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Degenerate Eisenstein series on symplectic groups*

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Abstract. *In this paper we describe the generalization of usual notion of Siegel Eisenstein series (see for example [9]) to give a simple and natural construction of some classes of square–integrable automorphic representations for symplectic groups. The construction of automorphic representations obtained in this paper is an automorphic version of the local construction of strongly negative unramified representations [5]* [1] *or of discrete series obtained by Tadić [11] in early* 90*'s (see also later work [8]). This is taken from our paper [7].*

As an application we show how one can obtain an automorphic realization of certain global spherical representations. This has an interesting consequence locally and globally. We adopt Arthur's point of view (see [2]).

Key words: *automorphic forms, Eisenstein series*

AMS subject classifications: 11F70, 11F72

1. Notation

Let $G = \mathrm{Sp}_{2n}$ be a split symplectic group of rank n over a number field k. Let $\mathbb{A}$ be the ring of adéles of k. We write $\{v\}$ for the set of places of k. For each place v of k, let k_v be its completion at that place. Let $|\ |_v$ be the normalized absolute value of k_v. Put $|\ | = \prod_v |\ |_v$. If v is finite (denoted by $v < \infty$), then let $\mathcal{O}_v$ be the ring of integers of k_v and ϖ_v a generator of the maximal ideal in $\mathcal{O}_v$. Then $\mathbb{A} \simeq \prod'_v k_v$ is a restricted product over all places $\{v\}$.

For a finite place v, we fix a maximal compact subgroup $K_v = G(\mathcal{O}_v)$ of $Sp_{2n}(k_v)$. Let $G_\infty = \prod_v G(k_v)$, where the product extends over all Archimedean places of k. It is a semisimple Lie group with the finite center. We fix a suitable maximal compact subgroup $K_\infty \subset G_\infty$. We let

$$K = K_\infty \times \prod_{v \text{ is finite}} K_v.$$

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1 An unramified representation is strongly negative (resp., negative) if its exponents relative to a Borel belong to (resp., the closure) in the obtuse Weyl chamber.

This is a maximal compact subgroup of $G(\mathbb{A})$.

We let $\mathfrak{g}_\infty$ be the Lie algebra of G_∞. Let $\mathcal{U}(\mathfrak{g}_\infty)$ be the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{g}_\infty \otimes_\mathbb{R} \mathbb{C}$. Let $\mathcal{Z}(\mathfrak{g}_\infty)$ be the center of $\mathcal{U}(\mathfrak{g}_\infty)$.

Let $G(\mathbb{A}_f)$ be the restricted product of all $G(k_v)$, v is finite. The group $G(\mathbb{A}_f)$ is a totally disconnected group, i.e. it has a basis of neighborhoods of 1 consisting of open compact subgroups. The group $G(k)$ is embedded diagonally as a discrete subgroup of $G(\mathbb{A})$. We have the following:

$$G(\mathbb{A}) \simeq G_\infty \times G(\mathbb{A}_f).$$

We say that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ is smooth if $f(\cdot, g_f) \in C^\infty(G_\infty)$ for all $g_f \in G(\mathbb{A}_f)$, and there exists an open compact subgroup $L \subset G(\mathbb{A}_f)$ such that $f(g_\infty, g_f \cdot l) = f(g_\infty, g_f)$ for all $(g_\infty, g_f) \in G_\infty \times G(\mathbb{A}_f)$ and $l \in L$. Here we consider f as a function of two variables $f(g) = f(g_\infty, g_f)$, where $g = (g_\infty, g_f)$. We write $C^\infty(G(\mathbb{A}))$ for the vector space of all smooth functions on $G(\mathbb{A})$. We let $C_c^\infty(G(\mathbb{A}))$ be the space of all smooth compactly supported functions on $G(\mathbb{A})$.

By definition, we let $C^\infty(G(k) \setminus G(\mathbb{A})) \subset C^\infty(G(\mathbb{A}))$ be the subspace consisting of all functions $f \in C^\infty(G(\mathbb{A}))$ such that $f(\gamma \cdot g) = f(g)$ for all $\gamma \in G(k)$ and $g \in G(\mathbb{A})$.

