Log-Euler's gamma multifractal scenario for products of Ornstein-Uhlenbeck type processes*

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Abstract. *We investigate the properties of multifractal products of the exponential of Ornstein-Uhlenbeck processes driven by Lévy motion. The conditions on the mean, variance and covariance functions of these processes are interpreted in terms of the moment generating functions. We provide an illustrative example of Euler's gamma distribution. We establish the corresponding log-Euler multifractal scenario for the limiting processes, including their Rényi function and dependence structure.*

Key words: *multifractal products, Ornstein-Uhlenbeck processes, Lévy processes, infinitely divisible distribution, Euler's gamma distribution*

AMS subject classifications: 60G57, 60G10, 60G17

1. Introduction

Multifractal models have been used in many applications in hydrodynamic turbulence, finance, genomics, computer network traffic, etc. (see Kolmogorov 1941, 1962, Kahane 1985, 1987, Gupta and Waymire 1993, Novikov 1994, Frisch 1995, Mandelbrot 1997, Falconer 1997). There are many ways to construct random multifractal measures ranging from the simple binomial cascade to measures generated by branching processes and the compound Poisson process (see Kahane 1985, 1987, Gupta and Waymire 1993, Falconer 1997, Barral and Mandelbrot 2002, Riedi 2003, Mörters and Shieh 2002, 2004, 2008, Shieh and Taylor 2002). Most of these multifractal models are not designed to cover other important features such as tractable dependence structure or a natural form of the singularity spectrum (see Novikov

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1994, Riedi 2003, for example). Jaffard (1999) showed that Lévy processes (except Brownian motion and Poisson processes) are multifractal; but since the increments of a Lévy process are independent, this class excludes the effects of tractable dependence structures. Moreover, Lévy processes have a linear singularity spectrum while real data often exhibit a strictly concave spectrum.

Anh, Leonenko and Shieh (2007, 2008a,b) follow a different approach. They considered multifractal products of stochastic processes as defined in Kahane (1985, 1987) and Mannersalo *et al.* (2002), but provided a new interpretation of the conditions on the mean, variance and covariance functions of the resulting cumulative processes in terms of the moment generating functions. They showed that the logarithms of the corresponding limiting processes have an infinitely divisible distribution such as the inverse Gaussian and normal inverse Gaussian distributions (yielding the log-inverse Gaussian and log-normal inverse Gaussian scenarios respectively). They also described the behaviour of the q-th order moments and Rényi functions, which are nonlinear, hence displaying the multifractality of the processes as constructed. However, an extension of these results to more general classes of distributions, such as the generalized hyperbolic distribution corresponding to the logarithm of geometric Ornstein-Uhlenbeck type processes, seems hard to establish in view of the generally complicated nature of their Lévy densities (see Barndorff-Nielsen and Shephard 2001, Subsection 5.1). This paper presents a new multifractal scenario based on Euler's gamma distribution and the corresponding theory of Ornstein-Uhlenbeck processes.

2. Multifractal products of stochastic processes

This section recaptures some basic results on multifractal products of stochastic processes as developed in Kahane (1985, 1987) and Mannersalo *et al.* (2002). We provide a new interpretation of their conditions based on the moment generating functions, which is useful for our exposition.

We introduce the following conditions:

A$'$. Let $\Lambda(t),\ \ t \in [0,1]$, be a measurable, separable, strictly stationary, positive stochastic process such that, for all $t, t_1, t_2 \in [0,1]$, the following assumptions hold:

$$\mathrm{E}\Lambda(t) = 1, \tag{2.1}$$

$$\mathrm{Var}\Lambda(t) = \sigma_\Lambda^2 < \infty, \tag{2.2}$$

$$\mathrm{Cov}(\Lambda(t_1), \Lambda(t_2)) = R_\Lambda(t_1 - t_2) = \sigma_\Lambda^2 \rho(t_1 - t_2), \quad \rho(0) = 1. \tag{2.3}$$

We call this process the mother process and consider the following setting:

A$''$. Let $\Lambda^{(i)},\ i = 0, 1, ...$ be independent copies of the mother process Λ, and $\Lambda_b^{(i)}$ be the rescaled version of $\Lambda^{(i)}$:

$$\Lambda_b^{(i)}(t) \stackrel{d}{=} \Lambda^{(i)}(tb^i), \quad t \in [0,1], \quad i = 0, 1, 2, \dots ,$$

where the scaling parameter $b > 1$, and $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions.

