Classifications coarser than shape*

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Abstract. *About thirty years ago, in the time of an intensive study of the shape theory, several classifications of metric compacta coarser than shape were introduced. Two of them have been of a special interest: The quasi-equivalence (K. Borsuk, 1976) and S-equivalence (S. Mardešić, 1978). In the last decade a much deeper view into these relations has been achieved. In attempt to characterize or, at least, describe them in a category framework on purpose of easier operative studying, several new classifications and new "shape" theories occurred. The most interesting among them are the coarse and weak shape theory. This paper is intended to be an exhaustive survey of new classifications and corresponding theories, their mutual relations and the most important results.*

Key words: *shape, quasi-equivalence, $\overline{q}$-equivalence, S-equivalence, S^*-equivalence, S_n-equivalence, coarse shape, weak shape*

AMS subject classifications: Primary 55P55, 57N25; Secondary 18A99

1. Introduction

About thirty years ago, in the time of intensive studying of the shape theory, several classifications of metric compacta coarser than shape were introduced. Two of them have been of a special interest: The quasi-equivalence (K. Borsuk, 1976, [2]) and S-equivalence (S. Mardešić, 1978, [10]). The first one was based on idea of a quantitative estimation of a difference between the shape types of a pair of very "alike" compacta. A recent result, however, has shown that this relation, in general, is not a classification (it is not transitive, [6]). Nevertheless, it still generates an important and useful equivalence relation on compacta. The second one was introduced relating to the problem of the shape types of fibres of a shape fibration, [3], [11], [12]. The fact is that all the fibres of a shape fibration (over a continuum) are mutually S-equivalent [10], while they need not to be of the same shape type, [7], [5]. Let us briefly recall the basic facts concerning the both relations.

The quasi-equivalence was originally defined and studied in [2] by means of fundamental sequences [1] and neighbourhoods in a pair of AR ambient spaces.

*Presented in part in invited talks at the 4th Croatian Mathematical Congress, Osijek, Croatia, June 2008.

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Afterwards, it was characterized by means of sequences of morphisms of compact ANR inverse sequences ([15], Section 4). Let X and Y be compact metric spaces, and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be associated with them compact ANR inverse sequences, respectively ($\lim \boldsymbol{X} = X$ and $\lim \boldsymbol{Y} = Y$). Let $(f, f_j), (f', f'_j) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of $(cANR)^{\mathbb{N}}$. Then, (f, f_j) is said to be *n-homotopic to* (f', f'_j), denoted by $(f, f_j) \simeq_n (f', f'_j)$, provided

$$f_n p_{f(n)i} \simeq f'_n p_{f'(n)i}.$$

Clearly, $(f, f_j) \simeq_n (f', f'_j)$ is equivalent to $(f, f_j) \simeq_j (f', f'_j)$ for every $j \leq n$. Further, $(f, f_j) \simeq (f', f'_j)$ if and only if $(f, f_j) \simeq_n (f', f'_j)$ for every $n \in \mathbb{N}$ Being n-homotopic is an equivalence relation on each set $(cANR)^{\mathbb{N}}(\boldsymbol{X}, \boldsymbol{Y})$. However, it is not compatible with composition (from the left), so there is no appropriate quotient category. It is sometimes convenient to consider $\boldsymbol{X}$ and $\boldsymbol{Y}$ having homotopy classes as bonding morphisms , i.e. being objects of $(HcANR)^{\mathbb{N}}$. Then $\simeq_n$ induces the appropriate equivalence relation on each set $(HcANR)^{\mathbb{N}}(\boldsymbol{X}, \boldsymbol{Y})$, which may be written as

$$(f, [f_j]) =_n (f', [f'_j]).$$

Further, by passing to the quotient category

$$tow\text{-}HcANR = (HcANR)^{\mathbb{N}}/(\simeq),$$

the relation $=_n$ induces the corresponding equivalence relation on each set *tow-*$HcANR(\boldsymbol{X}, \boldsymbol{Y})$, which can be denoted by $\boldsymbol{f} =_n \boldsymbol{f}'$, where $\boldsymbol{f} = [(f, [f_j])]$ and $\boldsymbol{f}' = [(f', [f'_j])]$.

Now, $\boldsymbol{X}$ is said to be *quasi-equivalent to* $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, provided there exists a pair of sequences $(\boldsymbol{f}^n)$, $(\boldsymbol{g}^n)$, where $\boldsymbol{f}^n \in HcANR(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{g}^n \in HcANR(\boldsymbol{Y}, \boldsymbol{X})$ such that, for every n,

$$\boldsymbol{g}^n \boldsymbol{f}^n =_n \mathbf{1}_{\boldsymbol{X}} \quad \text{and} \quad \boldsymbol{f}^n \boldsymbol{g}^n =_n \mathbf{1}_{\boldsymbol{Y}}.$$

Then, X is *quasi-equivalent to* Y, denoted by $X \overset{q}{\simeq} Y$, provided $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ for some (equivalently, any) associated pair $\boldsymbol{X}, \boldsymbol{Y}$). In [2], Borsuk proved that the quasi-equivalence is strictly coarser than the shape type, and that on the class of all compacta having the homotopy types of ANR's (or polyhedra), it reduces to the shape (thus, to the homotopy) type classification. Further, the movability and Betti numbers are invariants of the quasi-equivalence. However, Borsuk stated the following question ([2], Problem (7.13)): "Is the relation of quasi-equivalence transitive?". The question is recently answered in the negative, [6].

The S-equivalence was defined and studied in [10]. Two compact ANR inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ are said to be *S-equivalent*, denoted by $S(\boldsymbol{X}) = S(\boldsymbol{Y})$, provided, for every $n \in \mathbb{N}$, the following condition is fulfilled:

$$(\forall j_1)(\exists i_1)(\forall i'_1 \geq i_1)(\exists j'_1 \geq j_1)(\forall j_2 \geq j'_1)(\exists i_2 \geq i'_1) \cdots$$
$$\cdots (\forall i'_{n-1} \geq i_{n-1})(\exists j'_{n-1} \geq j_{n-1})(\forall j_n \geq j'_{n-1})(\exists i_n \geq i'_{n-1})$$

and there exist mappings $f_k \equiv f^n_{j_k} : X_{i_k} \to Y_{j_k}$, $k = 1, \ldots, n$, and $g_k \equiv g^n_{i'_k} : Y_{j'_k} \to X_{i'_k}$, $k = 1, \ldots, n-1$, making the following diagram

$$(*)\quad \begin{array}{ccccccccc} X_{i_1} & \leftarrow & X_{i'_1} & \leftarrow & \cdots & \leftarrow & X_{i'_{n-1}} & \leftarrow & X_{i_n} \\ \downarrow f_1 & & \uparrow g_1 & & \cdots & & \uparrow g_{n-1} & & \downarrow f_n \\ Y_{j_1} & \leftarrow & Y_{j'_1} & \leftarrow & \cdots & \leftarrow & Y_{j'_{n-1}} & \leftarrow & Y_{j_n} \end{array}$$

commutative up to homotopy.

