauge transformation

$$ig\, A_\mu(x) \to ig\, A'_\mu(x) = U(x)\, ig\, A_\mu(x)\, U^{-1}(x) - [\partial_\mu U(x)]\, U^{-1}(x) \tag{11}$$

transforms into

$$P \exp [-ig \int_{R_x}^{R_y} A'_\mu(z) dz^\mu] = U(R_y)\, P \exp [-ig \int_{R_x}^{R_y} A_\mu(z) dz^\mu]\, U^{-1}(R_x), \tag{12}$$

in this case remains unchanged. In other words, if we have a Wilson loop, a part of which lies on some hyperplane, that part will contribute in the axial gauge exactly the same amount as in some covariant gauge.

Since we are not able to specify this covariant gauge precisely beyond the fact that, at least in the classical sense, it exists, it is not obvious that we can profit from this observation. We do expect, at least in the temporal case, that the Wilson loop as a quantum operator possesses the property observed in the classical case, because the difference between the classical and the quantum Wilson loop amounts only to the additional ordering of the quantum fields according to time[9], and this ordering does not interfere with any step in the proof of the behaviour of a classical Wilson loop on a space-like hyperplane.

In the quantum case, the Lorentz condition (2) can only be considered as a limit $\alpha \to 0$ of the gauge-fixing part of the Lagrangian of the form

$$\frac{1}{2\alpha}(\partial.A)^2 \tag{13}$$

The freedom in the classical case of residual covariant gauge transformations, specified by gauge parameters Λ which satisfy Eq.(3), is now reflected in the unspecified nature of the double pole in k^2 of the gauge field propagator

$$D^{ab}_{\mu\nu} = \frac{i\delta^{ab}}{(2\pi)^4} [\frac{-g_{\mu\nu}}{k^2 + i\varepsilon} + (1 - \alpha)\frac{k_\mu k_\nu}{(k^2)^2}] \tag{14}$$

If there is some aspect of the Wilson line, which is independent of α, then this aspect may be used as a diagnosing tool for diseases in axial gauge Feynman rules.

The crucial property of the Wilson line operator, that allows straightforward analysis of its behaviour under gauge transformations, is represented by Eq.(12). Exactly the same change under gauge transformations experiences the following binomial of Dirac fields

$$\psi'(y)\overline{\psi'}(x) = U(y)[\, \psi(y)\, \overline{\psi}(x)]\, U^{-1}(x). \tag{15}$$

Therefore, for the sake of testing compatibility of axial with covariant Feynman rules, we may also study

$$S(x-y) = \langle 0|\, T[P \exp(-ig \int_x^{R_x} A_\mu dz^\mu)\, \psi(x)\overline{\psi}(y)\, \overline{P} \exp (+ig \int_y^{R_y} A_\mu dz^\mu)]|\, 0\rangle \tag{16}$$

A calculational simplification is achieved, if as large a portion of the contour as possible is parallel to the axial gauge-fixing vector, for on that part of the contour the path-ordered exponential is trivial in the axial gauge. Therefore, I choose both Wilson lines on the R.H.S. of Eq.(16) parallel to n. The expression for S(x-y), when calculated in perturbation theory with covariant Feynman rules to order g^2 (diagrammatically illustrated on Fig.1)

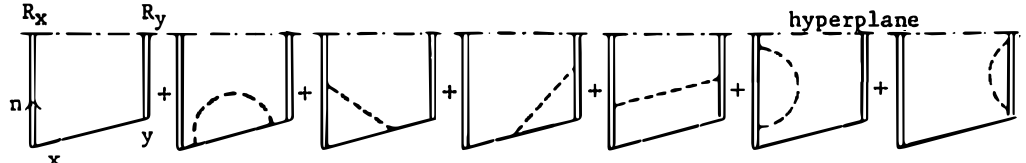

Fig.1: ═══ Wilson line, ——— fermion propagator, ------ gauge propagator

takes the simplest form for n.(x-y) = 0, which is

$$S(x-y) = S_o(x-y) + \frac{g^2}{(2\pi)^{4\omega}} \int d^{2\omega}p\, e^{-ip.(x-y)} \int d^{2\omega}k \frac{1}{k^2+i\varepsilon} [S_o(p)\gamma^\mu S_o(p-k)\gamma_\mu S_o(p)$$

$$+ (S_o(p-k) + S_o(p+k))\not n S_o(p) \frac{\cos k.nR - 1}{k.n} \cancel{+ i(S_o(p+k) - S_o(p-k))\not n S_o(p)} \frac{\sin k.nR}{k.n}$$

$$+ n^2 (S_o(p-k) - S_o(p))(2 \frac{1 - \cos k.nR}{(k.n)^2} + \frac{1-\alpha}{n^2\, k^2})] + O(g^4) \qquad (17)$$

In the axial gauge, with the gauge field propagator given by

$$D^{ab}_{\mu\nu} = \frac{i\,{}^{ab}}{(2\pi)^4 . k^2+i\varepsilon} \frac{1}{} (-g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{[k.n]} - n^2 \frac{k_\mu k_\nu}{[(k.n)^2]}) \qquad (18)$$

the expression for S(x-y) takes the form

$$S(x-y) = S_o(x-y) + \frac{g^2}{(2\pi)^{4\omega}} \int d^{2\omega}p\, e^{-ip.(x-y)} \int d^{2\omega}k \frac{1}{k^2+i\varepsilon} [S_o(p)\gamma^\mu S_o(p-k)\gamma_\mu S_o(p)$$

$$- (S_o(p-k) \not n S_o(p) + S_o(p) \not n S_o(p-k) + 2\, S_o(p) \not n S_o(p)) \frac{1}{[k.n]}$$

$$+ n^2 (S_o(p-k) - S_o(p) + S_o(p) \not k S_o(p)) \frac{1}{[(k.n)^2]}] + O(g^4) \qquad (19)$$

Introducing the Cauchy principal value prescription for $[k.n]^{-1}$ and $[(k.n)^2]^{-1}$ into Eq.(19), I find that the difference between the R.H.S. of Eq.(19) and the R.H.S. of Eq.(17) diverges like $\Gamma(\omega - 2)$, as $\omega \to 2$, for any finite value of α. Therefore, the Cauchy principal value prescription for spurious singularities of the axial gauge propagator (18) is incompatible with the Feynman rules in covariant gauges.

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