Moreover, in the example of Section 4, the stationary mother process satisfies the following conditions:

$\mathbf{A}'''$. For $t \in [0,1]$, let $\Lambda(t) = \exp\{X(t)\}$, where $X\ (t)$ is a stationary process with $\mathrm{E}X^2(t) < \infty$,

$$\mathrm{Cov}(X(t_1), X(t_2)) = R_X(t_1 - t_2) = \sigma_X^2 r_X(t_1 - t_2), \quad r_X(0) = 1.$$

We assume that there exist a marginal probability density function $p_\theta(x)$ and a bivariate probability density function $p_\theta(x_1, x_2; t_1 - t_2)$ such that the moment generating function

$$M(\zeta) = \mathrm{E}\exp\{\zeta X(t)\}$$

and the bivariate moment generating function

$$M(\zeta_1, \zeta_2; t_1 - t_2) = \mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\}$$

exist. Here, θ is the parameter vector of the distribution function of the process $X\ (t)$.

Under the conditions $\mathbf{A}'$-$\mathbf{A}'''$, the assumptions (2.1)-(2.3) can be rewritten as

$$\mathrm{E}\Lambda_b^{(i)}(t) = M(1) = 1;$$
$$\mathrm{Var}\Lambda_b^{(i)}(t) = M(2) - 1 = \sigma_\Lambda^2 < \infty;$$

$$\mathrm{Cov}(\Lambda_b^{(i)}(t_1), \Lambda_b^{(i)}(t_2)) = M(1, 1; (t_1 - t_2)b^i) - 1, \ \ b > 1.$$

We define the finite product processes

$$\Lambda_n(t) = \prod_{i=0}^{n} \Lambda_b^{(i)}(t) = \exp\left\{\sum_{i=0}^{n} X^{(i)}(tb^i)\right\},$$

and the cumulative processes

$$A_n(t) = \int_0^t \Lambda_n(s)ds, \quad n = 0, 1, 2, \dots,$$

where $X^{(i)}(t), i = 0, ..., n, ...,$ are independent copies of a stationary process $X(t), t \geq 0$.

We also consider the corresponding positive random measures defined on Borel sets B of $[0,1]$:

$$\mu_n(B) = \int_B \Lambda_n(s)ds, \quad n = 0, 1, 2, \dots$$

Kahane (1987) proved that the sequence of random measures μ_n converges weakly almost surely to a random measure μ. Moreover, given a finite or countable family of Borel sets B_j on $[0,1]$, it holds that $\lim_{n\to\infty} \mu_n(B_j) = \mu(B_j)$ for all j with probability one. The almost sure convergence of $A_n\ (t)$ in countably many points of $[0,1]$ can be extended to all points in $[0,1]$ if the limit process $A\ (t)$ is almost surely continuous. In this case, $\lim_{n\to\infty} A_n(t) = A(t)$ with probability one for all $t \in [0,1]$. As noted in Kahane (1987), there are two extreme cases: (i) $A_n(t) \to A(t)$ in L_1 for each given t, in which case $A(t)$ is not almost surely zero and and is said to be fully active (non-degenerate) on $[0,1]$; (ii) $A_n(1)$ converges to 0 almost surely, in which case $A(t)$ is said to be degenerate on $[0,1]$. Sufficient conditions for non-degeneracy and degeneracy in a general situation and relevant examples are provided in Kahane (1987) (Eqs. (18) and (19), respectively.) The condition for complete degeneracy is detailed in Theorem 3 of Kahane (1987).

The Rényi function, also known as the deterministic partition function, is defined as

$$\begin{aligned} T(q) &= \liminf_{n\to\infty} \frac{\log \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right)}{\log \left|I_k^{(n)}\right|} \\ &= \liminf_{n\to\infty} \left(-\frac{1}{n}\right) \log_2 \mathrm{E} \sum_{k=0}^{2^n-1} \mu^q \left(I_k^{(n)}\right), \end{aligned}$$

where $I_k^{(n)} = [k2^{-n}, (k+1)2^{-n}]$, $\quad k = 0, 1, \ldots, 2^n - 1$, $\left|I_k^{(n)}\right|$ is its length, and $\log_b$ is log to the base b.

Remark 1. *The multifractal formalism for random cascades can be stated in terms of the Legendre transform of the Rényi function:*

$$T^*(\alpha) = \min_{q\in\mathbb{R}} (q\alpha - T(q)).$$

In fact, let $f(\alpha)$ be the Hausdorff dimension of the set

$$C_\alpha = \left\{ t \in [0,1] : \lim_{n\to\infty} \frac{\log \mu\left(I_k^{(n)}(t)\right)}{\log \left|I_k^{(n)}\right|} = \alpha \right\},$$

where $I_k^{(n)}(t)$ is a sequence of intervals $I_k^{(n)}$ that contain t. The function $f(\alpha)$ is known as the singularity spectrum of the measure μ, and we refer to μ as a multifractal measure if $f(\alpha) \neq 0$ for a continuum of α (Lau 1999). In order to determine the function $f(\alpha)$, Hentschel and Procaccia(1983), Frisch and Parisi (1985) and Halsey et al. *(1986) for example proposed to use the relationship*

$$f(\alpha) = T^*(\alpha). \tag{2.4}$$

This relationship may not hold for a given measure (see, for example, Taylor 1995). When the equality (2.4) is established for a measure μ, we say that the multifractal formalism holds for this measure.