Two compacta X and Y are said to be S-*equivalent*, denoted by $S(X) = S(Y)$, provided $S(\boldsymbol{Y}) = S(\boldsymbol{X})$ for some (equivalently, any) associated $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively ([10], Remarks 1.,2. and Definition 2.). Obviously, the S-equivalence is an equivalence relation which is coarser than the shape type classification, while on the class $cHANR$ it reduces to the shape (thus, homotopy) type classification. In [10], Mardešić proved that all the fibres of a shape fibration, with the base space connected, have the same S-type. Further, he proved that the following shape invariants: connectedness, shape triviality, shape dimension $\leq n$, n-shape connectedness, movability, n-movability and strong movability (being an FANR) are actually properties of the S-type. Is the S-equivalence strictly coarser than shape remained, at that time, an open problem. Soon afterwards, J. Keesling and Mardešić proved that it is the case, [7]. They, namely constructed a certain shape fibration having fibres of different shape types. Moreover, R. Goodearl and T. B. Rushing proved in [5] that its fibres belong to uncountable many different shape types. Finally, in attempt to relate the quasi-equivalence and S-equivalence, N. Uglešić proved that they are mutually independent relations, [15].

The study of the above mentioned relations has followed the next general procedure. When a class of mathematical objects has to be considered, the main task is to classify them by a given equivalence relation ($\sim$). If the equivalence relation admits a description in terms of a category ($\mathcal{A}$), then the work becomes somewhat easier. If, in addition, a reinterpretation of $\sim$ by an equivalence relation ($\approx$) on each $\mathcal{A}$-morphism set is possible, then one usually says that the equivalence relation $\sim$ (on $Ob\mathcal{A}$) admits a *category characterization* by $\approx$ (on $Mor\mathcal{A}$). More precisely, $X \sim Y$ if and only there exist an $f : X \to Y$ and a $g : Y \to X$ such that $gf \approx 1_X$ and $fg \approx 1_Y$. The best possible case occurs when $\approx$ is compatible with the composition in $\mathcal{A}$. Then, namely, there exists the corresponding quotient category $\mathcal{A}/(\approx)$, implying that $X \sim Y$ if and only there exist an $\boldsymbol{f} = [f] : X \to Y$ and a $\boldsymbol{g} = [g] : Y \to X$ such that $\boldsymbol{gf} = \boldsymbol{1}_X$ and $\boldsymbol{fg} = \boldsymbol{1}_Y$, i.e. $X \cong Y$ (isomorphic) in $\mathcal{A}/(\approx)$, which is a *full category characterization* of the starting equivalence relation. In such a case one says that an object equivalence relation admits a *theory* modeled on an appropriate category. The standard examples are the classifications of topological spaces by homeomorphisms, by homotopy types and by shape types.

Our first aim was to investigate whether the quasi-equivalence and S-equivalence admit (full) category characterizations. In the last decade a much deeper view into these relations has been achieved. It is proved that they admit category characterizations, [15] and [4], which are not full. It is also proved that the quasi-equivalence is not transitive on the whole class of compacta, [6]. However, in attempt to characterize or, at least, describe them in a category framework on purpose of easier operative studying, several new classifications and new "shape" theories occurred. The significant ones are as follows:

- $\overline{q}$-equivalence and q^*-equivalence (of compacta and of any category sequences) together with the appropriate theories modeled on the constructed categories and

functors, [15], [16] and [18];
- S^*-equivalence (of compacta) with the corresponding theory - a category and a functor related to the shape category, [14];
- S^*-equivalence (of compacta), with a graded sequence of categories and functors, [18];
- coarse shape theory - abstract and standard (for topological spaces), [9];
- weak shape theory - abstract and standard, [20];
- sequence $S_0 \Leftarrow S_0^+ \Leftarrow \cdots \Leftarrow S_n \Leftarrow S_n^+ \Leftarrow \cdots \Leftarrow S$ of classifications (of compacta and of any category sequences) including category descriptions and characterizations, [19] and [4].

This paper is intended to be an exhaustive survey of the above classifications, their mutual relations and the most important results.

2. Quasi-shapes

In [15] is proved that the quasi-equivalence is transitive on the class of all quasi-stable compacta (including all 0-dimensional compacta). However, it is recently proved that the quasi-equivalence is not transitive in general, [6]. The example is as follows. Let X be an infinite countable one-point union of pointed tori converging to the limit torus, let Y be an infinite countable one-point union of pointed tori converging to the base point, and let Z be the one-point union of the pointed space X and a pointed circle. Observe that X, Y and Z are metric continua embedable in the Euclidean space $\mathbb{R}^3$. Then $X \overset{q}{\simeq} Y$ and $Y \overset{q}{\simeq} Z$ hold, while X is not quasi-equivalent to Z.

Recall now the category characterization of the quasi-equivalence obtained in [15].

A morphism of inverse sequences $(f, ([f_j])) : \boldsymbol{X} \to Y$ in $(HcANR)^{\mathbb{N}}$ is said to be *special*, provided f increases and, for every $j \in \mathbb{N}$,

$$[f_j][p_{f(j)f(j+1)}] = [q_{j,j+1}][f_{j+1}].$$

Consider the collection $\mathcal{K}$ consisting of the class $Ob(HcANR^{\mathbb{N}}) \equiv Ob\mathcal{K}$ of objects and of the class $Mor\mathcal{K}$ of all the sets

$$\{F = ((f^n, [f_j^n])) \mid (f^n, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y} \text{ special, } n \in \mathbb{N}\ \} \equiv \mathcal{K}(\boldsymbol{X}, \boldsymbol{Y})$$

of morphisms, where $\boldsymbol{X}, \boldsymbol{Y}$ are inverse sequences in $HcANR$, together with the coordinatewise compositions, i.e.

$$GF = ((g^n, [g_k^n]))((f^n, [f_j^n])) = ((f^n g^n, [g_k^n f^n_{g^n(k)}])).$$

By putting $1_{\boldsymbol{X}} = (1_{\boldsymbol{X}}^n)$, where $1_{\boldsymbol{X}}^n = 1_{\boldsymbol{X}}$ for each n, to be the identity morphism on every $\boldsymbol{X}$, one easily verifies that $\mathcal{K}$ is a category.

A morphism $F = ((f^n, [f_j^n])) \in \mathcal{K}(\boldsymbol{X}, \boldsymbol{Y})$ is said to be *quasi-homotopic to* a morphism $F' = ((f'^n, [f'^n_j])) \in \mathcal{K}(\boldsymbol{X}, \boldsymbol{Y})$, denoted by $F \overset{q}{\simeq} F'$, provided there exists an increasing unbounded sequence (s_n) in $\{0\} \cup \mathbb{N}$ such that

$$(\forall n \in \mathbb{N}) s_n \geq 1 \Rightarrow (f^n, [f_j^n]) =_{s_n} (f'^n, [f'^n_j]).$$

Clearly, if $F = ((f^n, [f_j^n])), F' = ((f'^n, [f_j'^n])) \in \mathcal{K}(\mathbf{X}, \mathbf{Y})$ such that $(f^n, [f_j^n])) \simeq ((f'^n, [f_j'^n])$ in $(HcANR)^{\mathbb{N}}$ for almost all n, then $F \overset{q}{\simeq} F'$. The following facts are the immediate consequences of the definition.