Let $X \in \mathfrak{g}_\infty$. Let $f \in C^\infty(G(\mathbb{A}))$. Then we let

$$X.f(g_\infty, g_f) = d/dt|_{t=0} f(g_\infty \exp(tX), g_f).$$

This gives the structure of a $\mathcal{U}(\mathfrak{g}_\infty)$–module on $C^\infty(G(\mathbb{A}))$. The subspace $C^\infty(G(k)\setminus G(\mathbb{A}))$ is a $\mathcal{U}(\mathfrak{g}_\infty)$–submodule. In fact, both are invariant under the action of $G(\mathbb{A})$ by the right translation.

The function $f \in C^\infty(G(\mathbb{A}))$ is K_∞–finite (on the right) if $span_\mathbb{C}\{(g_\infty, g_f) \to f(g_\infty k_\infty, g_f);\quad k_\infty \in K_\infty\}$ is finite dimensional. Similarly, $f \in C^\infty(G(\mathbb{A}))$ is $\mathcal{Z}(\mathfrak{g}_\infty)$–finite if the space spanned by $z.f$, $z \in \mathcal{Z}(\mathfrak{g}_\infty)$ is finite dimensional. In other words, the annhilator of f in $\mathcal{Z}(\mathfrak{g}_\infty)$ has finite codimension. By a well–known result, if $f \in C^\infty(G(\mathbb{A}))$ is K_∞–finite and $\mathcal{Z}(\mathfrak{g}_\infty)$–finite, then it is real–analytic in g_∞. We write $C^\infty(G(\mathbb{A}))_{K_\infty, \mathcal{Z}(\mathfrak{g}_\infty)-\text{finite}}$ for the space of all $f \in C^\infty(G(\mathbb{A}))$ which are K_∞–finite and $\mathcal{Z}(\mathfrak{g}_\infty)$–finite on the right. Similarly, we define $C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, \mathcal{Z}(\mathfrak{g}_\infty)-\text{finite}}$. The space $C^\infty(G(\mathbb{A}))_{K_\infty, \mathcal{Z}(\mathfrak{g}_\infty)-\text{finite}}$ is no longer $G(\mathbb{A})$–invariant but it is $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module. Furthermore, the space $C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, \mathcal{Z}(\mathfrak{g}_\infty)-\text{finite}}$ is its submodule.

An automorphic form is a function $f \in C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, \mathcal{Z}(\mathfrak{g}_\infty)-\text{finite}}$ which satisfies certain growth condition (see [3], 4.2). The space of all automorphic forms we denote by $\mathcal{A}(G(k) \setminus G(\mathbb{A}))$. It is a $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–submodule of $C^\infty(G(k) \setminus G(\mathbb{A}))_{K_\infty, \mathcal{Z}(\mathfrak{g}_\infty)-\text{finite}}$.

One of the important ways to construct automorphic forms is through the Eisenstein series. This is what we explain next.

2. Construction of degenerate Eisenstein series

We write $B_n = T_n U_n$ for the Borel subgroup of G, W for its Weyl group and Δ for the set of simple roots with respect to B_n.

Let $P = MN$ be a maximal standard k–parabolic subgroup of Sp_{2n}. Assume $M \simeq GL(m) \times \mathrm{Sp}_{2n'}$. Let

$$V \subset \mathcal{A}(\mathrm{Sp}_{2n'}(k) \setminus \mathrm{Sp}_{2n'}(\mathbb{A}))$$

be an irreducible subspace of the space of automorphic forms. Let us call the corresponding representation Π. Assume that V is concentrated on $B_{n'}$. (That is, there is a constant term along $B_{n'}$ that does not vanish.) Let V_0 be the space of constant terms along $B_{n'}$ of V. The map $V \to V_0$ defined by

$$\varphi \rightsquigarrow (g' \mapsto \int_{U_{n'}(k)\setminus U_{n'}(\mathbb{A})} \varphi(u'g')du')$$

is an intertwining operator. In particular, since V is irreducible and concentrated on $B_{n'}$, the map is an isomorphism.