Mannersalo *et al.* (2002) presented the conditions for L_2-convergence and scaling of moments:

Theorem 1 [Mannersalo, Norros and Riedi 2002]. *Suppose that the conditions $\boldsymbol{A}'$-$\boldsymbol{A}'''$ hold.*

If, for some positive numbers δ and γ,

$$\exp\{-\delta|\tau|\} \leqslant \rho(\tau) = \frac{M(1,1;\tau) - 1}{M(2) - 1} \leqslant |C\tau|^{-\gamma}, \tag{2.5}$$

then $A_n(t)$ converges in L_2 if and only if

$$b > 1 + \sigma_\Lambda^2 = M(2).$$

If $A_n(t)$ converges in L_1, then the limit process $A(t)$ satisfies the recursion

$$A(t) = \frac{1}{b}\int_0^t \Lambda(s) d\tilde{A}(bs), \tag{2.6}$$

where the processes $\Lambda(t)$ and $\tilde{A}(t)$ are independent, and the processes $A(t)$ and $\tilde{A}(t)$ have identical finite-dimensional distributions.

If $A(t)$ is non-degenerate, the recursion (2.6) holds, $A(1) \in L_q$ for some $q > 0$, and $\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty$, where $c(q,t) = \mathrm{E}\sup_{s\in[0,t]} |\Lambda^q(0) - \Lambda^q(s)|$, then there exist constants $\overline{C}$ and $\underline{C}$ such that, for all $t \in [0,1]$,

$$\underline{C} t^{q-\log_b \mathrm{E}\Lambda^q(t)} \leqslant \mathrm{E}A^q(t) \leqslant \overline{C} t^{q-\log_b \mathrm{E}\Lambda^q(t)}, \tag{2.7}$$

which will be written as

$$\mathrm{E}A^q(t) \sim t^{q-\log_b \mathrm{E}\Lambda^q(t)}.$$

If, on the other hand, $A(1) \in L_q$, $q > 1$, then the Rényi function is given by

$$T(q) = q - 1 - \log_b \mathrm{E}\Lambda^q(t) = q - 1 - \log_b M(q). \tag{2.8}$$

If $A(t)$ is non-degenerate, $A(1) \in L_2$, and $\Lambda(t)$ is positively correlated, then

$$\mathrm{Var} A(t) \geqslant \mathrm{Var}\int_0^t \Lambda(s) ds. \tag{2.9}$$

Hence, if $\int_0^t \Lambda(s)ds$ is strongly dependent, then $A(t)$ is also strongly dependent.

Remark 2. *The result (2.7) means that the process $A(t)$, $t \in [0,1]$ with stationary increments behaves as*

$$\log \mathrm{E}\left[A(t+\delta) - A(t)\right]^q \sim K(q)\log\delta + C_q \tag{2.10}$$

for a wide range of resolutions δ with a nonlinear function

$$K(q) = q - \log_b \mathrm{E}\Lambda^q(t) = q - \log_b M(q),$$

where C_q is a constant. In this sense, the stochastic process $A(t)$ is said to be multifractal. The function $K(q)$, which contains the scaling parameter b and all the parameters of the marginal distribution of the mother process $X(t)$, can be estimated by running the regression (2.10) for a range of values of q. For the example in Section 4, the explicit form of $K(q)$ is obtained. Hence these parameters can be estimated by minimizing the mean square error between the $K(q)$ curve estimated from data and its analytical form for a range of values of q. This method has been used for multifractal characterization of complete genomes in Anh et al. *(2001).*

3. Infinitely divisible distributions and geometric OU processes

This section reviews a number of known results on Lévy processes (see Skorokhod 1991, Bertoin 1996, Sato 1999, Kyprianou 2006) and Ornstein-Uhlenbeck type processes (see Barndorff-Nielsen 2001, Barndorff-Nielsen and Shephard 2001). As standard notation we will write

$$\kappa(z) = C\{z; X\} = \log \mathrm{E} \exp\{izX\}, \quad z \in \mathbb{R}$$

for the cumulant function of a random variable X, and

$$K\{\zeta; X\} = \log \mathrm{E} \exp\{\zeta X\}, \quad \zeta \in \mathbb{R}$$

for the Lévy exponent or Laplace transform or cumulant generating function of the random variable X. Its domain includes the imaginary axis and frequently larger areas.