- the quasi-homotopy relation $\overset{q}{\simeq}$ is an equivalence relation on each set $\mathcal{K}(\boldsymbol{X}, \boldsymbol{Y})$;
- if $F \overset{q}{\simeq} F'$ then $FH \overset{q}{\simeq} F'H$ for every $H \in \mathcal{K}(\boldsymbol{W}, \boldsymbol{X})$;
- $F \overset{q}{\simeq} F'$ does *not* imply $GF \overset{q}{\simeq} GF'$, $G \in \mathcal{K}(\boldsymbol{Y}, \boldsymbol{Z})$.

The main fact is that $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ if and only if there exist an $F \in \mathcal{K}(\mathbf{X}, \mathbf{Y})$ and a $G \in \mathcal{K}(\mathbf{Y}, \mathbf{X})$ such that $GF \overset{q}{\simeq} 1_{\boldsymbol{X}}$ and $FG \overset{q}{\simeq} 1_{\boldsymbol{Y}}$ ([15], Theorem 3). Consequently, the quasi-equivalence of compacta is characterized in the same way by using any pair of associated compact ANR sequences.

In order to get a better view into the quasi-equivalence, recently is defined a complete (ultra)metric structure on a pro-morphism set, [16]. First, by exploiting the previous idea, given a category $\mathcal{A}$, a pair of morphisms $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ of inv-$\mathcal{A}$ and a $\mu \in M$, (f, f_μ) is said to be *μ-homotopic* to (f', f'_μ), denoted by $(f, f_\mu) \simeq_\mu (f', f'_\mu)$, provided there is a $\lambda \geq f(\mu), f'(\mu)$ such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

Further, if $\boldsymbol{Y}$ (M) is cofinite, given an $n \in \mathbb{N}$, (f, f_μ) is said to be *n-homotopic* to (f', f'_μ), denoted by $(f, f_\mu) \simeq_n (f', f'_\mu)$, provided $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ for every $\mu \in M$ with $|\mu| < n$. (Hereby $|\mu|$ denotes the number of all strict predecessors of μ in M.)

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse systems in a category $\mathcal{A}$, where $\boldsymbol{Y}$ is cofinite. Then the function

$$\rho : \text{inv} - \mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \times \text{inv} - \mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}$$

is well defined by putting

$$\rho((f, f_\mu), (f', f'_\mu)) = \begin{cases} \inf\{\frac{1}{n+1} \mid (f, f_\mu) \simeq_n (f', f'_\mu), \, n \in \mathbb{N}\} \\ \qquad 1, \text{otherwise} \end{cases}.$$

It is readily seen that ρ is a pseudo(ultra)metric on inv-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. Since $(f, f_\mu) \simeq (g, g_\mu)$ and $(f', f'_\mu) \simeq (g', g'_\mu)$ imply $\rho((f, f_\mu), (f', f'_\mu)) = \rho((g, g_\mu), (g', g'_\mu))$, we infer that ρ induces an (ultra)metric

$$d : \boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Y}^{\boldsymbol{X}} \to \mathbb{R}, \quad d(\boldsymbol{f}, \boldsymbol{g}) = \rho((f, f_\mu), (f', f'_\mu)),$$

on the set $\boldsymbol{Y}^{\boldsymbol{X}} \equiv$ *pro*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. Moreover, the following fact holds ([16], Theorem 1):

For every $\boldsymbol{X}$ and every cofinite $\boldsymbol{Y}$, the ordered pair $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is a complete (ultra) metric space.

Applying this fact and the appropriate technique, one can estimate how far is the quasi-equivalence from the shape type ([16], Theorem 7 and Corollary 4):
Two inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ in a category $\mathcal{A}$ are isomorphic, $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow-$\mathcal{A}$, if and only if $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ and there exists a pair of Cauchy sequences realizing this quasi-equivalence.

Consequently, two metrizable compacta X, Y have the same shape, $Sh(X) = Sh(Y)$, if and only if $X \overset{q}{\simeq} Y$ and there exists a pair of Cauchy sequences realizing this quasi-equivalence via a pair of associated compact ANR inverse sequences.

2.1. The $\overline{q}$-shape

In order to obtain a better result, one can strengthen the quasi-equivalence, according to its category characterization, in the following way ([15], Section 5). First, consider the subcategory $\overline{\mathcal{K}} \subseteq \mathcal{K}$ on the same object class such that $\overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y}) \subseteq \mathcal{K}(\boldsymbol{X}, \boldsymbol{Y})$ consists of all morphisms $F = ((f^n, [f_j^n]))$, where all $(f^n, [f_j^n])$ have a unique common index function $f = f^n$, $n \in \mathbb{N}$. Then the quasi-homotopy relation $F \overset{q}{\simeq} F'$ on $\overline{\mathcal{K}}$ is an equivalence relation, which is compatible with the category composition. Thus, there exists the corresponding quotient category $\overline{\mathcal{K}}/(\overset{q}{\simeq}) \equiv \overline{\mathcal{Q}}$.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in $HcANR$. Then $\boldsymbol{X}$ is said to be *$\overline{q}$-equivalent to* $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$, provided $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ and there exists a pair $F = ((f^n, [f_j^n]))$, $G = ((g^n, [g_i^n]))$ of morphisms realizing this relation in the category $\mathcal{K}$ such that, for all $i, j \in \mathbb{N}$, the sequences $(f^n(j))$ and $(g^n(i))$ are bounded. Given a pair X, Y of compacta, we define $X \overset{\overline{q}}{\simeq} Y$ provided $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ for some (equivalently: any) associated pair $\boldsymbol{X}$, $\boldsymbol{Y}$.

The main benefit of the defined strenghtening is this fact:

The $\overline{q}$-equivalence of inverse sequences, as well as of compacta, is an equivalence relation which admits a full category characterization. More precisely, $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ if and only if $\boldsymbol{X} \cong \boldsymbol{Y}$ in $\overline{\mathcal{Q}}$, and similarly for compacta and the corresponding category $\mathcal{Q}$.

To complete this new "shape" theory for compacta, called *$\overline{q}$-shape theory* (modeled on the *$\overline{q}$-shape category* $\mathcal{Q}$), we add the fact concerning the corresponding functors ([15], Theorem 7]):

There exists the $\overline{q}$-shape functor $Q : HcM \to \mathcal{Q}$ (keeping the objects fixed) and there exists the quotient functor $\Gamma : Sh \to \mathcal{Q}$ such that $\Gamma S = Q$, where $S : HcM \to Sh$ is the standard shape functor. Thus, the following diagram commutes:

$$\begin{array}{ccc} & HcM & \\ S \swarrow & & \searrow Q \\ Sh & \underset{\Gamma}{\rightarrow} & \mathcal{Q} \end{array}.$$

Finally, the complete (ultra)metric structure of $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ admits to relate the shape type and $\overline{q}$-equivalence ([16], Theorem 10):

Two inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $HcANR$ are isomorphic, $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow-$HcANR$, if and only if $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ and there is a pair of realizing sequences (morphisms) such that one of them is a Cauchy sequence.

