The Bing-Borsuk and the Busemann conjectures*

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Abstract. *We present two classical conjectures concerning the characterization of manifolds: the Bing Borsuk conjecture asserts that every n-dimensional homogeneous ANR is a topological n-manifold, whereas the Busemann conjecture asserts that every n-dimensional G-space is a topological n-manifold. The key object in both cases are so-called generalized manifolds, i.e. ENR homology manifolds. We look at the history from the early beginnings to the present day. We also list several open problems and related conjectures.*

Key words: *Bing-Borsuk conjecture, homogeneity, ANR, Busemann G-space, Busemann conjecture, Moore conjecture, de Groot conjecture, generalized manifold, cell-like resolution, general position property, delta embedding property, disjoint disks property, recognition theorem*

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1. Introduction

In this paper we will survey two famous conjectures, their relationship to each other, as well as their relationship to other famous unsolved problems. The first, and most general, is the Bing-Borsuk conjecture [18] which states that any n-dimensional homogeneous ENR space is an n-manifold. The 3-dimensional Bing-Borsuk conjecture implies the celebrated Poincaré conjecture, recently proven by Perel'man [90]. Given the complexity of the proof of the Poincaré conjecture, it is understandable why in particular, the 3-dimensional Bing-Borsuk conjecture remains unsolved.

The Busemann conjecture [38]-[40] is a special case of the Bing-Borsuk conjecture and it states that Busemann G-spaces are manifolds. Busemann G-spaces are well known to be homogeneous. Therefore the truth of the Bing-Borsuk conjecture immediately implies the truth of the Busemann conjecture. The Busemann conjecture has been proven for G-spaces of dimensions ≤ 4 (see [39],[79],[118]).

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Three other famous conjectures related to the Busemann conjecture are the Moore conjecture and two de Groot conjectures. The Moore conjecture [52], [126] is to determine whether or not all resolvable generalized manifolds are codimension one manifold factors. Small metric balls in Busemann G-spaces are known to be cones over their boundaries, and hence have a local product structure with respect to their boundaries which are known to be generalized $(n-1)$-manifolds. An affirmative answer to the Moore conjecture together with the resolvability of Busemann G-spaces implies an affirmative answer to the Busemann conjecture.

The de Groot conjectures [63] are to determine whether or not all absolute cones are balls (resp. whether or not all absolute suspensions are spheres). Small metric balls in Busemann G-spaces are also absolute cones. It has recently been proven that absolute cones are n-cells in dimensions $n \leq 4$, but there are counter-examples in higher dimensions [69]. The solution to the $n = 4$ case relies upon the Poincaré conjecture. Unfortunately, this result does not provide a solution to the Busemann conjecture in dimension $n \geq 5$.

The purpose of this paper is to survey the work that has been done on these manifold recognition problems. In Section 2 we will delineate important properties that are known to be satisfied by manifolds. In Section 3 we will provide an overview of progress that has been made towards resolving the Bing-Borsuk conjecture. In Section 4 we will do the same for the Busemann conjecture. In Section 5 we will discuss three related problems: the Moore conjecture and two de Groot conjectures. In Sections 6 and 7 we will provide a list of relevant problems that remain unsolved.

2. Manifolds and manifold properties

An *n-manifold* is a separable metric space such that each point has a neighborhood homeomorphic to the Euclidean n-space $\mathbb{R}^n$. Although this is a simple definition to state, applying the definition to verify that a space is a manifold can be a difficult task. Thus it is desirable to find alternate methods of detecting manifolds. In this section we describe the properties and characteristics known to be possed by manifolds. The question is which property or combination of properties are sufficient to imply that a space is a manifold (see [41],[101],[104]).

A topological space X is said to be *homogeneous* if for any two points $x_1, x_2 \in X$, there is a homeomorphism of X onto itself taking x_1 to x_2. It is a classical result that closed (i.e. connected compact without boundary) manifolds are homogeneous. It is the Bing-Borsuk conjecture that asks whether a homogeneous space is necessarily a manifold [18].

A topological space X is said to have the *invariance of domain property* if for every pair of homeomorphic subsets $U, V \subset X$, U is open if and only if V is open. Brouwer [25],[26] proved a century ago that every topological n-manifold has the invariance of domain property. Unfortunately, the invariance of domain property is not sufficient by itself to characterize manifolds (see e.g. [60],[121],[122],[123]).

An n-dimensional compact metric space X is called an *n-dimensional Cantor manifold* if whenever X can be expressed as the union $X = X_1 \cup X_2$ of its proper closed subsets, then $\dim(X_1 \cap X_2) \geq n-1$. Urysohn [119],[120], who introduced this notion in 1925, proved that every topological n-manifold is a Cantor n-manifold.

More fundamental results were established by Aleksandrov [1] in 1928. Krupski [80] proved in 1993 a more general result, namely, that every generalized n-manifold is a Cantor n-manifold.