For $t \in T_{n'}(\mathbb{A})$, we let

$$V_0^t = l(t)V_0,$$

where

$$l(t)F(g') = F(t^{-1}g').$$

The representation of $\prod_{v<\infty} \mathrm{Sp}_{2n'}(k_v) \times (\mathfrak{g}'_\infty, K'_\infty)$ on V_0^t is irreducible and isomorphic to V_0 (and to V). The main point of that construction is that we can find $0 \neq F \in \sum_{t\in T_{n'}(\mathbb{A})} V_0^t$ and a character $\lambda' : T_{n'}(\mathbb{A}) \to \mathbb{C}^\times$, necessarily trivial on $T_{n'}(k)$, such that

$$F(t'g') = \delta_{B_{n'}}^{1/2}(t')\lambda'(t')F(g'),\ t' \in T_{n'}(\mathbb{A}),\ g' \in \mathrm{Sp}_{2n'}(\mathbb{A}).$$

Hence, we have the following:

$$F(t'u'g') = \delta_{B_{n'}}^{1/2}(t')\lambda'(t')F(g'),\ t' \in T_{n'}(\mathbb{A}),\ u' \in U_{n'}(\mathbb{A}),\ g' \in \mathrm{Sp}_{2n'}(\mathbb{A}). \tag{1}$$

The same identity holds for all functions in the subrepresentation–$\prod_{v<\infty} \mathrm{Sp}_{2n'}(k_v)\times (\mathfrak{g}'_\infty, K'_\infty)$ $V' \subset \sum_{t\in T_{n'}(\mathbb{A})} V_0^t$ generated by F. Clearly, V' is direct sum of irreducible representations all isomorphic to V. Therefore, we may assume that V' is itself irreducible. Then (1) implies the embedding

$$\Pi \hookrightarrow \mathrm{Ind}_{B_{n'}(\mathbb{A})}^{\mathrm{Sp}_{2n'}(\mathbb{A})}(\lambda'). \tag{2}$$

Let $\mu : k^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$ be a (unitary) grössencharacter. The representation $\mu \mathbf{1}_{GL(m,\mathbb{A})}$ is an automorphic representation of $GL(m,\mathbb{A})$ on the one dimensional space $W \subset A(GL(m,k) \setminus GL(m,\mathbb{A}))$. The computation of the constant term W_0 of W along Borel subgroup B_m^{GL} gives an embedding:

$$\mu \mathbf{1}_{GL(m,\mathbb{A})} \hookrightarrow \mathrm{Ind}_{B_m^{GL}(\mathbb{A})}^{GL_m(\mathbb{A})} \left(|\ |^{s-(m-1)/2}\mu \otimes \cdots \otimes |\ |^{s+(m-1)/2}\mu \right).$$

Fixing above data, we can realize the induction in stages:

$$\mathrm{Ind}_{M(\mathbb{A})N(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} |\det|^s \mu \mathbf{1}_{GL(m,\mathbb{A})} \otimes \Pi) \hookrightarrow$$
$$\mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} \left(|\ |^{s-(m-1)/2}\mu \otimes \cdots \otimes |\ |^{s+(m-1)/2}\mu \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n'} \right)$$

which enables us to fix a nice realization for $\mathrm{Ind}_{M(\mathbb{A})N(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(|\det|^s\mu\mathbf{1}_{GL(m,\mathbb{A})}\otimes\Pi)$ with analytic sections f_s. Then we define a *degenerate Eisenstein series* as follows:

$$E(f_s,g)=\sum_{\gamma\in P(k)\backslash \mathrm{Sp}_{2n}(k)} f_s(\gamma g) \tag{3}$$

as opposed to the usual Eisenstein series:

$$E(f_s,g)=\sum_{\gamma\in B_n(k)\backslash \mathrm{Sp}_{2n}(k)} f_s(\gamma g).$$

This series converges for $Re(s)$ sufficiently large and continues to a meromorphic function in s. Obviously it as an automorphic form in $\mathcal{A}(\mathrm{Sp}_{2n}(k)\backslash \mathrm{Sp}_{2n}(\mathbb{A}))$. Finally, its analytic behaviour is controlled by its constant term along Borel B_n. More precisely, they have the same set of poles (counting with multiplicity).