A random variable X is *infinitely divisible* if its cumulant function has the Lévy-Khintchine form

$$C\{z; X\} = iaz - \frac{d}{2}z^2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - izu\mathbf{1}_{[-1,1]}(u)\right)\nu(du), \tag{3.1}$$

where $a \in \mathbb{R}$, $d \geq 0$ and ν is the Lévy measure, that is, a non-negative measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}} \min\left(1, u^2\right)\nu(du) < \infty. \tag{3.2}$$

The triplet (a, d, ν) uniquely determines the random variable X. For a Gaussian random variable $X \sim N(a, d)$, the Lévy triplet takes the form $(a, d, 0)$.

A random variable X is *self-decomposable* if, for all $c \in (0,1)$, the characteristic function $f(z)$ of X can be factorized as $f(z) = f(cz)\, f_c(z)$ for some characteristic function $f_c(z)$, $z \in \mathbb{R}$. Recall that a homogeneous Lévy process $Z = \{Z(t),\, t \geq 0\}$ is a continuous (in probability), càdlàg stochastic process with independent and stationary increments and $Z(0) = 0$. For such processes we have $C\{z; Z(t)\} = tC\{z; Z(1)\}$ and $Z(1)$ has the Lévy-Khintchine representation (3.1).

Let $f(z)$ be the characteristic function of a random variable X. If X is self-decomposable, then there exists a stationary stochastic process $\{X(t),\, t \geq 0\}$, such that $X(t) \overset{d}{=} X$ and

$$X(t) = e^{-\lambda t}X(0) + \int_{(0,t]} e^{-\lambda(t-s)}dZ(\lambda s) \tag{3.3}$$

for all $\lambda > 0$ (see Barndorff-Nielsen 1998). Conversely, if $\{X(t), t \geq 0\}$ is a stationary process and $\{Z(t), t \geq 0\}$ is a Lévy process, independent of $X(0)$, such that $X(t)$ and $Z(t)$ satisfy the Itô stochastic differential equation

$$dX(t) = -\lambda X(t)\, dt + dZ(\lambda t) \tag{3.4}$$

for all $\lambda > 0$, then $X(t)$ is self-decomposable. A stationary process $X(t)$ of this kind is said to be an Ornstein-Uhlenbeck type process or an OU-type process, for short.

The process $Z(t)$ is termed the background driving Lévy process (BDLP) corresponding to the process $X(t)$. In fact (3.3) is the unique (up to indistinguishability) strong solution to Eq. (3.4) (Sato 1999). The meaning of the stochastic integral in (3.3) was detailed in Applebaum (2004, p. 214).

A necessary and sufficient condition for (3.4) to have a stationary solution is that

$$\mathrm{E}(\log(1+|Z(1)|)) < \infty.$$

The stationary process $\{X(t), t \geq 0\}$ can be extended to a stationary process on the whole real line. To do this, we introduce an independent copy of the process $\{Z(t), t \geq 0\}$ but modify it to be càdlàg, thus obtaining a process $\{\overline{Z}(t), t \geq 0\}$, say. Now, for $t < 0$, define $Z(t) = \overline{Z}(-t)$, and for $t \in \mathbb{R}$ let

$$X(t) = e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} dZ(\lambda s).$$

Then, $\{Z(t), t \in \mathbb{R}\}$ is a homogeneous càdlàg Lévy process; and $\{X(t), t \in \mathbb{R}\}$ is a strictly stationary process of OU-type.

Let $X(t)$ be a square integrable OU process. Then $X(t)$ has the correlation function

$$r_X(t) = \exp\{-\lambda |t|\}, \quad t \in \mathbb{R}. \tag{3.5}$$

The cumulant transforms of $X = X(t)$ and $Z(1)$ are related by

$$C\{z; X\} = \int_0^\infty C\left\{e^{-s}z; Z(1)\right\} ds = \int_0^z C\{\xi; Z(1)\} \frac{d\xi}{\xi}$$

and

$$C\{z; Z(1)\} = z\frac{\partial C\{z; X\}}{\partial z}.$$

Suppose that the Lévy measure ν of X has a density function $p(u), u \in \mathbb{R}$, which is differentiable. Then the Lévy measure $\tilde{\nu}$ of $Z(1)$ has a density function $q(u), u \in \mathbb{R}$, and p and q are related by

$$q(u) = -p(u) - up'(u) \tag{3.6}$$

(see Barndorff-Nielsen 1998).

The logarithm of the characteristic function of a random vector $(X(t_1), ..., X(t_m))$ is of the form

$$\log \mathrm{E} \exp\{i(z_1 X(t_1) + ... + z_m X(t_m)\}$$
$$= \int_{\mathbb{R}} \kappa(\sum_{j=1}^{m} z_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))ds, \tag{3.7}$$

where

$$\kappa(z) = \log \mathrm{E} \exp\{izZ(1)\} = C\{z; Z(1)\},$$

and the function (3.7) has the form (3.1) with Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$.

