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FERMIONIZATION OF THE COULOMB GAS

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ABSTRACT

A new object, namely a *real weight* chiral $b-c$ system, is introduced. It is shown that on the sphere it is equivalent to (the chiral part of) a free boson with a charge at infinity. The charge at infinity acquires a topological meaning. The Coulomb gas prescription to compute correlation functions of conformal minimal models can be implemented in this framework. The main advantage of the $b-c$ system w.r.t. the standard formulation lies on the fact that it can be generalized in a completely straightforward way to study conformal field theories on higher genus Riemann surfaces. Geometrical factors such as curvature and holonomy appears naturally.

1. As it was shown by Belavin, Polyakov and Zamolodchikov [1] the conformal invariance, together with the associativity of the operator algebra, fixes in a univoque way the correlation functions of the minimal model theories. To solve the partial differential equations so implied is however, from a practical point of view, not an easy task. The Coulomb gas method elaborated in [2] gives an integral representation of the PDE. The basic idea underlying this method is the existence of a certain free field theory whose correlation functions, suitably constrained, have to be interpreted as correlation functions for the minimal model under consideration. The free field theory considered is that of a free bosonic field Φ, with a background charge $-2\alpha_0$ placed at ∞.

The Coulomb gas can be rephrased emphasizing another point of view, namely its algebraic aspect. There are two such kind of versions. One is a bosonic [3] realization in terms of oscillators; the other goes back to the original work of Feigin-Fuchs [4] and it appears in terms of semiinfinite forms constructed starting from bases of real weight tensors. All these versions are equivalent. In particular the equivalence between the two algebraic formulations goes under the name of boson-fermion correspondence. On the other hand the duality between bosons and fermions in two dimensions is a well-established property.[5]

So far however it was lacking a field-theoretic fermionic realization of the Coulomb gas. This gap was filled in [6], where a free chiral anticommuting $b-c$ system of suitable real weight has been introduced to describe any given minimal model.

The interest in considering such a system is not merely an academic one; nothing prevents the $b-c$ system from being formulated on higher genus Riemann surfaces. Therefore a Coulomb gas prescription can be given, as it was proven in [7], also for conformal theories in higher genus.

The $b-c$ approach can also be applied [8] to other theories which admit a free field interpretation, like the WZW models described in [9].

In this talk I am interested in illustrating how this method works, therefore I limit myself to discussing the $b-c$ system on the sphere, getting informations about the minimal models and proving the equivalence with the standard Coulomb gas approach.

2. The real weight chiral $b-c$ system is described in terms of two anticommuting fields b and c of real weight λ and $(1\text{-}\lambda)$ respectively. The energy momentum tensor of the system is

$$T = (1-\lambda)\partial b c - \lambda b \partial c. \tag{1}$$

A system of bases for the b fields on the sphere is given by: [(1)]

$$f_j^{(\lambda,l)}(z) = (P_+ - P_-)^{j+\lambda}(z-P_+)^{j-\lambda}(z-P_-)^{-j-\lambda}(dz)^\lambda, \tag{2}$$

where $j \in \mathbf{Z} + \lambda l$ and $l \in \mathbf{Z}^+$ is a sector index, namely when going once around P_+

$$f_j^{(\lambda,l)} \rightarrow e^{2\pi i\lambda l} f_j^{(\lambda,l)}. \tag{3}$$

When λ is rational, i.e. $\lambda = \frac{m}{n}$, with m,n relatively prime integers, then there are only n distinct sectors $l = 1, 2, ..., n$. When λ is a half-integer we recover the Neveu-Schwarz $(l=1)$ and the Ramond $(l=2)$ sectors. The bases $f^j_{(1-\lambda,l)}(z)$ for the c fields are fixed by the duality requirement:

$$\frac{1}{2\pi i}\oint_C f^i_{(1-\lambda,l)}(z) f_j^{(\lambda,l)}(z) = \delta^i_j \tag{4}$$

(1) This non-standard notation is used as an introduction to the higher genus case. The usual notation is recovered by sending P_+ to 0 and P_- to infinity.