The Eisenstein series given by (3) is concentrated on the Borel subgroup, and its constant term along B_n is given by

$$E_0(f_s,g)=\int_{U_n(k)\backslash U_n(\mathbb{A})} E(f_s,ug)du=\sum_{w\in W,\ w(\Delta\backslash\{\alpha\})>0} M(\lambda(\underline{s}),w)f_s(g). \tag{4}$$

Here α is the unique simple root in N and we write

$$\lambda(s)=|\ |^{s-(m-1)/2}\mu\otimes\cdots\otimes|\ |^{s+(m-1)/2}\mu\otimes\lambda_1\otimes\cdots\otimes\lambda_{n'}.$$

We remind the reader that $M(\lambda(\underline{s}),w)$ that is the standard intertwining operator $\mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(\lambda(\underline{s}))\to \mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(w(\lambda(\underline{s})))$. For $Re(s)>>0$, it is given by the integral:

$$M(\lambda(\underline{s}),w)f_s(k)=\int_{U_n(\mathbb{A})\cap wU_n(\mathbb{A})w^{-1}\backslash U_n(\mathbb{A})} f_s(\widetilde{w}^{-1}uk),\quad k\in K, \tag{5}$$

which does not depend on the choice of the representative $\widetilde{w}$ for w in $\mathrm{Sp}_{2n}(k)$.

This intertwining operators factors into a product of the local intertwining operators that needs to be suitable normalized:

$$M(\lambda(\underline{s}),w)f_s=\otimes_v A(\lambda(\underline{s})_v,\widetilde{w})f_{s,v},$$

for factorizable $f_s=\otimes_v f_{s,v}$.

The expression (4) is studied using local methods of [5] by normalizing intertwining operators as in [4] or [10].

The normalizing factor is defined by

$$r(\lambda(\underline{s})_v,w)=\prod_{\alpha\in\Sigma^+,w(\alpha)<0}\frac{L(1,\lambda(\underline{s})_v\circ\alpha^\vee)\epsilon(0,\lambda(\underline{s})_v\circ\alpha^\vee,\psi_v)}{L(0,\lambda(\underline{s})_v\circ\alpha^\vee)},$$

where L and ϵ-factors are the usual one. We let

$$\mathcal{N}(\lambda(\underline{s})_v,\widetilde{w})=r(\lambda(\underline{s})_v,w)A(\lambda(\underline{s})_v,\widetilde{w}).$$

Since $f_s = \otimes_v f_{s,v}$ is unramified (that is, K_v–invariant) outside of finite set S of places of k (containing all Archimedean places), we have

$$M(\lambda(\underline{s}), w) f_s = \left(\prod_{\substack{\alpha \in \Sigma^+ \\ w(\alpha) < 0}} \frac{L(0, \lambda(\underline{s}) \circ \alpha^\vee)}{L(1, \lambda(\underline{s}) \circ \alpha^\vee) \epsilon(0, \lambda(\underline{s}) \circ \alpha^\vee)} \right) \otimes_{v \in S} \mathcal{N}(\lambda(\underline{s})_v, \widetilde{w}) f_{s,v} \otimes_{v \notin S} \widetilde{f}_{s,v}. \quad (6)$$

Here, for $v \notin S$,

$$\widetilde{f}_{s,v} \in \mathrm{Ind}_{B_n(k_v)}^{\mathrm{Sp}_{2n}(k_v)} \left(w(\lambda(\underline{s})_v) \right)$$

is unramified and normalized.

Having written the expression for the global intertwining operator in the form (3-6), we have local and global contribution to the poles of the intertwining operator and hence to the poles of the degenerate Eisenstein series.

The global contribution that comes from the term in the parenthesis

$$\prod_{\alpha \in \Sigma^+, w(\alpha) < 0} \frac{L(0, \lambda(\underline{s}) \circ \alpha^\vee)}{L(1, \lambda(\underline{s}) \circ \alpha^\vee) \epsilon(0, \lambda(\underline{s}) \circ \alpha^\vee)}$$

is easily analyzed using some basic facts about automorphic L–functions $L(s, \mu)$ ($s \in \mathbb{C}$) attached to characters $\mu : k^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$.