Further ([16], Corollary 7), *being an FANR is an invariant of the $\overline{q}$-equivalence*, i.e. *if $X \overset{\overline{q}}{\simeq} Y$ and X is an* FANR, *then Y is an* FANR *and $Sh(X) = Sh(Y)$.*

Remark 1. *There exists an equivalence relation on compacta, which is strictly finer than the quasi-equivalence and strictly coarser than $\overline{q}$-equivalence. It is characterized by by requirement that at least one of realizing sequences admits a unique index function. In order to verify this, first observe that in such a case the quasi-equivalence is transitive ([16], Remark 4). However, it is not sufficient to become the $\overline{q}$-equivalence. Namely, there exist quasi-equivalent compacta X and Y, that are not $\overline{q}$-equivalent, which admit only one realizing sequence having a unique index function ([16], Example 3).*

2.2. The q^*-shape

Let us observe that the definition of being quasi-homotopic admits a slight strengthening in the category $\overline{\mathcal{K}}$ as follows. A morphism $F \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$ is said to be *uniformly quasi-homotopic to* a morphism $F' \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$, denoted by $F \overset{q^*}{\simeq} F'$, provided $F \overset{q}{\simeq} F'$ and there exists a sequence (i_j) in $\mathbb{N}$, $i_j \geq f(j), f'(j)$, such that

$$(\forall n \in \mathbb{N})(\forall j \in [1, s_n]_{\mathbb{N}})\ [f_j^n][p_{f(j)i_j}] = [f_j'^n][p_{f'(j)i_j}].$$

It is easy to verify that $\overset{q^*}{\simeq}$ is an equivalence relation on $Mor\overline{\mathcal{K}}$, which is compatible with the composition. Let $[F]^*$ and $\overline{\mathcal{Q}}^*$ denote the corresponding class of F and the quotient category $\overline{\mathcal{K}}/(\overset{q^*}{\simeq})$ respectively. Let $\mathcal{Q}^*$ be the category on compacta represented by the category $\overline{\mathcal{Q}}^*$, which may be called the *q^*-shape category.* Further, in the same way as for the category $\mathcal{Q}$, one can obtain the *q^*-shape functor* Q^* : $Hc\mathcal{M} \to \mathcal{Q}^*$ (keeping the objects fixed) and the quotient functor $\Gamma^* : \mathcal{S}h \to \mathcal{Q}^*$ such that $\Gamma^* S = Q^*$. Since $[F]^* \subseteq [F]$, for every $F \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$, there exists the appropriate quotient functor $\overline{\Pi} : \overline{\mathcal{Q}}^* \to \overline{\mathcal{Q}}$, $\overline{\Pi}(\boldsymbol{X}) = \boldsymbol{X}$ and $\overline{\Pi}([F]^*) = [F]$. Then the functor $\overline{\Pi}$ induces the functor $\Pi : \mathcal{Q}^* \to \mathcal{Q}$, $\Pi(X) = X$ and $\Pi(\phi^*) = \overline{\phi}$, where ϕ^* is represented by $[F]^*$, $\overline{\phi}$ is represented by $[F]$ and $F = ((f, [f_j^n])) \in \overline{\mathcal{K}}(\boldsymbol{X}, \boldsymbol{Y})$. Moreover, the $\overline{q}$-shape functor Q factorizes through Q^*, $Q = \Pi Q^*$ and the quotient functor $\Gamma : Sh \to \mathcal{Q}$ factorizes through Γ^*, $\Gamma = \Pi\Gamma^*$. Thus, the following diagram commutes:

$$\begin{array}{ccccc} & & HcM & & \\ & S \swarrow & \downarrow Q^* & \searrow Q & \\ Sh & \underset{\Gamma^*}{\rightarrow} & \mathcal{Q}^* & \underset{\Pi}{\rightarrow} & \mathcal{Q} \end{array}.$$

Consequently, there exists a certain *q^*-shape theory* for metrizable compacta, lying between shape and $\overline{q}$-shape. We do not know yet whether the q^*-shape ($\overline{q}$-shape) is indeed *strictly* coarser than shape (q^*-shape).

3. S-shapes

In this section we bring a review of several "shape" theories arisen from the S-equivalence.

3.1. The S^*-shape

In attempt to provide a category characterization of the S-equivalence, Mardešić and Uglešić considered in [14] a slight strengthening of the S-equivalence, called *S^*-equivalence*, which is defined by requiring that the choice of indices i_k and j_k' does not depend on a given $n \in \mathbb{N}$ (while the mappings $f_{j_k}^n : X_{i_k} \to Y_{j_k}$ and $g_{i_k'}^n : Y_{j_k'} \to X_{i_k'}$ still depend on n). Thus, an equivalent definition may be as follows:

Given $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow\text{-}HcANR)$, we define $S^*(\boldsymbol{X}) = S^*(\boldsymbol{Y})$ provided

$$(\forall j_1 \in \mathbb{N})(\exists i_1 \in \mathbb{N})(\forall i_1' \geq i_1)(\exists j_1' \geq j_1)(\forall j_2 \geq j_1')(\exists i_2 \geq i_1') \cdots$$
$$\cdots (\forall i_k' \geq i_k)(\exists j_k' \geq j_k)(\forall j_{k+1} \geq j_k')(\exists i_{k+1}' \geq i_k) \cdots$$

and, for every $n \in \mathbb{N}$, there exist mappings $f^n_{j_k} : X_{i_k} \to Y_{j_k}$, $k = 1, \cdots, n$, and $g^n_{i'_k} : Y_{j'_k} \to X_{i'_k}$, $k = 1, \cdots, n-1$, such that the diagram

$$\begin{array}{ccccccccc} X_{i_1} & \leftarrow & X_{i'_1} & \leftarrow & X_{i_2} & \leftarrow \cdots \leftarrow & X_{i'_{n-1}} & \leftarrow & X_{i_n} \\ [f^n_{j_1}]\downarrow & & [g^n_{i'_1}]\uparrow & & \downarrow [f^n_{j_2}] & \cdots & [g^n_{i'_{n-1}}]\uparrow & & \downarrow [f^n_{j_n}] \\ Y_{j_1} & \leftarrow & Y_{j'_1} & \leftarrow & Y_{j_2} & \leftarrow \cdots \leftarrow & Y_{j'_{n-1}} & \leftarrow & Y_{j_n} \end{array}$$

commutes. The S^*-equivalence retains all the properties already proved for the S-equivalence. For instance, all the fibres of a shape fibration (base connected) have the same S^*-type. So, we do not know yet whether the S^*-equivalence is indeed strictly finer than S-equivalence. However, we shall show that the S^*-equivalence yields infinitely (countable) many appropriate "shape" theories. The first one is obtained by Mardešić and Uglešić for compact metric spaces as follows, [14].