A metric space (X, ρ) is said to have the *disjoint* (k, m)*-cells property* $(k, m \in \mathbb{N})$ if for each pair of maps $f : B^k \to X$ and $g : B^m \to X$ and every $\varepsilon > 0$ there exist maps $f' : B^k \to X$ and $g' : B^m \to X$ such that

$$\rho(f, f') < \varepsilon, \quad \operatorname{dist}(g, g') < \varepsilon \quad \text{and} \quad f'(B^k) \cap g'(B^m) = \emptyset.$$

It is well known that topological manifolds of dimension n have the disjoint (k, m)-cells property for $k + m + 1 \leq n$ (see [108]). The disjoint $(2, 2)$-cells property is often referred to the *disjoint disks property* and plays a key role in characterizing manifolds of dimension $n \geq 5$.

A space X is said to be *locally* k*-connected,* LC^k $(k \geq 0)$, if for every point $x \in X$ and every neighborhood $U \subset X$ of x, there exists a neighborhood $V \subset U$ of x such that the inclusion-induced homeomorphisms $\pi_{i \leq k}(V) \to \pi_{i \leq k}(U)$ are trivial. Clearly, locally contractible spaces, such as manifolds and polyhedra, are locally k connected for all k.

Let Y be a metric space. Then Y is said to be an *absolute neighborhood retract (ANR)* provided for every closed embedding $e : Y \to Z$ of Y into a metric space Z, there is an open neighborhood U of the image $e(Y)$ which retracts to $e(Y)$. That is, there is a continuous surjection $r : U \to e(Y)$ with $r(x) = x$ for all $x \in e(Y)$. It is a classical result that finite-dimensional spaces that are ANR's are characterized as the locally contractible separable metric spaces [19]. A *Euclidean neighborhood retract* (ENR) is a finite-dimensional, locally compact, locally contractible subset X of the Euclidean n-space $\mathbb{R}^n$. It follows immediately from local contractibility that every topological manifold is an ENR (hence an ANR).

An n-dimensional $(n \in \mathbb{N})$ locally compact Hausdorff space X is called a $\mathbb{Z}$*-homology* n*-manifold* (n-hm$_{\mathbb{Z}}$) if for every point $x \in X$ and all $k \in \mathbb{N}$, $H_k(X, X \setminus \{x\}; \mathbb{Z}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$. Trivially, every topological manifold is a homology manifold. An n-dimensional topological space X is called a *generalized* n*-manifold* $(n \in \mathbb{N})$ if X is an ENR $\mathbb{Z}$-homology n-manifold. It follows that every topological n-manifold is a generalized n-manifold. Every generalized $(n \leq 2)$-manifold is known to be a topological n-manifold [126]. On the other hand, for every $n \geq 3$ there exist *totally singular* generalized n-manifolds X, i.e. X is not locally Euclidean at any point (see [41], [52], [101], [104]).

A natural way in which a generalized manifold may arise is as the image of a cell-like map defined on a manifold. A proper onto map $f : M \to X$ is said to be *cell-like* if for every point $x \in X$ the point-inverse $f^{-1}(x)$ contracts in any neighborhood of itself (i.e., $f^{-1}(x)$ has the shape of a point)[81]. A space X that is the proper image of a cell-like map is said to be *resolvable.* Trivially, every topological manifold is resolvable.

The following classical result attests to the crucial importance of cell-like maps in geometric topology (it was proved for $n \leq 2$ by Wilder [126], for $n = 3$ by Armentrout [4], for $n = 4$ by Quinn [96], and for $n \geq 5$ by Siebenmann [111]):

Theorem 2.1 [Cell-like Approximation Theorem]. *For every* $\varepsilon > 0$*, every* $n \in \mathbb{N}$*, and every cell-like map* $f : M^n \to N^n$ *between topological* n*-manifolds* M^n

and N^n, there exists a homeomorphism $h : M^n \to N^n$ such that $d(f(x), h(x)) < \varepsilon$ for every $x \in M^n$.

The fact that not all resolvable generalized manifolds are manifolds has been known since the mid 1950's, when Bing [16],[17] constructed his famous Dogbone space as the cell-like image of a map defined on $\mathbb{R}^3$. Generalized manifolds have been a subject of intense studies since 1960's [104]. In the mid 1970's Cannon recognized that the disjoint $(2,2)$-cells property, often referred to as the disjoint disks property (DDP), plays a key role in characterizing manifolds of dimension $n \geq 5$. Recall that a metric space X is said to have the *disjoint disks property* (DDP) if for every $\varepsilon > 0$ and every pair of maps $f, g : B^2 \to X$ there exist ε-approximations $f', g' : B^2 \to X$ with disjoint images $f'(B^2) \cap g'(B^2) = \emptyset$.

Cannon [43] utilized the DDP property to solve the celebrated Double suspension problem [41], [82], [101] which asks if the double suspension $\Sigma^2(H^n)$ of an arbitrary homology n-sphere H^n, $n \geq 3$, is the $(n+2)$-sphere S^{n+2}. In 1978, Edwards [61] [62] generalized Cannon's results by proving the famous Cell-like approximation theorem (for a detailed proof for $n = 5$ see [54] and for $n \geq 6$ see [52]), and at the same time also gave an affirmative answer of the high-dimensional case of the Manifold Recognition Problem (which asks if every resolvable generalized $(n \geq 3)$-manifold with the "appropriate amount of general position" is a topological n-manifold [41], [42], [48], [51], [101]-[104]):