We can write μ as a restricted tensor product $\mu = \otimes_v \mu_v$ of local characters $\mu_v : k_v^\times \to \mathbb{C}^\times$. We choose a non–trivial additive character $\psi : k \setminus \mathbb{A} \to \mathbb{C}^\times$ and write it as a restricted product of local characters $\psi = \prod_v' \psi_v$. In the above set-up, Tate associated local factors $L(s, \mu_v)$ and $\epsilon(s, \mu_v, \psi_v)$ such that the product $\epsilon(s, \mu) = \prod_v \epsilon(s, \mu_v, \psi_v)$ is independent of ψ and is a finite product since for all but finite places ψ_v and μ_v are unramified, and, in that case, $\epsilon(s, \mu_v, \psi_v) = 1$. Moreover, we have that the product $L(s, \mu) = \prod_v L(s, \mu_v)$ initially converges for $Re(s) > 1$ and extends to the whole complex plane as a meromorphic function satisfying the following functional equation:

$$L(s, \mu) = \epsilon(s, \mu) L(1 - s, \mu^{-1}).$$

Moreover, if $\mu \neq \mathbf{1}$, then $L(s, \mu)$ is holomorphic. $L(s, \mathbf{1})$ has simple poles at $s = 0$ and $s = 1$ and no other poles.

To controll the local contribution we can use local representation theory. This is what we explain next.

3. Construction of automorphic representations

Let W_k be the Weil group of k. Let $\hat{G}(\mathbb{C}) = SO(2n+1, \mathbb{C})$ be the complex dual group of $G = \mathrm{Sp}_{2n}$.

We say that an Arthur parameter $\varphi : W_k \times SL(2, \mathbb{C}) \to \hat{G}(\mathbb{C})$ is <u>spherical unipotent</u> if it is trivial on W_k. Thus, it is of the form $\varphi : SL(2, \mathbb{C}) \to \hat{G}(\mathbb{C})$. We can find a unique increasing sequence of positive integers $(m_1, \ldots, m_k)$,

$$\sum_{i=1}^{k} 2m_i + 1 = 2n + 1,$$

such that

$$\varphi = \oplus_{i=1}^{k} V_{2m_i+1}. \tag{7}$$

We remark that k must be odd.

Let $\lambda(\varphi) : T_{min}(k) \setminus T_{min}(\mathbb{A}) \to \mathbb{C}^\times$ be defined by:

$$(|\ |^{-m_k} \otimes |\ |^{-m_k+1} \otimes \cdots \otimes |\ |^{m_{k-1}}) \otimes \cdots \otimes (|\ |^{-m_3} \otimes |\ |^{-m_3+1} \otimes \cdots \otimes |\ |^{m_2}) \otimes \\ \otimes (|\ |^{-m_1} \otimes |\ |^{-m_1+1} \otimes \cdots \otimes |\ |^{-1}).$$

We remark that for $k = 1$, the trivial representation $\mathbf{1}_{\mathrm{Sp}_{2m_1}(\mathbb{A})}$ has a unique automorphic realization

$$j(V_{2m_1+1}) : \mathbf{1}_{\mathrm{Sp}_{2m_1}(\mathbb{A})} \to A_2(\mathrm{Sp}_{2m_1}(k) \setminus \mathrm{Sp}_{2m_1}(\mathbb{A})).$$

The usual embedding

$$\mathbf{1}_{\mathrm{Sp}_{2m_1}(\mathbb{A})} \hookrightarrow \mathrm{Ind}_{B_{m_1}(\mathbb{A})}^{\mathrm{Sp}_{2m_1}(\mathbb{A})}(|\ |^{-m_1} \otimes \cdots \otimes |\ |^{-1})$$

is obtained computing the constant term along B_{m_1} on the space of constant functions $Image(j(V_{2m_1+1}))$.

First, we describe the construction of the spherical component.

Theorem 1. *Let $k > 0$ be an odd integer. Under above assumptions, the unique irreducible K–spherical subquotient $\sigma(\varphi)$ of the globally induced representation $\mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(\lambda(\varphi))$ is its subrepresentation, and there is a non–zero embedding $j(\varphi) : \sigma(\varphi) \to A_2(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$ constructed recursively as follows. Let $k \geq 3$. Put $\varphi' = \oplus_{i=1}^{k-2} V_{2m_i+1}$ and $2n' + 1 = \sum_{i=1}^{k-2} 2m_i + 1$. Consider the global induced representation*