The logarithms of the moment generation functions (if they exist) take the forms

$$\log M(\zeta) = \log \mathrm{E} \exp \{\zeta X(t)\} = \zeta a + \frac{d}{2}\zeta^2 + \int_{\mathbb{R}} (e^{\zeta u} - 1 - \zeta u \mathbf{1}_{[-1,1]}(u))\nu(du),$$

where the triplet (a, d, ν) is the Lévy triplet of $X(0)$, or in terms of the Lévy triplet $(\tilde{a}, \tilde{d}, \tilde{\nu})$ of $Z(1)$

$$\begin{aligned}\log M(\zeta) &= \tilde{a} \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))ds + \frac{\tilde{d}}{2}\zeta^2 \int_{\mathbb{R}} (\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s))^2 ds \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp\left\{u\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\right\} - 1 \\ &- u\left(\zeta e^{-\lambda(t-s)} \mathbf{1}_{[0,\infty)}(t-s)\right) \mathbf{1}_{[-1,1]}(u)]\tilde{\nu}(du)\, ds, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\log M(\zeta_1, \zeta_2; t_1 - t_2) = \log \mathrm{E} \exp\{\zeta_1 X(t_1) + \zeta_2 X(t_2)\} \\ &= \tilde{a} \int_{\mathbb{R}} (\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))ds + \frac{\tilde{d}}{2}\zeta^2 \int_{\mathbb{R}} (\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s))^2 ds \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} [\exp\left\{u \sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s)\right\} - 1 \\ &\qquad - u\left(\sum_{j=1}^{2} \zeta_j e^{-\lambda(t_j - s)} \mathbf{1}_{[0,\infty)}(t_j - s)\right) \mathbf{1}_{[-1,1]}(u)]\tilde{\nu}(du)\, ds. \end{aligned} \tag{3.9}$$

The following result plays a key role in mutifractal analysis of geometric OU processes.

Theorem 2. *Let $X(t), t \in [0,1]$ be an OU-type stationary process (3.3) such that the Lévy measure ν in (3.1) of the random variable $X(t)$ satisfies the condition that for some $q \in Q \subseteq \mathbb{R}$,*

$$\int_{|x|\geq 1} g(x)\nu(dx) < \infty, \tag{3.10}$$

where $g(x)$ denotes any of $e^{2qx}, e^{qx}, e^{qx}|x|$. Then, for the geometric OU-type process $\Lambda_q(t) := e^{qX(t)}$,

$$\sum_{n=0}^{\infty} c(q, b^{-n}) < \infty,$$

where $c(q,t) = \mathrm{E}\sup_{s\in[0,t]} |\Lambda_q(0)^q - \Lambda_q(s)^q|$.

To prove that our geometric OU-type process satisfies the covariance decay condition (2.5) in Theorem 1, the expression given by (3.9) is not ready to yield the decay as $t_2 - t_1 \to \infty$. The following proposition gives a general decay estimate which the driving Lévy processes Z in Section 4 below indeed satisfy.

Proposition 1. *Consider an OU-type process X given by*

$$dX(t) = -\lambda X(t)dt + dZ(\lambda t),$$

where the BDLP Z is without Gaussian part (that is, $\tilde{d} = 0$ in (3.8)), and the Lévy measure $\tilde{\nu}(dx)$ of Z has the density $g(x)$ for which there exists some $\beta > 0$ so that $g(x) \leq const \times e^{-\beta|x|}$ for all $|x| > 1$. Then there exist positive constants c, C such that

$$E[e^{X(t)}e^{X(0)}] \leq Ce^{-ct}$$

for all t sufficiently large.

Remark 3. *The constant c is given by $\tilde{\nu}(|x| > 1)$. If the region for the boundedness assumption on $g(x)$ is $|x| > a, a > 1$, then c is determined by $\tilde{\nu}(|x| > a)$.*

The proofs of Theorem 2 and Proposition 1 are given in Anh, Leonenko and Shieh (2007), denoted below as ALSh07.

Very often the correlation structure found in applications is more complex than the exponential decreasing autocorrelation of the form (3.5) (see, for example Anh and Leonenko 1999). Barndorff-Nielsen (1998) (see also Barndorff-Niesen and Sheppard 2001, Barndorff-Nieslen and Leonenko 2005) proposed to consider the following class of autocovariance functions:

$$R_{\sup}(t) = \sum_{j=1}^{m} \sigma_j^2 \exp\{-\lambda_j |t|\}, \tag{3.11}$$

which is flexible and can be fitted to many autocovariance functions arising in applications.

