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UNIVERSAL INVARIANT RENORMALIZATION AND CUT-OFF SCHEMES IN THEORIES WITH HIGH INTERNAL SYMMETRIES

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In this report I'll describe shortly the so called universal invariant ultraviolet regularization designed to preserve an internal symmetry of a quantum field theory in non-anomalous situations. It is universal in the sense tha it is applicable to any model which can be formulated in terms of a trilinear interaction. It is invariant in the sense that it posesses as many nice intrinsic properties as are needed normally in order to derive Ward identities. As it has been proved in [1,2] it works all right in gauge theories and in theories with global supersymmetry. One might think as well that contrary to dimensional regularization [3] it leads to no troubles in locally supersymmetric theories.

The comlete recipe is rather lengthy, so here we'll just present the idea and give examples. The detailed formulation for Abelian Yang-Mills theories is contained in [2], the extended discussion of the recipe for the general case will be published elsewhere.

To begin with let us recall the simplest regularization scheme - the Pauli-Villars one. In a massless theory it amounts to the change of the initial propagator $1/k^2$ into a combination $1/k^2 - 1/(k^2-\mu^2)$ which has improved ultraviolet properties since the main asymptotic term is independent of the mass parameter μ. If one wants to obtain still better asymptotic behaviour for the propagator one makes the change

$$\frac{1}{k^2} \to \sum_i \frac{c_i}{k^2 - \mu_i^2} \qquad (1)$$

where $\mu_0 = 0$, $c_0 = 1$ and

$$\sum_i c_i = 0, \sum_i c_i \mu_i^2 = 0, \ldots, \sum_i c_i \mu_i^{2N} = 0. \qquad (2)$$

The removal of the cut-off corresponds then to the limiting process $\mu_i \longrightarrow \infty$ ($i > 0$) provided the coefficients c_i remain finite. Relations (1) and (2) can be handled also in another form, namely

$$\frac{1}{k^2} \to \int d\mu \; \rho(\mu) \; \frac{1}{k^2 - \mu^2} \qquad (3)$$

where $\rho(\mu) = \Sigma\, c_i\, \delta(\mu - \mu_i)$. Conditions (2) may then be written as

$$\int d\mu \; \rho(\mu) = 0, \quad \int d\mu \; \mu^2 \; \rho(\mu) = 0, \ldots, \int d\mu \; \mu^{2N} \; \rho(\mu) = 0. \qquad (4)$$

In fact this proves to be the most general form of a regularized propagator. Note that in order to provide an effective cut-off the integration density $\rho(\mu)$ need not at all be a linear combinaion of δ-functions. It might well be continuous or contain derivatives of δ-function etc. What really matters is conditions (4) which tell that a certain amount of lower moments of the function $\rho(\mu)$ should be equal to zero. Under these conditions the regularized propagator (3) will posess the improved asymptotic behaviour. We dwell that choosing the function $\rho(\mu)$ appropriately one can reproduce any standard regularization of a propagator - e.g. the analytic regularization, the momentum cut-off, the cut-off in the α-parametric space etc.

The regularization (3) refers to individual propagators. An immediate generalization is so to say a 'collective' regularization.

Namely let $G(k_1,\dots,k_n)$ denote the formal massless 'bare' Feynman amplitude corresponding to some one-particle irreducible scalar diagram. Let $G_{\mu_1\dots\mu_l}(k_1,\dots,k_n)$ be the same formal amplitude but with non-zero masses $\mu_1,\dots,\mu_l$ on the lines $1,\dots,l$ of the diagram. At last let $\rho(\mu_1,\dots,\mu_l)$ be some function whose several first moments are zero:

$$\int d\mu_1\dots d\mu_l\ \mu_1^{N_1}\dots\mu_l^{N_l}\rho(\mu_1,\dots,\mu_l) = 0, \quad (\ N_1+\ \dots\ +N_l \le 2N\ ) \qquad (5)$$

Then the most general ultraviolet cut-off will be achieved in result of the change

$$G\ (k_1,\dots,k_n) \longrightarrow \int d\mu_1\dots d\mu_l\ \rho\ (\mu_1,\dots,\mu_l) \times$$
$$\times\ G_{\mu_1\dots\mu_l}(k_1,\dots,k_n). \qquad (6)$$

Choosing the weight function ρ appropriately one can reproduce different known regularization schemes e.g. dimensional regularization etc. The above mentioned 'individual propagator ' regularization corresponds to the degenerate case when the function ρ splits into the product of 'individual' factors: $\rho = \rho_1(\mu_1)\rho_2(\mu_2)\dots\rho(\mu_l)$.