Theorem 2.2 [Edwards]. *For $n \geq 5$, topological n-manifolds are precisely the n-dimensional resolvable spaces with the disjoint disks property.*

An analogous result for 3-manifolds was proved in the early 1980's by Daverman and Repovš [56],[57] (whereas only partial results are known in dimension 4, see [15], [56]). A metric space X is said to have the *Spherical simplicial approximation property* (SSAP) if for each $\mu : S^2 \to X$ and each $\varepsilon > 0$, there exist a map $\psi : S^2 \to X$ and a finite topological 2–complex $K_\psi \subset X$ such that: (1) $d(\psi, \mu) < \varepsilon$; (2) $\psi(S^2) \subset K_\psi$; and (3) $X \backslash K_\psi$ is 1–FLG in X. (The 1–FLG condition is known to characterize timely embedded 2–complexes K_ψ in 3–manifolds M^3, see [92]).

Theorem 2.3 [Daverman-Repovš]. *Topological 3-manifolds are precisely the 3-dimensional resolvable spaces with the simplicial spherical approximation property.*

It had been a long time a question as to whether all generalized manifolds are resolvable - this was the famous Resolution conjecture [32]-[36], [41], [44]-[47] [75], [88], [101]-[104]:

Conjecture 2.4 [Generalized Manifolds Resolution conjecture]. *Every generalized $(n \geq 3)$-manifold has a resolution.*

In dimension 3, the Generalized Manifolds Resolution conjecture 2.4 implies the Poincaré conjecture [101] and only special cases are known [23], [24], [33], [34], [48], [58], [100], [102], [105], [115]-[117]. In higher dimensions the Generalized Manifolds Resolution conjecture 2.4 turns out to be false. By the results of Bryant et al. [30] from 1996 which provide the construction (together with the work of Pedersen et al. [93] from 2003 which provided some key details on the surgery exact sequence used in the original construction), it is now known that there exist non-resolvable generalized n-manifolds, for every $n \geq 6$. In 2007 Bryant et al. [31] further strengthened their result to the following DDP Theorem:

Theorem 2.5 [Bryant-Ferry-Mio-Weinberger]. *There exist non-resolvable generalized n-manifolds with the disjoint disks property, for every $n \geq 7$.*

Hence, generalized manifolds may possess nice general position properties. Moreover, Krupski has shown that all generalized manifolds are Cantor manifolds (see Proposition 1.7 of [80]). Thus, the majority of properties listed above are known to be insufficient by themselves to characterize manifolds. Homogeneity is the remaining single candidate. Is this property strong enough to characterize manifolds? Are there other combinations of these properties that characterize manifolds which have not yet been discovered? [104]

3. The Bing-Borsuk conjecture

Bing and Borsuk [18] proved in 1965 that for $n < 3$ every n-dimensional homogeneous ANR is a topological n-manifold. They also conjectured that this holds in all dimensions:

Conjecture 3.1 [Bing-Borsuk conjecture]. *Every n-dimensional, $n \in \mathbb{N}$, homogeneous ANR is a topological n-manifold.*

Jakobsche [77] proved in 1978 that in dimension $n = 3$ the Bing-Borsuk conjecture 3.1 implies the Poincaré conjecture (see also [78]). Given the difficulty of the proof of the Poincaré conjecture [90], it is understandable why the Bing-Borsuk conjecture 3.1 remains unsolved.

3.1. Partial results

Although there is much work to be done before the Bing-Borsuk conjecture 3.1 is be solved, there are several partial results. In 1970 Bredon [21],[22] showed the following:

Theorem 3.2 [Bredon]. *If X is an n-dimensional homogeneous ENR ($n \in \mathbb{N}$) and for some (and, hence all) points $x \in X$, the groups $H_k(X, X - \{x\}; \mathbb{Z})$ are finitely generated, then X is a $\mathbb{Z}$-homology n-manifold.*

This theorem was reproved by Bryant [27] in 1987 with a more geometric argument. In 1976 Lysko [83] showed:

Theorem 3.3 [Lysko]. *Let X be a connected finite-dimensional homogeneous ANR-space. Then X is a Cantor manifold and it possesses the invariance of the domain property.*

In 1985 Seidel [110] proved a similar result in the case of locally compact, locally homogeneous separable ANR's.

Next, we quote the following result by Krupski [80] from 1993:

Theorem 3.4 [Krupski]. *Let X be a homogeneous locally compact space. Then: (1) If X is an ANR of dimension > 2, then X has the disjoint $(0, 2)$-cells property. (2) If $\dim X = n > 0$, X has the disjoint $(0, n-1)$-cells property and X is an LC^{n-1}-space, then local homologies satisfy $H_k(X, X - \{x\}) = 0$ for $k < n$ and $H_n(X, X - \{x\}) \neq 0$.*

A topological space Y is said to be *acyclic* in dimension $n \in \mathbb{N}$ if $\check{H}^n(Y; \mathbb{Z}) = 0$. In 2003 Yokoi [124] established the following algebraic property of n-dimensional homogeneous ANR's which is also possessed by topological n-manifolds:

Theorem 3.5 [Yokoi]. *Let X be an n-dimensional homogeneous ANR continuum which is cyclic in dimension n. Then no compact subset of X, acyclic in dimension $n-1$, separates X.*

These partial results, demonstrating that homogeneity implies several of the other manifold properties, indicate why the Bing-Borsuk conjecture 3.1 could be true.