In order to obtain models with dependence structure (3.11) and given marginal density with finite variance, we consider stochastic processes defined by

$$dX_j(t) = -\lambda_j X_j(t)\,dt + dZ_j(\lambda_j t),\ j = 1, 2, ..., m,$$

and their superposition

$$X_{\sup}(t) = X_1(t) + ... + X_m(t),\ t \geq 0, \tag{3.12}$$

where $Z_j,\ j = 1, 2, ..., m$, are mutually independent Lévy processes. Then the solution $X_j = \{X_j(t), t \geq 0\}$, $j = 1, 2, ..., m$, is a stationary process. Its correlation function is of the exponential form (assuming finite variance).

The superposition (3.12) has its marginal density given by that of the random variable

$$X_{\sup}(0) = X_1(0) + ... + X_m(0), \tag{3.13}$$

autocovariance function (3.11) (where σ_j^2 are now variances of X_j), and spectral density

$$f_{\sup}(\lambda) = \frac{2}{\pi}\sum_{j=1}^{m} \sigma_j^2 \frac{\theta_j}{\theta_j + \lambda^2},\ \lambda \in \mathbb{R}.$$

We are interested in the case when the distribution of (3.13) is tractable, for instance when $X_{\sup}(0)$ belongs to the same class as $X_j(0), j = 1, ..., m$ (see the example in Section 4 below).

Note that an infinite superposition ($m \to \infty$) gives a complete monotone class of covariance functions

$$R_{\sup}(t) = \int_0^\infty e^{-tu}dU(u), t \geq 0,$$

for some finite measure U, which display long-range dependence (see Barndorff-Nielsen 1998, 2001, Barndorff-Nielsen and Leonenko 2005 for possible covariance structures and spectral densities).

4. Log-Euler's gamma multifractal scenario

This section presents a new scenario which is based on Euler's gamma distribution (see, for example, Grigelionis 2003). We consider the random variable Y with the gamma distribution $\Gamma(\beta, \alpha)$ having probability density function (pdf)

$$\pi(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \mathbf{1}_{[0,\infty)}(x), \alpha > 0, \beta > 0,$$

and the random variable

$$X_\gamma = \gamma \log Y, \gamma \neq 0,$$

which has the pdf

$$\pi(x) = \frac{\alpha^\beta}{|\gamma| \Gamma(\beta)} \exp\left\{ \frac{\beta x}{\gamma} - \alpha e^{\frac{x}{\gamma}} \right\}, \quad x \in \mathbb{R}, \tag{4.1}$$

where the parameters satisfy

$$\alpha > 0, \beta > 0, \gamma \neq 0.$$

The characteristic function of random variable X with pdf (4.1) is

$$\mathrm{E} e^{izX_\gamma} = \frac{\Gamma(\beta + i\gamma z)}{\Gamma(\beta) \alpha^{i\gamma z}}, z \in \mathbb{R}.$$

Grigelionis (2003) proved that for

$$\delta > 0, \alpha > 0, \beta > 0, \gamma \neq 0, \tag{4.2}$$

the function

$$\mathrm{E} e^{izX} = \left(\frac{\Gamma(\beta + i\gamma z)}{\Gamma(\beta) \alpha^{i\gamma z}} \right)^\delta, z \in \mathbb{R} \tag{4.3}$$

is a self-decomposable characteristic function. We denote the distribution of the random variable X by $\Gamma(\gamma, \alpha, \beta, \delta)$.

We note that $\Gamma(\gamma, e^{-\frac{\theta}{\gamma}}, 1, 1), \theta \in \mathbb{R}$, is the Gumbel distribution with location parameter θ and scale parameter $|\gamma|$, since with

$$\Lambda(x) = \exp\left\{-e^{-x}\right\}, \bar{\Lambda}(x) = 1 - \Lambda(-x), x \in \mathbb{R},$$

$$P\{X \leq x\} = \begin{cases} \Lambda\left(\frac{x-\theta}{|\gamma|}\right), \gamma < 0, \\ \bar{\Lambda}\left(\frac{x-\theta}{\gamma}\right), \gamma > 0 \end{cases}, x \in \mathbb{R}.$$