So in fact the problem of introduction of invariant regularization can be reduced to the problem of introduction of massive parameters μ_l (which will be called 'soft masses' in what follows) into the diagrams of a theory without breaking its intrinsic symmetry (or better say the corresponding Ward identities). Then integration over these μ_l with an arbitrary weight function ρ (with several zero moments) provides the necessary invariant regularization. On the other hand invariant introduction of massive parameters μ_l solves as well the problem of invariant renormalization since these μ_i can be used as 'soft masses' in the corresponding soft-mass quantization methods [4].

After these general pleriminaries suppose that we are challenged to introduce an invariant regularization into a quantum field theory. Let the field content of the theory be labelled by a condenced symbol φ_x with φ standing for all sorts of fields (including vector, spinor etc) and x denoting all indices (isotopic and Lorentz ones, coordinates etc). The first step should be to handle the theory in terms of only trilinear intraction. This is usually achieved by introduction of auxiliary fields with trivial contractions. For example the quartic Yang-Mills vertex $\frac{g^2}{4}[A_\mu, A_\nu]*[A_\mu, A_\nu]$ has to be changed into the trilinear vertex $\frac{g}{2}[A_\mu, A_\nu]\ \sigma^{\mu\nu}$ where $\sigma^{\mu\nu}$ is the antisymmetric auxiliary tenzor field with the trivial propagator $<T\ \sigma^{\mu\nu}(x)\ \sigma^{\lambda\gamma}(y)> = = \delta(x-y)\ \delta^{\mu\lambda}\ \delta^{\nu\gamma}$ (of course both formulations are equivalent).

The second step will be to separate all one-particle irreducible diagrams into equivalence classes. Each equivalence class is specified by the so called *skeleton*. A skeleton is an arbitrary vacuum diagram of the φ^3-theory. The diagrams of an equivalence class are obtained from the corresponding skeleton by means of 'painting' i.e. by means of adding a certain amount of external legs to the skeleton and of converting the lines of the so obtained φ^3-diagram into the lines of the theory in question. After 'painting' is finished we can put every loop of every diagram into correspondence to some loop of some skeleton.

Next you fix (in any way you like) the sets of independent loops in every skeleton, adjust loop momenta $q_1, q_2, \ldots, q_n$ to them and choose (for each skeleton) some cut-off function $\rho(\mu_1, \mu_2, \ldots, \mu_n)$ with a desired number of zero moments. The same loop momenta $q_1, q_2, \ldots, q_n$ are ascribed to the corresponding loops of all diagrams from the equivalence class. The 'soft masses' $\mu_1, \mu_2, \ldots, \mu_n$ appear in the theory in result of simultaneous 'lengthening' of loop momenta in all diagrams of equivalence classes. That means that in the momentum-space Feynman integrand for the diagram you change each variable q_1^2 into

$q_1^2 - \mu_1^2$ while all other variables remain unchanged. At last you have to integrate the new Feynman amplitude over $\mu_1, \mu_2, \ldots, \mu_n$ with the weight function $\rho(\mu_1, \mu_2, \ldots, \mu_n)$ of the corresponding skeleton. (In fact the procedure of extracting the variables q_1^2 out of the set of all scalar combinations $q_i q_j$, $q_i A_j$ etc is strictly speaking ambiguous.

But here we shall not comment this ambiguity since there can be no confusion in lower-loop examples we are going to consider below.)

The above universal rules define the operator T_{reg} of the cut-off chronological ordering which posesses the following basic properties.

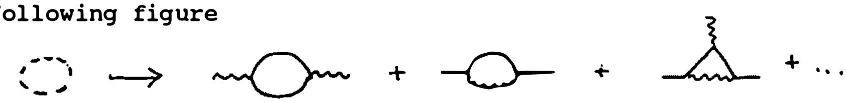
a) It commutes with functional derivatives: $T_{reg} \frac{\delta}{\delta\varphi_x} = \frac{\delta}{\delta\varphi_x} T_{reg}$.

b) The 'naive' relation $\Box \langle T\ \varphi_x\ \varphi_y \rangle = \delta\ (x-y)$ can be still used to analyze the structure of regularized diagrams despite the fact

that T_{reg} might have little to do with the conventional operator T. (The symbol $\Box$ denotes the kernal of the kinetic bilinear term in the Lagrangian. The conventional propagator is inverse with respect to $\Box$).

c) Integration by parts is possible under the sign of T_{reg}.