3.2. A special case

In 1996 Repovš et al. [107] (see [106] for a very geometric proof of the 2-dimensional case) proved the following result which in some sense can be considered as a *smooth version* of the Bing-Borsuk conjecture 3.1. Recall that a subset $K \subset \mathbb{R}^n$ is said to be C^1*–homogeneous* if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset \mathbb{R}^n$ of x and y, respectively, and a C^1*–diffeomorphism*

$$h : (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y),$$

i.e. h and h^{-1} have continuous first derivatives.

Theorem 3.6 [Repovš-Skopenkov-Ščepin]. *Let K be a locally compact (possibly nonclosed) subset of $\mathbb{R}^n$. Then K is C^1-homogeneous if and only if K is a C^1–submanifold of $\mathbb{R}^n$.*

This theorem clearly does not work for arbitrary *homeomorphisms*, a counterexample is the *Antoine Necklace* [3] - a wild Cantor set in $\mathbb{R}^3$ which is clearly *homogeneously* (but not C^1*-homogeneously*) embedded in $\mathbb{R}^3$. In fact, *Theorem 3.6* does not even work for *Lipschitz* homeomorphisms, i.e. the maps for which $d(f(x), f(y)) < \lambda\ d(x, y)$, for all $x, y \in X$. Namely, Malešič and Repovš [84] proved in 1999 that there exists a Lipschitz homogeneous wild Cantor set in $\mathbb{R}^3$. Their result was later strengthened by Garity et al. [68]:

Theorem 3.7 [Garity-Repovš-Željko]. *There exist uncountably many rigid Lipschitz homogeneous wild Cantor sets in $\mathbb{R}^3$.*

3.3. Alternate statement

Daverman and Husch [55] were able to determine an equivalent conjecture to the Bing-Borsuk conjecture. In order to state this conjecture, recall that a surjective map $p : E \to B$ between locally compact, separable metric ANR's E and B is said to be an *approximate fibration* if p has the *approximate homotopy lifting property* for every space X. Equivalently, whenever $h : X \times I \to B$ and $H : X \times \{0\} \to E$ are maps such that $p \circ H = h \mid X \times \{0\}$ and ε is a cover of B, h extends to a map $H : X \times I \to E$ such that h and $p \circ H$ are ε-close. The alternate statement of the Bing-Borsuk conjecture 3.1 is given as follows [28],[55],[85],[125]:

Conjecture 3.8 [Alternate statement of the Bing-Borsuk conjecture]. *Suppose that X is nicely embedded in $\mathbb{R}^{m+n}$, for some $m \geq 3$, so that X has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi : \partial N \to X$, with mapping cylinder projection $\pi : N \to X$. Then $\pi : N \to X$ is an approximate fibration.*

3.4. Modified Bing-Borsuk conjecture

Recall that for $n \geq 6$, Bryant et al. [30] proved in 1996 that there exist nonresolvable generalized n-manifolds, for every $n \geq 6$. Based on earlier work by Quinn [97], these nonresolvable generalized manifolds must be *totally singular*, i.e., have no points with Euclidean neighborhoods (we may assume these examples are connected). Moreover, in 2007 Bryant et al. [31] strengthened their result to show that there exist nonresolvable generalized n-manifolds with the disjoint disks property, for every $n \geq 7$. Based on these results the following conjecture was proposed:

Conjecture 3.9 [Bryant-Ferry-Mio-Weinberger]. *Every generalized n-manifold ($n \geq 7$) satisfying the disjoint disks property, is homogeneous.*

Note that if conjecture 3.9 is true, then the Bing-Borsuk conjecture 3.1 is false for $n \geq 7$. In 2002 Bryant [28] suggested the following modified Bing-Borsuk conjecture:

Conjecture 3.10 [Modified Bing-Borsuk conjecture]. *Every homogeneous ($n \geq 3$)-dimensional ENR is a generalized n-manifold.*

A further modification was posed by Quinn [98] at the 2003 Oberwolfach workshop on exotic homology manifolds. It is based on a perturbation of the homogeneity property itself. A space X is *homologically arc-homogeneous* provided that for every path $\alpha : [0,1] \to X$, the inclusion induced map

$$H_*(X \times 0, X \times 0 - (\alpha(0), 0)) \to H_*(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of α. The following is the conjecture proposed by Quinn [98] which was proved in 2006 by Bryant [29].

Theorem 3.11 [Bryant]. *Every n-dimensional homologically arc-homogeneous ENR is a generalized n-manifold.*

This is arguably the strongest result so far relating to the Bing-Borsuk conjecture 3.1.