We will use a stationary OU-type process (3.4) with marginal distribution $\Gamma(\gamma,\alpha,\beta,\delta)$, which is self-decomposable, and hence infinitely divisible. It means that the characteristic function of $X(t), t\in[0,1]$ is of the form (4.3) under the set of parameters (4.2). Note that, for $\beta>0$, we have

$$\Gamma(\beta+iz)=\Gamma(\beta)\exp\{iz\int_0^\infty\left(\frac{e^{-x}}{x}-\frac{e^{-\beta x}}{1-e^{-x}}1_{\{0\leq x\leq 1\}}\right)dx$$

$$+\int_{-\infty}^0(e^{izx}-1-izx1_{\{-1\leq x<0\}})\frac{e^{\beta x}}{|x|\,(1-e^{x})}dx\},$$

and thus the distribution corresponding to the characteristic function (4.3) has the Lévy triplet $(\delta a,0,\nu)$, where

$$a=\gamma\int_0^{\frac{1}{|\gamma|}}\left(\frac{e^{-x}}{x}-\frac{e^{-\beta x}}{1-e^{-x}}\right)dx+\gamma\int_{\frac{1}{|\gamma|}}^\infty\frac{e^{-x}}{x}dx-\gamma\log\alpha,$$

and

$$\nu(du)=\delta b(u)du, \tag{4.4}$$

$$b(u)=\begin{cases}\frac{e^{\frac{\beta}{\gamma}u}}{|u|\left(1-e^{\frac{1}{\gamma}u}\right)}, & u<0,\gamma>0,\\ \frac{e^{\frac{\beta}{\gamma}u}}{u\left(1-e^{\frac{1}{\gamma}u}\right)}, & u>0,\gamma<0.\end{cases}$$

Thus, if $X_j(t),\ \ j=1,...,m$, are independent so that $X_j(t)\sim\Gamma(\gamma,\alpha,\beta,\delta_j)$, $j=1,...,m$, then we have that

$$X_1(t)+...+X_m(t)\sim\Gamma(\gamma,\alpha,\beta,\delta_1+...+\delta_m)$$

and if $X_j(t),\ \ j=1,...,m$, are independent so that $X_j(t)\sim\Gamma(\gamma,\alpha_j,\beta,\delta)$, $j=1,...,m$, then

$$X_1(t)+...+X_m(t)\sim\Gamma(\gamma,\prod_{j=1}^m\alpha_j,\beta,\delta).$$

The BDLP $Z(t)$ in (3.5) has a Lévy triplet $(\tilde{a},0,\tilde{\nu})$, where

$$\tilde{a}=\gamma\lambda\delta\frac{\frac{d}{d\beta}\Gamma(\beta)}{\Gamma(\beta)}+\gamma\lambda\delta\log\alpha-\lambda\delta\int_{|x|>1}x\omega(x)dx,$$

with the density of $\tilde{\nu}$ given by

$$\tilde{\nu}(du)=\lambda\delta\omega(u)du, \tag{4.5}$$

$$\omega(u)=\begin{cases}\frac{\beta}{\gamma}e^{\frac{\beta}{\gamma}u}\left(1-e^{\frac{1}{\gamma}u}+\frac{1}{\beta}e^{\frac{1}{\gamma}u}\right)\frac{1}{\left(1-e^{\frac{1}{\gamma}u}\right)^2}, & \gamma>0,u<0,\\ \frac{\beta}{|\gamma|}e^{\frac{\beta}{\gamma}u}\left(1-e^{\frac{1}{\gamma}u}+\frac{1}{\beta}e^{\frac{1}{\gamma}u}\right)\frac{1}{\left(1-e^{\frac{1}{\gamma}u}\right)^2}, & \gamma<0,u>0.\end{cases}$$

The correlation function of the stationary process $X(t)$ then takes the form

$$r_X(t) = \exp\{-\lambda|t|\}, t \in [-2,2].$$

Note that

$$\mathrm{E}X(t) = \gamma\delta\frac{\frac{d}{d\beta}\Gamma(\beta)}{\Gamma(\beta)} - \gamma\delta\log\alpha, \mathrm{Var}X(t) = \delta\gamma^2\int_0^\infty \frac{xe^{-\beta x}}{1-e^{-x}}dx.$$

B′. Consider a mother process of the form

$$\Lambda(t) = \exp\{X(t) - c_X\},$$

with

$$c_X = \delta\log\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)e^{\gamma}}, \beta > 0, \gamma < 0, \beta > -\gamma.$$

where $X(t), t \in [0,1]$ is a stationary $\Gamma(\gamma,\alpha,\beta,\delta)$ OU-type stochastic process with covariance function

$$R_X(t) = \mathrm{Var}X(t)\exp\{-\lambda|t|\}, t \in [-2,2].$$

The logarithm of the moment generating function of $\Gamma(\gamma,\alpha,\beta,\delta)$ is

$$\log M_\theta(\zeta) = \delta\log\frac{\Gamma(\beta+\gamma\zeta)}{\Gamma(\beta)e^{\gamma\zeta}}, 0 < \zeta < -\frac{\beta}{\gamma}, \beta > 0, \gamma < 0.$$