d) It still remains possible to make 'translations' of momenta variables in integrands of diagrams regularized by means of T_{reg}. Usually (if a theory has no inavoidable intrinsic anomalies) the properties a) - d) prove to be sufficient to derive Ward identities. For simple examples we use QED. The only skeleton for one-loop diagrams is presented in the left-hand side of the following figure



(and looks like a single circular line). A part of diagrams generated by this skeleton form the right-hand side of the figure. Their contributions are respectively $\int A^{\mu}(k)\ \Pi_{\mu\nu}(k)\ A^{\nu}(-k)\ dk$, $\int\bar{\psi}(p)\ \Sigma(p)\ \psi(-p)\ dp$, $\int \bar{\psi}\ (p)\ A^{\nu}(k)\ \Gamma_{\nu}\ (k,p)\ \psi\ (-p-k)$ where (formally)

$$\Pi_{\mu\nu}\ (k) = 4\int dq\ \frac{(k_{\mu}+q_{\mu})\ q_{\nu} - g_{\mu\nu}\ [(kq)+q^2] + (k_{\nu}+q_{\nu})\ q^{\mu}}{(k+q)^2\ \ q^2}, \quad (7)$$

$$\Sigma\ (p) = \int dq\ \frac{\hat{p}+\hat{q}}{(p+q)^2\ \ q^2}, \quad (8)$$

$$\Gamma_{\nu}(k,p) = 2\int dq \frac{\gamma^{\nu}\ \hat{k}\ (\hat{p}+\hat{q}) + \gamma^{\nu}\ (p+q)^2 - 2\ (k^{\nu}+p^{\nu}+q^{\nu})\ (\hat{p}+\hat{q})}{(p+q)^2\ (p+k+q)^2\ q^2}. \quad (9)$$

Here q is of course the only loop momentum of the skeleton. Now according to the 'universal rules' we have to 'lengthen momenta' (i.e. to make the change $q^2 \Rightarrow q^2 - \mu^2$) in the integrands of (7) - (9) and to integrate the so obtained expression with an appropriate function $\rho(\mu)$. What we'll obtain for the regularized amplitudes is

$$\Pi_{\mu\nu}\ (k) = 4\int dq \int d\mu\ \rho(\mu)\ \times$$
$$\times\ \frac{(k_{\mu}+q_{\mu})\ q_{\nu} - g_{\mu\nu}\ [(kq)+q^2-\mu^2] + (k_{\nu}+q_{\nu})\ q^{\mu}}{[(k+q)^2-\mu^2]\ (q^2-\mu^2)},$$

$$\Sigma\ (p) = \int dq \int d\mu\ \rho(\mu)\ \frac{\hat{p} + \hat{q}}{[(p+q)^2 - \mu^2]\ [q^2 - \mu^2]},$$

$$\Gamma_\nu(k,p) = 2\int dq \int d\mu\ \rho(\mu)\ \times$$

$$\times \frac{\gamma^\nu\ \hat{k}\ (\hat{p} + \hat{q}) + \gamma^\nu[(p+q)^2 - \mu^2] - 2\ (k^\nu + p^\nu + q^\nu)(\hat{p} + \hat{q})}{[(p+q)^2 - \mu^2]\ [(p+k+q)^2 - \mu^2]\ [q^2 - \mu^2]}.$$

One can check that these integrals are really finite provided the first two moments of the function ρ are zero. It is easy to see also that the Ward identities really remain true in the regularized theory. Namely the polarization operator $\Pi^{\mu\nu}$ proves to be transversal: $k^\mu \Pi_{\mu\nu} = 0$, and the vertex function Γ_ν proves to be related to the electron self-energy Σ by the well known identity

$$\Gamma_\nu\ (0,\ p) = \frac{\partial\ \Sigma\ (p)}{\partial\ p_\nu}.$$

We dwell once again that lower orders diagrams are considered here only for illustrations and in fact invariance of the 'universal regularization' is rigorously proved in the general framework of gauge theories.

Literature:

1. Zavialov O.I., Theor. Math. Phys. 76, 1 (1986) (in Russian)
2. Zavialov O.I. Renormalized Quantum Field Theory, Kluwer Academic Publishers (1989);
Proceedings of Bechyně Castle Conference, Chechoslovakia (1988), 292.
3. Lowenstein J.H., Nucl. Phys. B96, 189 (1985);
Maison D., Nuov. Cim. 26, 360 (1975)
4. Vladimirov A.N., Nucl. Phys. B100, 203 (1983)