Let $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow\text{-}HcANR)$. An *$S^*$-mapping* $(f, [f^n_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing unbounded function $f : \mathbb{N} \to \mathbb{N}$ and of a family of homotopy classes of mappings $f^n_j : X_{f(j)} \to Y_j$, $n \in \mathbb{N}$, $j \in \mathbb{N}$, such that there exists an increasing unbounded function $\gamma : \mathbb{N} \to \mathbb{N}$ (*the commutativity radius*), which has the property that, for every $n \in \mathbb{N}$, the following diagram is commutative:

$$\begin{array}{ccccc} X_{f(1)} & \leftarrow & X_{f(2)} & \leftarrow \cdots \leftarrow & X_{f(\gamma(n))} \\ [f^n_1]\downarrow & & \downarrow [f^n_2] & \cdots & \downarrow [f^n_{\gamma(n)}] \\ Y_1 & \leftarrow & Y_2 & \leftarrow \cdots \leftarrow & Y_{\gamma(n)} \end{array}.$$

The *identity* S^*-mapping is defined to be $(1_{\mathbb{N}}, [1_{X_i}])$. If $(f, [f^n_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, [g^n_k]) : \boldsymbol{Y} \to \boldsymbol{Z}$ are S^*-mappings, then their *composition* is defined by $(fg, [g^n_k f^n_{g(k)}])$, which is an S^*-mapping of $\boldsymbol{X}$ to $\boldsymbol{Z}$. All the S^*-mappings on $Ob(tow\text{-}HcANR)$ make a category.

Two S^*-mappings $(f, [f^n_j]), (f', [f'^n_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ are said to be *equivalent* (*homotopic*), $(f, [f^n_j]) \simeq (f', [f'^n_j])$, provided there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ (*shift function*) and there exists an increasing unbounded function $\chi : \mathbb{N} \to \{0\} \cup \mathbb{N}$ (*homotopy radius*) such that, for every $n \in \mathbb{N}$ and every $1 \le j \le \chi(n)$,

$$[f^n_j][p_{f(j)\sigma(j)}] = [f'^n_j][p_{f'(j)\sigma(j)}].$$

This homotopy relation is an equivalence relation on the sets of S^*-mappings, and it is compatible with the category composition. Let $\underline{\mathcal{S}}^*$ be the corresponding quotient category, i.e. $Ob\underline{\mathcal{S}}^* = Ob(tow\text{-}HcANR)$ and

$$\underline{\mathcal{S}}^*(\boldsymbol{X}, \boldsymbol{Y}) = \{\boldsymbol{f}^* \mid \boldsymbol{f}^* = [(f, [f^n_j])] : \boldsymbol{X} \to \boldsymbol{Y}\}.$$

The corresponding category on compacta (via appropriate expansions $\boldsymbol{p} : X \to \boldsymbol{X}$) is denoted by $\mathcal{S}^*$. Now, the main fact is ([14], Theorem 2 and Corollary 1):

Two inverse sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$ of tow-HcANR (two metric compacta X and Y) are S^*-equivalent if and only if they are isomorphic objects of $\underline{\mathcal{S}}^*$ ($\mathcal{S}^*$).

To complete the obtained *S^*-shape theory* for compacta (modeled on the category $\mathcal{S}^*$), let us mention the corresponding functors:

There exists the S^*-shape functor $S^* : HcM \to \mathcal{S}^*$ (keeping the objects fixed) and there exists the "quotient" functor $J : Sh \to \mathcal{S}^*$ such that $JS = S^*$, i.e. the following diagram commutes:

$$\begin{array}{ccc} & HcM & \\ S \swarrow & & \searrow S^* \\ Sh & \underset{J}{\rightarrow} & \mathcal{S}^* \end{array}.$$

3.2. The coarse shape

Recently N. Koceić Bilan and the author succeeded to generalize the S^*-shape theory for compacta to arbitrary topological spaces and, moreover, to any abstract case $(\mathcal{C}, \mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is a dense subcategory (in the sense of $\mathcal{D}$-expansions of the $\mathcal{C}$-objects), [9]. The key facts are the next characterizations of an S^*-mapping and of the homotopy relation ([9], Theorem 3.1 and Theorem 3.2).
(i) Let $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow\text{-}HcANR)$, let $f : \mathbb{N} \to \mathbb{N}$ be an increasing and unbounded function and let, for every $n \in \mathbb{N}$ and every $j \in \mathbb{N}$, $f_j^n : X_{f(j)} \to Y_j$ be a mapping. Then $(f, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ is an S^*-mapping if and only if the following condition is fulfilled:

$$(\forall j, j' \in \mathbb{N}, j \leq j')(\exists n \in \mathbb{N})(\forall n' \geq n)[f_j^{n'}][p_{f(j)f(j')}] = [q_{jj'}][f_{j'}^{n'}],$$

$$\begin{array}{ccc} X_{f(j)} & \leftarrow & X_{f(j')} \\ [f_j^{n'}] \downarrow & & \downarrow [f_{j'}^{n'}] \\ Y_j & \leftarrow & Y_{j'} \end{array}$$

(ii) An S^*-mapping $(f, [f_j^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ is homotopic to an S^*-mapping $(f', [f_j'^n]) : \boldsymbol{X} \to \boldsymbol{Y}$ if and only if

$$(\forall j \in \mathbb{N})(\exists i \in \mathbb{N}, i \geq f(j), f'(j))(\exists n \in \mathbb{N})(\forall n' \geq n),\ [f_j^{n'}][p_{f(j)i}] = [f_j'^{n'}][p_{f'(j)i}],$$

$$\begin{array}{ccc} X_{f(j)} & X_{f'(j)} & \leftarrow X_i \\ [f_j^{n'}] \downarrow & \swarrow [f_j'^{n'}] & \\ Y_j & & \end{array}$$

This admits to construct an abstract analogue of the category $\underline{\mathcal{S}}^*$, denoted by tow^*-$\mathcal{A}$, for any category $\mathcal{A}$ and the corresponding tow-$\mathcal{A}$. Clearly, tow^*-$HcANR$ is isomorphic to $\underline{\mathcal{S}}^*$. Since there exists a faithful functor $\underline{J} : tow\text{-}\mathcal{A} \to tow^*\text{-}\mathcal{A}$ keeping the objects fixed, one may consider tow-$\mathcal{A}$ to be a subcategory of tow^*-$\mathcal{A}$.