4. The Busemann conjecture

The Busemann conjecture is also a manifold recognition problem, and is in fact a special case of the Bing-Borsuk conjecture 3.1. Beginning in 1942, Herbert Busemann [38],[39] developed the notion of a G-space as a way of putting a Riemannian like geometry on a metric space (and also in an attempt to obtain a "synthetic description" of Finsler's spaces [65]). A Busemann G-space is a metric space that satisfies four basic axioms on a metric space. These axioms imply the existence of geodesics, local uniqueness of geodesics, and local extension properties. These axioms also infer homogeneity and a cone structure for small metric balls. In 1943, Busemann [39] proved:

Theorem 4.1 [Busemann]. *Busemann G-spaces of dimension $n = 1, 2$ are manifolds.*

Busemann then proposed the following conjecture [40]:

Conjecture 4.2 [Busemann conjecture]. *Every n-dimensional G-space ($n \in \mathbb{N}$) is a topological n-manifold.*

When Busemann [40] proposed conjecture 4.2 in 1955, he predicted: *Although this (the Busemann conjecture) is probably true for any G-space, the proof, if the* *conjecture is correct, seems quite inaccessible in the present state of topology.* As we shall see, this prediction proved true.

4.1. Definitions

We now formally define Busemann G-spaces and state several of the classical properties of Busemann G-spaces.

Definition 4.3. *Let (X, d) be a metric space. X is said to be a Busemann G-space provided it satisfies the following Axioms of Busemann:*

(i) Menger Convexity: *Given distinct points $x, y \in X$, there is a point $z \in X - \{x, y\}$ so that $d(x, z) + d(z, y) = d(x, y)$.*

(ii) Finite Compactness*: Every d-bounded infinite set has accumulation points.*

(iii) Local Extendibility*: To every $w \in X$, there is a positive radius ρ_w, such that for any pair of distinct $x, y \in B(w, \rho_w)$, there is $z \in int\, B(w, \rho_w) - \{x, y\}$ such that $d(x, y) + d(y, z) = d(x, z)$.*

(iv) Uniqueness of the Extension*: Given distinct $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both*

$$d(x, y) + d(y, z_i) = d(x, z_i) \quad \textit{for } i = 1, 2,$$

and

$$d(y, z_1) = d(y, z_2)$$

hold, then $z_1 = z_2$.

From these basic properties, a rich structure on a G-space can be derived. If (X, d) is a G-space and $x \in X$, then (X, d) satisfies the following properties:

- *Complete Inner Metric:* (X, d) is a complete inner metric space which is locally compact.
- *Existence of Geodesics:* Any two points in X are joined by a geodesic segment.
- *Local Uniqueness of Joins:* There is a positive radius r_x such that any two points $y, z \in B_{r_x}(x)$ in the closed ball are joined by a unique segment in X.
- *Local Cones:* There is a positive radius ϵ_x for which the closed metric ball $B_{\epsilon_x}(x)$ is homeomorphic to the cone over its boundary. That is, $B_{\epsilon_x} \cong \Delta(S_{\epsilon_x}(x))$ where $S_{\epsilon_x}(x)$ denotes the metric sphere about x.
- *Homogeneity:* Every G-space is homogeneous. Moreover, homogeneity homeomorphisms can be chosen so that each is isotopic to the identity.

It is this last property that makes the Busemann conjecture 4.2 a special case of the Bing-Borsuk conjecture 3.1. The truth of the Bing-Borsuk conjecture would imply the truth of the Busemann conjecture. Equivalently, if examples of non-manifold Busemann G-spaces could be constructed, the Bing-Borsuk conjecture would be settled in the negative.

4.2. Results in higher dimensions

The first success in resolving the Busemann conjecture 4.2 in higher dimensions occured in 1968 when Krakus [79] proved it in dimension $n = 3$.

Theorem 4.4 [Krakus]. *Busemann G-spaces of dimension* $n = 3$ *are manifolds.*

Krakus applied Borsuk's 2-sphere recognition criterion [20] to show that small metric spheres in 3-dimensional Busemann spaces are topological 2-spheres. The truth of the conjecture follows immediately from the local product structure on small metric spheres and homogeneity. Unfortunately, this strategy cannot be extended to higher dimensions due to the lack of similar characterizations of topological spheres in higher dimensions.

Starting in dimension $n = 4$, it can now be seen that Busemann's prediction of the difficulty of the problem was remarkably accurate. For example, the case $n = 4$ required several modern results and techniques including sheaf theory [21], epsilon surgery [47], resolution theory [97], decomposition theory [52], and theory of 4-manifolds [66],[67],[109]. The major breakthrough in dimension $n = 4$ and partial results applicable to higher dimensions were made by Thurston [118] in 1996:

Theorem 4.5 [Thurston]. *Busemann G-spaces of dimension* $n = 4$ *are manifolds. Moreover, every finite dimensional G-space is a generalized* n*-manifold. More precisely, let* (X, d) *be a G-space, dim* $X = n < \infty$. *Then for all sufficiently small* $r > 0$ *and* $x \in X$, $B_r(x)$ *is a homology* n*-manifold with boundary* $\partial B_r(x) = S_r(x)$ *and* $S_r(x)$ *is a homology* $(n-1)$*-manifold with empty boundary.*

In 2002 Berestovskii [13] proved the special case of the Busemann conjecture 4.2 for Busemann G-spaces that have Alexandrov curvature bounded above. A Busemann G-space (X, d) has Alexandrov curvature $\leq K$ if geodesic triangles in X are at most as "fat" as corresponding triangles in a surface S_K of constant curvature K, i.e. the length of a bisector of the triangle in X is at most the length of the corresponding bisector of the corresponding triangle in S_K. For example, the boundary of a convex region in $\mathbb{R}^n$ has a nonnegative Alexandrov curvature (see also [2] [10]-[12], [14], [37], [76], [94], [95]).