with $i, j \in Z + \lambda l$. The contour integral winds once around P_+.
The fields b and c are expanded in each sector l:

$$b^l(z) = \sum_j b^{j,l} f_j^{(\lambda,l)}(z) \qquad c^l(z) = \sum_j c_j^l f_{(1-\lambda,l)}^j(z), \tag{5}$$

and the following anticommutation relations are assumed:

$$\{b^{i,l}, c_j^l\} = \delta_j^i \qquad \{b^{i,l}, b^{j,l}\} = \{c_{i,l} c_{j,l}\} = 0. \tag{6}$$

For each sector the vacuum is defined as follows [10]:

$$\begin{aligned} b^{j,l}|0>_l = {}_l<0|c_j{}^l &= 0, \qquad for\ j \le \lambda + \overline{\lambda(l-1)} - 1; \\ c_j^l|0>_l = {}_l<0|b^{j,l} &= 0, \qquad for\ j \ge \lambda + \overline{\lambda(l-1)}. \end{aligned} \tag{7}$$

The bar denotes the non-integer part of $\lambda(l-1)$. It is assumed that ${}_l<0||0>_l = 1$.

The propagator is defined as $S^{(\lambda,l)}(z,w) = {}_l<0|R\left(b^l(z)c^l(w)\right)|0>_l$ (R denotes the radial ordering w.r.t. the euclidean time). Then the expression for the propagator $S^{(\lambda,l)}(z,w)$ is:

$$S^{(\lambda,l)}(z,w) = \frac{1}{(z-w)} \left(\frac{z-P_+}{w-P_+}\right)^{\overline{\lambda(l-1)}} \left(\frac{z-P_-}{w-P_-}\right)^{-2\lambda+1-\overline{\lambda(l-1)}} (dz)^{\lambda}(dw)^{1-\lambda}. \tag{8}$$

The sum $1-2\lambda$ of the exponents plays the role of a total charge (it can be seen as a sort of generalized Riemann-Roch index; in higher genus[7] one gets $(2\lambda-1)(g-1)$).

3. The basic idea now consists in modifying the bases (2) in order to mimic the insertion of primary fields $V_i(P_i)$ which in the bosonized version are represented by the chiral part of the operators $: e^{i\alpha_i \Phi(P_i)} :$. For this purpose a "fat" $b-c$ system (denoted as $B-C$ system) will be introduced. The bases of the B and C fields will no longer be holomorphic at the points P_i where the vertex operators are inserted. For the B fields one is motivated to perform the expansion

$$B(z) = \sum_j B^j g_j^{(\lambda)}(z) \qquad g_j^{(\lambda)}(z) = (P_+ - P_-)^{j-\lambda+1} \frac{(z-P_+)^{j-\lambda}}{(z-P_-)^{j-\lambda+1}} \prod_{i=1}^{n} (z-P_i)^{\tilde{\alpha}_i} (dz)^{\lambda} \tag{9}$$

(from now on only the sector $l = 1$ will be considered).
The corresponding expansion for the fields C is fixed by duality. A total charge conservation

$$\sum_i \tilde{\alpha}_i = 1 - 2\lambda \tag{10}$$

is required as a geometrical consistency condition for the B and C bases. The propagator $S(z,w)$ is given by:

$$S(z,w) = \frac{1}{(z-w)} \prod_i \left(\frac{z-P_i}{w-P_i}\right)^{\tilde{\alpha}_i} (dz)^{\lambda}(dw)^{1-\lambda}. \tag{11}$$

The basic relation which allows to identify this formalism with the standard one is the following

$$S(z,w) = {}_1 < 0|R(B(z)C(w))\,|0>_1 = \frac{<\prod_k V^k(P_k)b(z)c(w)>}{<\prod_k V^k(P_k)>}. \qquad (12)$$