The next step was to generalize the construction of tow^*-$\mathcal{A}$ to the corresponding "pro"-category pro^*-$\mathcal{A}$. Briefly, an S^*-morphism of inverse systems $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is subjected to the requirement

$$(\forall \mu, \mu' \in M, \mu \leq \mu')(\exists \lambda \in \Lambda, \lambda \geq f(\mu), f(\mu'))(\exists n \in \mathbb{N})(\forall n' \geq n)$$
$$f_\mu^{n'} p_{f(\mu)\lambda}] = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda},$$

while the equivalence ("homotopy") relation $(f, f_\mu^n) \sim (f', f_\mu'^n)$ means:

$$(\forall \mu \in M)(\exists \lambda \in \Lambda, \lambda \geq f(\mu), f'(\mu))(\exists n \in \mathbb{N})(\forall n' \geq n)$$
$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}].$$

Again, one may consider that $\underline{J} : pro\text{-}\mathcal{A} \to pro^*\text{-}\mathcal{A}$ is an "inclusion" functor. From now on, given a category pair $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{D} \subseteq \mathcal{C}$ is dense, the construction follows the well known procedure of constructing the (abstract) shape category $Sh_{(\mathcal{C},\mathcal{D})}$ and shape functor $S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$. Hereby, of course, the role of pro-$\mathcal{D}$ takes *pro**-$\mathcal{D}$. As a resume:

There exists a category $Sh^*_{(\mathcal{C},\mathcal{D})}$, *called* coarse shape category, *and there exists a functor* $S^* : \mathcal{C} \to Sh^*_{(\mathcal{C},\mathcal{D})}$, *called* coarse shape functor, *such that* $S^* = JS$, *i.e. the following diagram commutes*:

$$\begin{array}{ccc} & \mathcal{C} & \\ S \swarrow & & \searrow S^* \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{J}{\to} & Sh^*_{(\mathcal{C},\mathcal{D})} \end{array} .$$

The underlying theory is called the *abstract coarse shape theory.* The most important example is $\mathcal{C} = HTop$ and $\mathcal{D} = HPol$ (or $HANR$). In that case, we speak about the *standard* (or *ordinary*) coarse shape theory modeled on the standard coarse shape category $Sh^* \equiv Sh^*_{(HTop,HPol)}$. The restriction to metrizable compacta yields the coarse shape theory modeled on $Sh^*(cM) \equiv Sh^*_{(HcM,HcPol)} \cong Sh^*_{(HcM,HcANR)}$. Recall that coarse shape classification on compacta and the S^*-equivalence coincide. Thus, according to [7] or [9] (Examples 7.1 and 7.2), the coarse shape type classification is indeed strictly coarser than the shape type classification.

3.3. The subshape spectrum

In [18], B. Červar and the author have fully characterized the S^*-equivalence in a quite different manner comparing to [14]. They constructed a new kind of morphisms of inverse sequences started with the notion of an n-ladder, $n \in \mathbb{N}$. Given any $j_1 < \cdots < j_{n+1}$ in $\mathbb{N}$, let us denote the corresponding $(n+1)$-tuple $(j_1, \ldots, j_{n+1})$ by $\boldsymbol{j}^n$, and the set of all such $\boldsymbol{j}^n$ by $\boldsymbol{J}(n)$. The limit case $n \to \infty$, $\boldsymbol{j}^\omega \in \boldsymbol{J}(\omega)$, holds as well. For instance,

$$\boldsymbol{J}(1) = \{\boldsymbol{j}^1 = (j_1, j_2) \mid j_1, j_2 \in \mathbb{N},\ j_1 < j_2\},$$

while every $\boldsymbol{j}^\omega$ is a strictly increasing sequence (j_l) in $\mathbb{N}$. Given any $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow$-$HcANR)$, a 1-*ladder* $\boldsymbol{f}_{\boldsymbol{j}^1} = (f, [f_j])$ of $\boldsymbol{X}$ to $\boldsymbol{Y}$ over a $\boldsymbol{j}^1 \in \boldsymbol{J}(1)$, denoted by $\boldsymbol{f}_{\boldsymbol{j}^1} : \boldsymbol{X} \to \boldsymbol{Y}$, consists of an increasing function f whose domain is either empty or an initial segment $[j_1, \alpha_1]_{\mathbb{N}} \subseteq [j_1, j_2 - 1]_{\mathbb{N}}$, $j_1 \leq \alpha_1 < j_2$,

$$f : [j_1, \alpha_1]_{\mathbb{N}} \to [j_1, j_2 - 1]_{\mathbb{N}},$$

and, in the later case, of homotopy classes of mappings

$$f_j : X_{f(j)} \to Y_j, \quad j = 1, \ldots, \alpha_1,$$

such that

$$(\forall j \leq j')\, [f_j][p_{f(j)f(j')}] = [q_{jj'}][f_{j'}].$$

A more general is the notion of an n-ladder which is obtained by fitting together n 1-ladders. Thus, an n-*ladder* $\boldsymbol{f}_{\boldsymbol{j}^n} = (f, [f_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ over a $\boldsymbol{j}^n \in \boldsymbol{J}(n)$ consists of an increasing (index) function

$$f : \bigcup_{l=1}^{n} [j_l, \alpha_l]_{\mathbb{N}} \to [j_1, j_{n+1} - 1]_{\mathbb{N}}, \quad j_l \leq \alpha_l < j_{l+1},$$

and of a set of the homotopy classes of mappings

$$f_j : X_{f(j)} \to Y_j, \quad j \in \bigcup_{l=1}^{n} [j_l, \alpha_l]_{\mathbb{N}},$$

such that the following two conditions are satisfied:

$(L(n)_1)$ $(\forall l \in [1,n]_{\mathbb{N}})\ f(j_l) \geq j_l \wedge f(\alpha_l) < j_{l+1}$;

$(L(n)_2)$ $(\forall j, j' \in \bigcup_{l=1}^{n} [j_l, \alpha_l]_{\mathbb{N}})\ j \leq j' \Rightarrow [f_j][p_{f(j)f(j')}] = [q_{jj'}][f_{j'}]$.

An n-ladder $\boldsymbol{f}_{\boldsymbol{j}^n}$ having an *empty l-block*, i.e. with no mapping for any $j \in [j_l, j_{l+1} - 1]_{\mathbb{N}}$, is allowed. We also admit the *empty n-ladder* of $\boldsymbol{X}$ to $\boldsymbol{Y}$ over a $\boldsymbol{j}^n$, i.e. the empty set of homotopy classes of mappings for a given $\boldsymbol{j}^n$.

Observe that every special mappings of inverse sequences $\boldsymbol{f} = (f, [f_j]) : \boldsymbol{X} \to \boldsymbol{Y}$, with $f \geq 1_{\mathbb{N}}$, induces an appropriate n-ladder $\boldsymbol{f}_{\boldsymbol{j}^n}$ for each $n \in \mathbb{N} \cup \{\omega\}$ and each $\boldsymbol{j}^n \in \boldsymbol{J}(n)$. Especially, the identity mapping $\boldsymbol{1}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ induces the *identity n-ladder* $\boldsymbol{1}_{\boldsymbol{X}\boldsymbol{i}^n} : \boldsymbol{X} \to \boldsymbol{X}$ over $\boldsymbol{i}^n \in \boldsymbol{J}(n)$.