Theorem 4.6 [Berestovskii]. *Busemann G-spaces of dimension* $n \geq 5$ *having bounded Aleksandrov curvature bounded above are* n*-manifolds.*

The general case of the Busemann conjecture 4.2 for $n \geq 5$ remains unsolved: there are many Busemann G-spaces which do not satisfy the condition of Aleksandrov curvature bounded from above (or below). However, all such examples are known to be topological manifolds. The simplest example of Busemann G-space which is not "covered" by Berestovskii's proof [13] is the finite-dimensional vector space $(\mathbb{R}^n, |\cdot|)$ in which the closed unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ is a strongly convex centrally symmetric convex body in $\mathbb{R}^n$ which is not an ellipse.

Conjecture 4.7 [Busemann G-Spaces Resolution conjecture]. *Every* $(n \geq 5)$*-dimensional Busemann G-space has a resolution.*

5. Related problems

Related to the Busemann conjecture 4.2 are three other famous problems: the two de Groot conjectures and the Moore conjecture.

5.1. The de Groot conjectures

The de Groot conjectures are two manifold recognition problems for spaces that are absolute suspensions or absolute cones. A compact finite-dimensional metric space X is called an *absolute suspension* (AS) if it is a suspension with respect to any pair of distinct points and is called an *absolute cone* if it is a cone with respect to any point. Any space topologically equivalent to S^n is an absolute suspension and any space topologically equivalent to B^n is an absolute cone. The question is whether the converse statements are true. At the 1971 Prague Symposium, de Groot [63] made the following two conjectures:

Conjecture 5.1 [Absolute Suspension conjecture]. *Every n-dimensional absolute suspension is homeomorphic to the n-sphere.*

Conjecture 5.2 [Absolute Suspension conjecture]. *Every n-dimensional absolute cone is homeomorphic to the n-cell.*

The fact that small metric balls in Busemann G-spaces are absolute cones follows from the local cone structure and homogeneity. Therefore the truth of the Absolute Cone conjecture 5.2 would imply the truth of the Busemann conjecture 4.2.

In 1974 Szymański [114] proved the Absolute Suspension conjecture 5.1 for dimensions $n \leq 3$. In 1978, Mitchell [86] gave an alternate proof to Szymański result and showed that every n-dimensional absolute suspension is an ENR homology n-manifold homotopy equivalent to the n-sphere. In 2005 Bellamy and Lysko [9] proved the generalized Schönflies theorem for absolute suspensions. Nadler [91] gave a proof of the Absolute Cone conjecture 5.2 in dimensions $n = 1, 2$. However, in 2007 Guilbault [69] completely clarified the status of the Absolute Cone conjecture 5.2:

Theorem 5.3 [Guilbault]. *The Absolute Cone conjecture 5.2 is true for $n \leq 4$ and false for $n \geq 5$.*

Guilbaut proved this result in dimensions $n \geq 5$ by constructing counter-examples. For the special case $n = 4$, Guilbaut shows the Absolute Cone conjecture 5.2 is true modulo the 3-dimensional Poincaré conjecture, which indeed follows by Perelman's proof of the Poincaré conjecture in dimension $n = 3$ [90]. Although the solution of the Absolute Cone conjecture 5.2 leaves the status of the Busemann conjecture 4.2 unresolved, it does cast some suspicion the validity of 4.2.

5.2. The Moore conjecture

A space X is said to be a *codimension one manifold factor* if $X \times \mathbb{R}$ is a topological manifold. In 1955 Bing constructed his infamous Dogbone space [16]. Bing's Dogbone space is the image of a cell-like map $\pi : \mathbb{R}^3 \to X$. Bing [17] showed that the the Dogbone space X is not homeomorphic to $\mathbb{R}^3$, however $X \times \mathbb{R}^1$ is homeomorphic to $\mathbb{R}^4$. This result led to the Moore conjecture:

Conjecture 5.4 [Moore conjecture]. *Every resolvable generalized manifold is a codimension one manifold factor.*

The Moore conjecture 5.4 is also related to the Busemann conjecture 4.2. Every Busemann G-space is a manifold if and only if small metric spheres are codimension one manifold factors. Equivalently, in dimensions $n \geq 5$, every Busemann G-space

X is a manifold if and only if small metric spheres $\Sigma \subset X$ are resolvable and have the property that $\Sigma \times \mathbb{R}$ has the disjoint disks property.

Although it is unknown whether small metric spheres are resolvable, Thurston showed that they are generalized $(n-1)$-manifolds. Also, according to the properties of the Quinn index number which measures the obstruction of a space being resolvable, the resolvability of Σ is equivalent to the resolvability of X (see [97]). Moreover, Mitchell [86] proved in 1978 that any n-dimensional absolute suspension X is a regular generalized n-manifold homotopy equivalent to S^n; all its links are generalized $(n-1)$-manifolds homotopy equivalent to S^{n-1}.