The correlation functions $<\prod_i V^i(P_i)>$ can now be computed by means of a standard procedure,[11] which makes use of the O.P.E. between the stress-energy tensor and any given weight h primary field $V(P)$:

$$T(z)V(P) = \frac{hV(P)}{(z-P)^2} + \frac{1}{(z-P)}\partial_P V(P). \qquad (13)$$

The result one gets is the following:

$$<\prod_i V^i(P_i)> = const \cdot \delta\left(\sum_i \tilde{\alpha}_i - (1-2\lambda)\right)\prod_{i<j}(z_{P_i} - z_{P_j})^{\tilde{\alpha}_i\tilde{\alpha}_j}. \qquad (14)$$

The conformal weight of the vertex operator in P_i is given by:

$$h_{\tilde{\alpha}_i} = \frac{1}{2}\tilde{\alpha}_i(\tilde{\alpha}_i + 2\lambda - 1). \qquad (15)$$

The vertex operators satisfy a duality relation since the same conformal weight is obtained both from $\tilde{\alpha}_i$ and from $2\tilde{\alpha}_0 - \tilde{\alpha}_i$ (here $2\tilde{\alpha}_0 \equiv 1 - 2\lambda$). The central charge of the fermionic system is:

$$c(\lambda) = -12\lambda^2 + 12\lambda - 2 = 1 - 12\tilde{\alpha}_0^2. \qquad (16)$$

It is clear at this point that a $b-c$ system of real weight λ shares the same features of a bosonic field Φ with a background charge at infinity.

The electromagnetic duality [12] in this framework is expressed by the transformation $\lambda \rightarrow (1-\lambda)$.

The relation between these results and those given in ref. [2] is established by setting:

$$\tilde{\alpha}_i = \sqrt{2}\alpha_i \qquad \tilde{\alpha}_0 = \sqrt{2}\alpha_0. \qquad (17)$$

From now on the two formalisms proceed in a parallel way. To fulfil the condition 10 one must introduce screening charges in the correlators, recovering the Kač quantization condition on the conformal weight of the primary field operators.

A minimal model of the unitary series

$$c = 1 - \frac{m}{m(m+1)} \qquad m \quad integer \quad \geq 1 \qquad (18)$$

is represented by a $b-c$ system of real weight (in general irrational):

$$\lambda = \frac{1}{2} \pm \frac{1}{\sqrt{2m(m+1)}}. \qquad (19)$$

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References

[1] A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl. Phys. B 241 (1984), 333.

[2] V.S.Dotsenko and V.A.Fateev, Nucl.Phys. B240 (1984),312.

[3] M.Kato and S.Matsuda, in Adv. Stud.in Pure Math. 16 (88) CFT and Solv. Latt. Mod. (1987),205.

[4] B.L.Feigin and D.B.Fuchs, Funct. Anal. Appl. 16 (1982),114.

[5] L.Alvarez-Gaumè, G.Moore and C.Vafa, Comm. Math. Phys. 106 (1986),1.
L.Alvarez-Gaumè,J.B.Bost,G.Moore,P.Nelson and C.Vafa, Comm. Math. Phys. 112 (1987),503.
T.Eguchi and H.Ooguri, Phys. Lett. B187 (1987),127.
E.Verlinde and H.Verlinde, Nucl.Phys. B288 (1987),357.

[6] L.Bonora,M.Matone,F.Toppan and K.Wu, Phys. Lett. B224 (1989),115.

[7] L.Bonora,M.Matone,F.Toppan and K.Wu, Preprint ISAS 55-89 (1989), to appear in Nucl. Phys. B [FS].

[8] L.Bonora and F.Toppan, in preparation.

[9] A.Gerasimov,A.Marshakov,A.Morozov,M.Olshanetsky and S.Shatasvili, Preprint Moscow, February 1989.

[10] M.Matone, in Proceedings C.I.M.E. Conference, Montecatini 1988.

[11] L.Dixon, D.Friedan, E.Martinec and S.Shenker, Nucl.Phys. B282(1987),13.

[12] R.Dijkgraaf,E.Verlinde and H.Verlinde, Preprint THU 87/17 (1987).