Under condition **B′**, we obtain the following moment generating function

$$M_\theta(\zeta) = \mathrm{E}\exp\{\zeta(X(t) - c_X)\} = e^{-c_X\zeta}e^{M_\theta(\zeta)}, \quad 0 < \zeta < -\frac{\beta}{\gamma}, \tag{4.6}$$

and bivariate moment generating function

$$\begin{aligned} M_\theta(\zeta_1,\zeta_2;(t_1-t_2)) &= \mathrm{E}\exp\{\zeta_1(X(t_1)-c_X) + \zeta_2(X(t_2)-c_X)\} \\ &= e^{-c_X(\zeta_1+\zeta_2)}\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2(X(t_2)\}, \end{aligned} \tag{4.7}$$

where $\theta = (\gamma,\alpha,\beta,\delta)$, and $\mathrm{E}\exp\{\zeta_1 X(t_1) + \zeta_2(X(t_2)\}$ is given by (3.8) with Lévy measure $\tilde{\nu}$ having density (4.5). Thus, the correlation function of the mother process takes the form

$$\rho(\tau) = \frac{M_\theta(1,1;\tau) - 1}{M_\theta(2) - 1}, \tag{4.8}$$

where $M_\theta(2)$ is given by (4.6) and $M_\theta(1,1;\tau)$ is given by (4.7).

Theorem 3. *Suppose that condition **B′** holds and*

$$q \in Q = \left\{q;\ 0 < q < -\frac{\beta}{\gamma}, \beta > 0, \gamma < 0, \beta > -\gamma\right\}.$$

Then, for any

$$b > \delta\log\frac{\Gamma(\beta)\Gamma(\beta+2\gamma)}{\Gamma^2(\beta+\gamma)} > 1,$$

the stochastic processes

$$A_n(t) = \int_0^t \prod_{j=0}^{n} \Lambda^{(j)}\left(sb^j\right) ds, t \in [0,1]$$

converge in L_2 *to the stochastic process* $A(t)$, $t \in [0,1]$ *as* $n \to \infty$ *such that, if* $A(1) \in L_q$, $q > 1$ *and* $q \in Q$, *then*

$$\mathrm{E}A^q(t) \sim t^{q\left(1+\frac{\delta}{\log b}\log\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)}\right)-\frac{\delta}{\log b}\log\Gamma(\beta+q\gamma)+\frac{1}{\log b}\delta\log\Gamma(\beta)},$$

The Rényi function is given by

$$T(q) = q\left(1+\frac{\delta}{\log b}\log\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)}\right) - \frac{\delta}{\log b}\log\Gamma(\beta+q\gamma) + \frac{1}{\log b}\delta\log\Gamma(\beta) - 1.$$

Moreover,

$$VarA(t) \geqslant \int_0^t\int_0^t \left[M_\theta(1,1;u-w)-1\right] du dw,$$

where M_θ *is given by (4.7).*

Proof. Theorem 3 follows from Theorems 1&2 and Proposition 1. □

Remark 4. *For* $q \in Q \cap [1,2]$, *the condition* $A(1) \in L_q$, $q > 1$ *is not needed; thus, at least for this range, the above result holds true. However, for q outside this range, the condition is partly required for the validity of the multifractal moment scaling.*

We can construct log-$\Gamma(\gamma,\alpha,\beta,\delta)$ scenario for a more general class of finite superpositions of stationary OU-type processes:

$$X_{\sup}(t) = \sum_{j=1}^{m} X_j(t), t \in [0,1],$$

where $X_j(t), j = 1,...,m$, are independent stationary processes with mariginals $X_j(t) \sim \Gamma(\gamma,\alpha,\beta,\delta_j), j = 1,...,m$, and parameters $\delta_j,\ j = 1,...,m$. Then $X_{\sup}(t), t \in [0,1]$ has the marginal distribution $\Gamma(\gamma,\alpha,\beta,\sum_{j=1}^{m}\delta_j)$, and covariance function

$$R_{\sup}(t) = \left[\gamma^2\int_0^\infty x\frac{e^{-\beta x}}{1-e^{-x}}dx\right]\sum_{j=1}^{m}\delta_j\exp\left\{-\lambda_j|t|\right\}, t \in [-2,2].$$

The generalization of Theorem 2 and Proposition 1 to this situation is straightforward and the statement of Theorem 3 can be reformulated for the process of superposition $X_{\sup}$ with $\delta = \sum_{j=1}^{m}\delta_j$, and

$$M_\theta(\zeta_1,\zeta_2;(t_1-t_2)) = \prod_{j=1}^{m} M_{\theta_j}(\zeta_1,\zeta_2;(t_1-t_2)),\ \theta = (\gamma,\alpha,\beta,\delta),\ \theta_j = (\gamma,\alpha,\beta,\delta_j),$$

and λ must be replaced by λ_j in the expression (4.7) for $M_{\theta_j}(\zeta_1,\zeta_2;(t_1-t_2))$.

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