He furthermore showed that an n-dimensional absolute cone X is a regular generalized n-manifold proper homotopy equivalent to $\mathbb{R}^n$; all its links are generalized $(n-1)$-manifolds homotopy equivalent to S^{n-1}. Note that if in Mitchell's theorem "homotopy equivalent" could be replaced with "fine homotopy equivalent", Mitchell's theorem would imply resolvability [49] (see also [81],[87],[88]).

Although it is also still unknown whether small metric spheres X in Busemann G-spaces satisfy the disjoint disks properties, there have been several results determining useful general position properties of an ANR X that characterize $X \times \mathbb{R}$ as having the disjoint disks property. In particular, these properties include:

(i) *The disjoint arc-disk property* (Daverman [50]). A space X has the disjoint arc-disk property if any pair of maps $\alpha : I \to X$ and $f : D^2 \to X$ can be approximated by paths with disjoint images (i.e. X has the disjoint $(1,2)$-cells property). If X has the disjoint arc-disk property, then $X \times \mathbb{R}$ has the disjoint disks property.

(ii) *The disjoint homotopies property* (Edwards [62], Halverson [71]). A space X has the disjoint homotopies property if every pair of path homotopies $f, g : D \times I \to X$, where $D = I = [0,1]$, can be approximated by homotopies $f', g' : D \times I \to X$ so that $f_t(D) \cap g_t(D) = \emptyset$ for all $t \in I$. If X has the disjoint homotopies property, then $X \times \mathbb{R}$ has the disjoint disks property.

(iii) *The plentiful 2-manifolds property* (Halverson [71]). An ANR X has the plentiful 2-manifolds property if every path $\alpha : I \to X$ can be approximated by a path $\alpha' : I \to N \subset X$ where N is a 2-manifold embedded in X. If an ANR X of dimension $n \geq 4$ has the plentiful 2-manifolds property, the X has the disjoint homotopies property.

(iv) *The method of δ-fractured maps* (Halverson [72]). A map $f : D \times I \to X$ is said to be δ-fractured over a map $g : D \times I \to X$, where $D = I = [0,1]$, if there are disjoint balls $B_1, B_2, \ldots, B_m$ in $D \times I$ such that

1. $diam(B_i) < \delta$;
2. $f^{-1}(im(g)) \subset \bigcup_{i=1}^{m} int(B_i)$; and
3. $diam(g^{-1}(f(B_i))) < \delta$

If X is a space such that for any two path homotopies $f, g : D \times I \to X$, where g is a constant path homotopy, and $\delta > 0$, there are approximations $f', g' : D \times I \to X$ such that f' is δ-fractured over g', then X has the disjoint homotopies property.

(v) *The 0-stitched disks property* (Halverson [73]). A space X has the *0-stitched disks property* if any two maps $f, g : D^2 \to X$ can be approximated by maps $f', g' : D^2 \to X$ such that there are infinite 1-skeleta $(K^\infty)^{(1)}$ and $(L^\infty)^{(1)}$ of D^2

and 0-dimensional F_σ sets $A \subset \mathrm{int}(D^2) - (K^\infty)^{(1)}$ and $B \subset \mathrm{int}(D^2) - (L^\infty)^{(1)}$ such that $f'|_{(K_1^\infty)^{(1)}} \cup g'|_{(K_2^\infty)^{(1)}}$ is $1-1$ and $f'(D^2 - A) \cap g'(D^2 - B) = \emptyset$. If X has the 0-stitched disks property, then X has DHP.

(vi) *The disjoint concordances property* (Daverman and Halverson [53]). A *path concordance* in a space X is a map $F : D \times I \to X \times I$, where $D = I = [0,1]$, such that $F(D \times e) \subset X \times e, e \in \{0,1\}$. A metric space (X, ρ) satisfies the *Disjoint Path Concordances Property (DCP)* if, for any two path homotopies $F_i : D \times I \to X$ $(i = 1, 2)$ and any $\epsilon > 0$, there exist path concordances $F_i' : D \times I \to X \times I$ such that

$$F_1'(D \times I) \cap F_2'(D \times I) = \emptyset$$

and $\rho(F_i, \mathrm{proj}_X F_i') < \epsilon$. An ANR X has the disjoint concordances property if and only if $X \times \mathbb{R}$ has the disjoint disks property.

Due to homogeneity, if a Busemann G-space X has a single metric sphere satisfying any one of these properties, then X has the disjoint disks property.

6. Summary and questions

In summary, the following relationships hold between the conjectures and problems discussed in this survey.

- Bing-Borsuk conjecture 3.1 $\Rightarrow$ Busemann conjecture 4.2
- de Groot conjecture 5.2 $\Rightarrow$ Busemann conjecture 4.2
- Moore conjecture 5.4 & Busemann G-Spaces Resolution conjecture 4.7 $\Rightarrow$ Busemann conjecture 4.2
- Bryant-Ferry-Mio-Weinberger conjecture 3.9 $\Rightarrow$ failure of the Bing-Borsuk conjecture 3.1

So far, the validity of only one of these implications has been determined. Recall that the de Groot conjecture 5.2 was shown to be false for all $n \geq 5$ (see [69]). Note that the failure of the Busemann conjecture 4.2 would settle the Bing-Borsuk conjecture 3.1 in the negative. If the Busemann G-Spaces Resolution conjecture 4.7 were proved and the Busemann conjecture 4.2 proved to be false, then the Moore conjecture 5.4 would be settled in the negative.