If $\boldsymbol{f}_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}_{\boldsymbol{k}^n} = (g, [g_k]) : \boldsymbol{Y} \to \boldsymbol{Z}$ are n-ladders, then we *compose* them only in the case $\boldsymbol{j}^n = \boldsymbol{k}^n$ by using the ordinary rule, i.e.

$$\boldsymbol{g}_{\boldsymbol{k}^n} \boldsymbol{f}_{\boldsymbol{k}^n} \equiv \boldsymbol{u}_{\boldsymbol{k}^n} = (u, [u_k]),$$

such that $u = fg$ (wherever it is defined) and $u_k = g_k f_{g(k)}$, $k \in \bigcup_{l=1}^{n} [k_l, \gamma_l]_{\mathbb{N}}$, $\gamma_l \leq \beta_l$. Clearly, $\boldsymbol{g}_{\boldsymbol{k}^n} \boldsymbol{f}_{\boldsymbol{k}^n} : \boldsymbol{X} \to \boldsymbol{Z}$ is an n-ladder of $\boldsymbol{X}$ to $\boldsymbol{Z}$ over $\boldsymbol{k}^n$. Notice that its l-block is empty whenever the corresponding block of $\boldsymbol{f}_{\boldsymbol{k}^n}$ or $\boldsymbol{g}_{\boldsymbol{k}^n}$ is empty, or $g(k_l) > \alpha_l$. It is obvious that the composition of n-ladders is associative, and that

$$\boldsymbol{f}_{\boldsymbol{j}^n} \boldsymbol{1}_{\boldsymbol{X}\boldsymbol{j}^n} = \boldsymbol{f}_{\boldsymbol{j}^n},$$
$$\boldsymbol{1}_{\boldsymbol{X}\boldsymbol{i}^n} \boldsymbol{g}_{\boldsymbol{i}^n} = \boldsymbol{g}_{\boldsymbol{i}^n}$$

hold for all n-ladders $\boldsymbol{f}_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}_{\boldsymbol{i}^n} : \boldsymbol{Z} \to \boldsymbol{X}$. Therefore, for each $\boldsymbol{j}^n \in \boldsymbol{J}(n)$, there exists a certain category whose class of objects is *Ob*(*tow-HcANR*), and the sets of morphisms consist of all the corresponding n-ladders.

Let $\boldsymbol{f}_{\boldsymbol{j}^1}, \boldsymbol{f}'_{\boldsymbol{j}^1} = (f', [f'_j]) : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-ladders over the same $\boldsymbol{j}^1$. Then $\boldsymbol{f}_{\boldsymbol{j}^1}$ is said to be *homotopic to* $\boldsymbol{f}'_{\boldsymbol{j}^1}$ provided they both are empty or there exists a $j_1^* \in [j_1, \min\{\alpha_1, \alpha'_1\}]_{\mathbb{N}}$ such that

$$(\forall j \in [j_1, j_1^*]_{\mathbb{N}})(\exists i = i(j) \in [\max\{f(j), f'(j)\}, j_2 - 1]_{\mathbb{N}})$$
$$[f_j][p_{f(j)i}] = [f'_j][p_{f'(j)i}].$$

In the general case of a pair of n-ladders, the definition of being m-homotopic, $m \leq n$, is as follows:

Let $n, m \in \mathbb{N} \cup \{\omega\}$, $m \leq n$, and let $\boldsymbol{f}_{\boldsymbol{j}^n}, \boldsymbol{f}'_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$ be n-ladders over the same $\boldsymbol{j}^n$. Then, $\boldsymbol{f}_{\boldsymbol{j}^n}$ is said to be *m-homotopic to* $\boldsymbol{f}'_{\boldsymbol{j}^n}$, denoted by $\boldsymbol{f}_{\boldsymbol{j}^n} \simeq_m \boldsymbol{f}'_{\boldsymbol{j}^n}$, provided, for every $l \in [1,m]_{\mathbb{N}}$, the both $\boldsymbol{f}_{\boldsymbol{j}^n}$ and $\boldsymbol{f}'_{\boldsymbol{j}^n}$ have the l-block empty or there exists a $j_l^* \in [j_l, \min\{\alpha_l, \alpha'_l\}]_{\mathbb{N}})$ such that

$$(\forall j \in [j_l, j_l^*]_{\mathbb{N}})(\exists i = i(j) \in [\max\{f(j), f'(j)\}, j_{l+1} - 1]_{\mathbb{N}})$$
$$[f_j][p_{f(j)i}] = [f'_j][p_{f'(j)i}].$$

Notice that $\boldsymbol{f}_{\boldsymbol{j}^n} \simeq_{m'} \boldsymbol{f}'_{\boldsymbol{j}^n}$ implies $\boldsymbol{f}_{\boldsymbol{j}^n} \simeq_m \boldsymbol{f}'_{\boldsymbol{j}^n}$ whenever $m \leq m'$. Clearly, the m-homotopy relation of n-ladders is an equivalence relation on the corresponding set. In the case of $m = n$, we simply write $\boldsymbol{f}_{\boldsymbol{j}^n} \simeq \boldsymbol{f}'_{\boldsymbol{j}^n}$ and say that $\boldsymbol{f}_{\boldsymbol{j}^n}$ and $\boldsymbol{f}'_{\boldsymbol{j}^n}$ are *homotopic*.

Let us now recall the main notion. First the simplest case $n = 1$. A 1-*hyperladder* of $\boldsymbol{X}$ to $\boldsymbol{Y}$ is a certain family F_1 of 1-ladders (of $\boldsymbol{X}$ to $\boldsymbol{Y}$) indexed by all pairs $\boldsymbol{j}^1 = (j_1, j_2) \in \boldsymbol{J}(1)$. We require that every two elements $j_1 \leq j'_1$ of $\mathbb{N}$ admit an $i^1 \in \mathbb{N}$, $i^1 \geq j'_1$, such that, for every $j_2 > i^1$, the 1-ladder $\boldsymbol{f}_{\boldsymbol{j}^1} = (f, [f_j]) \in F_1$, assigned to the pair $\boldsymbol{j}^1 = (j_1, j_2) \in \boldsymbol{J}(1)$, has the following two properties:
- the domain $[j_1, \alpha_1]_{\mathbb{N}}$ of the index function f contains $[j_1, j'_1]_{\mathbb{N}}$;
- the image $f[j_1, j'_1]_{\mathbb{N}}$ is contained in $[j_1, i^1]_{\mathbb{N}}$.
Briefly, a family $F_1 = (\boldsymbol{f}_{\boldsymbol{j}^1})$ of 1-ladders $\boldsymbol{f}_{\boldsymbol{j}^1} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{j}^1 \in \boldsymbol{J}(1)$, is said to be a 1-hyperladder of $\boldsymbol{X}$ to $\boldsymbol{Y}$, provided

$$(\forall j_1 \in \mathbb{N})(\forall j'_1 \geq j_1)(\exists i^1 \geq j'_1)(\forall j_2 > i^1)$$

the index function of the corresponding $\boldsymbol{f}_{\boldsymbol{j}^1} = (f, [f_j]) \in F_1$ fulfills the following two conditions:

$$\alpha_1 \geq j'_1 \quad \text{and} \quad f(j'_1) \leq i^1.$$

Notice that, since f increases, the second condition implies $f(j) \leq i^1$ for every $j \in [j_1, j'_1]_{\mathbb{N}}$.