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})}\left(|\det|^s \mathbf{1}_{\mathrm{GL}(2m_{k-1}+1,\mathbb{A})} \otimes Image(j(\varphi'))\right), \tag{8}$$

where P is a standard parabolic subgroup of $\mathrm{Sp}_{2n'+4m_{k-1}+2}$ with Levi factor $\mathrm{GL}(2m_{k-1}+1) \times \mathrm{Sp}_{2n'}$. (At $s = 0$ this representation is unitary and therefore semisimple (of infinite length).) Then the map obtained from a degenerate Eisenstein series

$$f_s \rightsquigarrow E(f_s, \cdot)|_{s=0} \tag{9}$$

is an intertwining operator

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})}\left(\mathbf{1}_{\mathrm{GL}(2m_{k-1}+1,\mathbb{A})} \otimes Image(j(\varphi'))\right) \to \\ A(\mathrm{Sp}_{2n'+4m_{k-1}+2}(k) \setminus \mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})),$$

which is non–trivial on the unique irreducible K–spherical subrepresentation of (8) for $s = 0$; let us write X for the image of the K–spherical subrepresentation. Taking the constant term of X along Borel $B_{n'+2m_{k-1}+1}$ we obtain the following embedding:

$$X \hookrightarrow \mathrm{Ind}_{B_{n'+2m_{k-1}+1}(\mathbb{A})}^{\mathrm{Sp}_{2n'+4m_{k-1}+2}(\mathbb{A})} \left(|\ |^{-m_{k-1}} \otimes |\ |^{-m_{k-1}+1} \otimes \cdots \otimes |\ |^{m_{k-1}} \otimes \lambda(\varphi')\right) \tag{10}$$

which we use to construct degenerate Eisenstein series

$$f_s \rightsquigarrow E(f_s, g) = \sum_{\gamma \in P(k) \backslash \mathrm{Sp}_{2n}(k)} f_s(\gamma g)$$

attached to the global induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} \left(|\det|^s \mathbf{1}_{\mathrm{GL}(m_k - m_{k-1}, \mathbb{A})} \otimes X\right), \tag{11}$$

where P is a standard parabolic subgroup of Sp_{2n} with Levi factor $\mathrm{GL}(m_k - m_{k-1}) \times \mathrm{Sp}_{2n'+4m_{k-1}+2}$. Then the map

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})} \left(|\det|^{\frac{m_{k-1}+m_k+1}{2}} \mathbf{1}_{\mathrm{GL}(m_k - m_{k-1}, \mathbb{A})} \otimes X\right)^K \to A(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$$

given by

$$f_{\frac{m_{k-1}+m_k+1}{2}} \rightsquigarrow \left(s - \frac{m_{k-1} + m_k + 1}{2}\right)^2 E(f_s, \cdot)\big|_{s = \frac{m_{k-1}+m_k+1}{2}} \tag{12}$$

is well–defined and non–trivial. Let E be a $\prod_{v<\infty} \mathrm{Sp}_{2n}(k_v) \times (\mathfrak{g}_\infty, K_\infty)$-subrepresentation of $A(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$ generated by the image of the space of K–invariants. Then E is irreducible, contained in the space of square–integrable automorphic forms $A_2(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$, and it induces the required embedding $j(\varphi) : \sigma(\varphi) \simeq E \subset A_2(\mathrm{Sp}_{2n}(k) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$. Finally, the embedding $\sigma(\varphi) \hookrightarrow \mathrm{Ind}_{B_n(\mathbb{A})}^{\mathrm{Sp}_{2n}(\mathbb{A})}(\lambda(\varphi))$ is obtained computing the constant term of E along B_n.

Second, we can describe the generalization of this theorem for non–spherical representations. This requires the construction of a particular local normalized intertwining operator. (See [7], Theorem 6-21 for details.)

We end by the following remark which follows from *Theorem 1.* Let $k = \mathbb{Q}$ and ∞ be the unique Archimedean place of $\mathbb{Q}$. Then $\sigma(\varphi)_\infty$ is automorphic and in fact

$$\sigma(\varphi)_\infty \hookrightarrow L^2(\mathrm{Sp}_{2n}(\mathbb{Q}) \setminus \mathrm{Sp}_{2n}(\mathbb{A}))$$

with the image contained in the space of residual representations.

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