Below is a summary of relevant questions that remain unsolved:

1. Are all Busemann G-spaces resolvable?
2. Do all Busemann G-spaces X of dimension $n \geq 5$ have the disjoint disks property? (or equivalently, does X contain some metric sphere Σ that has a general position property that implies $X \times \mathbb{R}$ has DDP)?
3. Are all finite-dimensional Busemann G-spaces manifolds?
4. Are all absolute cones resolvable?

5. Are all finite-dimensional homogeneous connected compact metric spaces resolvable?
6. Are all resolvable generalized manifolds codimension one manifold factors?
7. Are all generalized manifolds with the disjoint disks property homogeneous?

7. Epilogue: Homogeneity and group actions

The Bing-Borsuk conjecture 3.1 belongs to a wide group of difficult open problems related to homogeneity and group actions. The nearest one is an old problem: Is the Hilbert cube the only homogeneous compact AR?

All problems of this sort can be seen in the following framework: Given a topological group G and a closed subgroup H, describe the topological structure of the coset space G/H assuming that it has some extra properties (local contractibility, finite-dimensionality, local compactness, etc.) Model results here concern the structure of topological groups [59],[89]:

Theorem 7.1 [Montgomery-Zippin]. *Each locally compact locally contractible topological group is a Lie group and hence a manifold.*

A *Polish group* is a topological group which is also a Polish space [8]:

Theorem 7.2 [Dobrowolski-Torunczyk]. *Each Polish ANR-group is a Hilbert manifold (finite or infinite-dimensional).*

The last theorem suggests the following problem which was partially solved by Banakh and Zarichnyi [7]:

Problem 7.3. *Is each complete metric ANR-group a Hilbert manifold?*

Now let us turn to homogeneous spaces.

Problem 7.4 [Banakh]. *Let G be a Polish group and H a closed subgroup such that G/H is an ANR. Is then G/H a manifold modeled on: (i) the Euclidean n-space $\mathbb{R}^n$; (ii) the Hilbert cube Q; or (iii) the Hilbert sequence space l_2? What if G is an ANR-group? What if the quotient map $G \to G/H$ is a locally trivial bundle?*

Problem 7.4(i) is exactly the Bing-Borsuk Problem while the second part is related to the question mentioned above on homogeneous compact AR's. For connected locally compact topological groups, Problem 7.4 was answered by Szenthe [112],[113].

Theorem 7.5 [Szenthe]. *Let G be a locally compact topological group such that the quotient group G/G_0 of G by its (connected) identity component G_0 is compact (the quotient group G/G_0 is called the group of components and is denoted by $\pi_0(G)$.) Then for any closed subgroup $H \subset G$, the coset space G/H is a disjoint union of topological manifolds if and only if it is locally contractible.*

This theorem has some interesting consequences for homogeneous metric spaces. We define a metric space X to be *metrically homogeneous* if for any two points $x, y \in X$ there is an isometry $f : X \to X$ such that $f(x) = y$. Note that every C^∞-smooth Riemannian manifold M is a Busemann G-space, hence M is homogeneous. However, M is in general not metrically homogeneous. We also observe that every transitive group G of isometries of every locally compact metric space M admits a natural metric with respect to which G is locally compact and acts continuously on

M. It follows from *Theorem 7.5* that the "isometric version" of the Bing-Borsuk conjecture 3.1 is true:

Corollary 7.6. *A metrically homogeneous compact metric space X is a topological n-manifold if and only if X is locally compact and locally contractible.*

Results from [112],[113] also imply that any locally compact connected (possibly metrizable) locally contractible topological space M with a locally compact transitive continuous group G of homeomorphisms, necessarily admits a structure of a C^∞-manifold and a compatible Riemannian metric tensor g such that G acts by isometries on (M, g).

This implies two facts: (i) such a manifold M is necessarily smoothable. As a corollary, no nonsmoothable compact 4-manifold (recall that most compact 4-manifolds are nonsmoothable [67],[109]) admits any locally compact continuous transitive group of homeomorphisms; and (ii) there are many smooth manifolds which admit no metrically homogeneous Riemannian metric (for example, the 2-sphere with two handles). Therefore no such manifold admits any locally compact transitive continuous group of homeomorphisms.

A topological space X is said to be *continuously homogeneous* if for every $x, y \in X$ there is a homeomorphism $h_{x,y} : X \to X$ such that $h_{x,y}(x) = y$ and $h_{x,y}$ continuously depends on the points x, y in the sense that the map $H : X^3 \to X$, $H : (x, y, z) \mapsto h_{x,y}(z)$, is continuous.

It is easy to see that each topological group (endowed with a left-invariant metric) is continuously homogeneous (as a metric space). Continuously homogeneous spaces were introduced and studied by Banakh et al. [5],[6]. It can be shown that a topological space X is continuously homogeneous if and only if it is rectifiable in the sense of Gulko [70]).

Problem 7.7 [Banakh]. *Let X be a continuously homogeneous Polish ANR-space. Is X a Hilbert manifold?*

It light of this problem one should mention that continuously homogeneous spaces cannot be Hilbert cube manifolds [6] (which distinguishes Problem 7.7 from a more general Problem 7.4).

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