A family $F_n = (\boldsymbol{f}_{\boldsymbol{j}^n})$ of n-ladders $\boldsymbol{f}_{\boldsymbol{j}^n} : \boldsymbol{X} \to \boldsymbol{Y}$, indexed by all $\boldsymbol{j}^n \in \boldsymbol{J}(n)$, is said to be an n-*hyperladder* of $\boldsymbol{X}$ to $\boldsymbol{Y}$, denoted by $F_n : \boldsymbol{X} \to \boldsymbol{Y}$, provided
$(\forall m \leq n)$
$(\forall j_1 \in \mathbb{N})(\forall j'_1 \geq j_1)(\exists i^1 \geq j'_1)(\forall j_2 > i^1) \cdots$
$(\forall j_m > i^{m-1})(\forall j'_m \geq j_m)(\exists i^m \geq j'_m)(\forall j_{m+1} > i^m)$
$(\forall j_{m+2} > j_{m+1}) \ldots (\forall j_{n+1} > j_n)$
the index function of the corresponding $\boldsymbol{f}_{\boldsymbol{j}^n} = (f, [f_j]) \in F_n$ fulfills the following two conditions:
$(S(n,m)_1)$ $(\forall l \in [1,m]_{\mathbb{N}})$ $\alpha_l \geq j'_l$;
$(S(n,m)_2)$ $(\forall l \in [1,m]_{\mathbb{N}})$ $f(j'_l) \leq i^l$.
The set of all n-hyperladders $F_n : \boldsymbol{X} \to \boldsymbol{Y}$ is denoted by $\underline{L}_n(\boldsymbol{X}, \boldsymbol{Y})$.

Notice that every special mapping of inverse sequences $\boldsymbol{f} = (f, [f_j]) : \boldsymbol{X} \to \boldsymbol{Y}$, with $f \geq 1_{\mathbb{N}}$, induces an appropriate n-hyperladder $F_n = (\boldsymbol{f}_{\boldsymbol{j}^n})$, for each $n \in \mathbb{N} \cup \{\omega\}$. In particular, the identity mapping $\boldsymbol{1}_{\boldsymbol{X}} = (1_{\mathbb{N}}, ([1_{X_i}]))$ induces *the identity* n-*hyperladder* $1_{\boldsymbol{X}n} = (\boldsymbol{1}_{\boldsymbol{X}\boldsymbol{i}^n}) : \boldsymbol{X} \to \boldsymbol{X}$, $\boldsymbol{i}^n \in \boldsymbol{J}(n)$.

If $F_n = (\boldsymbol{f}_{\boldsymbol{j}^n}) : \boldsymbol{X} \to \boldsymbol{Y}$ and $G_n = (\boldsymbol{g}_{\boldsymbol{k}^n}) : \boldsymbol{Y} \to \boldsymbol{Z}$, $\boldsymbol{k}^n \in \boldsymbol{J}(n)$, are n-hyperladders, then we *compose* them by composing the appropriate n-ladders $\boldsymbol{f}_{\boldsymbol{j}^n}$ and $\boldsymbol{g}_{\boldsymbol{k}^n}$ such that $\boldsymbol{j}^n = \boldsymbol{k}^n$. Hence,

$$G_n F_n \equiv U_n = (\boldsymbol{u}_{\boldsymbol{k}^n}),$$

where $\boldsymbol{u}_{\boldsymbol{k}^n} \equiv \boldsymbol{g}_{\boldsymbol{k}^n} \boldsymbol{f}_{\boldsymbol{k}^n}$, $\boldsymbol{k}^n \in \boldsymbol{J}(n)$. One straightforwardly proves that the composition of two n-hyperladders is an n-hyperladder, that it is associative and that the identities are the induced ones. Thus, for every $n \in \mathbb{N} \cup \{\omega\}$, there exists a category $\underline{\mathcal{L}}(n)$ consisting of the object class $Ob\underline{\mathcal{L}}(n) = Ob(tow\text{-}HcANR)$ and of the class $\mathrm{Mor}\underline{\mathcal{L}}(n)$ of all the morphism sets $\underline{L}_n(\boldsymbol{X}, \boldsymbol{Y})$.

In order to define a certain equivalence (homotopy) relation on each set $\underline{L}_n(\boldsymbol{X},\boldsymbol{Y})$, let us first consider the simplest case $n=1$. Let $F_1=(\boldsymbol{f}_{\boldsymbol{j}^1}), F_1'=(\boldsymbol{f}'_{\boldsymbol{j}^1}):\boldsymbol{X}\to\boldsymbol{Y}$ be a pair of 1-hyperladders. Then F_1 is said to be *homotopic to* F_1', provided every two elements $j_1\leq j_1'$ of $\mathbb{N}$ admit an $i_*^1\in\mathbb{N}$, $i_*^1\geq j_1'$, such that, for every $j_2>i_*^1$, the corresponding 1-ladders $\boldsymbol{f}_{\boldsymbol{j}^1}\in F_1$ and $\boldsymbol{f}'_{\boldsymbol{j}^1}\in F_1'$ (assigned to the pair $\boldsymbol{j}^1=(j_1,j_2)\in\boldsymbol{J}(1)$) are homotopic, $\boldsymbol{f}_{\boldsymbol{j}^1}\simeq\boldsymbol{f}'_{\boldsymbol{j}^1}$ and, in addition, the occurring $j_1^*\geq j_1'$ and $i=i(j_1')\leq i_*^1$.

Briefly, $F_1\simeq F_1'$ provided

$$(\forall j_1\in\mathbb{N})(\forall j_1'\geq j_1)(\exists i_*^1\geq j_1')(\forall j_2>i_*^1)$$

the corresponding $\boldsymbol{f}_{\boldsymbol{j}^1}\in F_1$ and $\boldsymbol{f}'_{\boldsymbol{j}^1}\in F_1'$ are homotopic, $\boldsymbol{f}_{\boldsymbol{j}^1}\simeq\boldsymbol{f}'_{\boldsymbol{j}^1}$, such that $j_1^*\geq j_\lambda'$ and $i=i(j_1')\leq i_*^1$. Notice that the last condition implies that $i=i(j)\leq i_*^1$ for every $j\in[j_1,j_1']_{\mathbb{N}}$. The definition in general is as follows.

Let $n\in\mathbb{N}\cup\{\omega\}$ and let $F_n=(\boldsymbol{f}_{\boldsymbol{j}^n}), F_n'=(\boldsymbol{f}'_{\boldsymbol{j}^n}):\boldsymbol{X}\to\boldsymbol{Y}$ be n-hyperladders. Then F_n is said to be *homotopic to* F_n', denoted by $F_n\simeq F_n'$, provided

$(\forall m\leq n)$
$(\forall j_1\in\mathbb{N})(\forall j_1'\geq j_1)(\exists i_*^1\geq j_1')(\forall j_2>i_*^1)\cdots$
$(\forall j_m>i_*^{m-1})(\forall j_m'\geq j_m)(\exists i_*^m\geq j_m')(\forall j_{m+1}>i_*^m)$
$(\forall j_{m+2}>j_{m+1})\ldots(\forall j_{n+1}>j_n)$

the corresponding n-ladders $\boldsymbol{f}_{\boldsymbol{j}^n}\in F_n$ and $\boldsymbol{f}'_{\boldsymbol{j}^n}\in F_n'$ satisfy the following condition:
$(H(n,m))$ $\boldsymbol{f}_{\boldsymbol{j}^n}\simeq_m\boldsymbol{f}_{\boldsymbol{j}^n}$,
i.e. for every $l\in[1,m]_{\mathbb{N}}$ there exists a $j_l^*\in[j_l,\min\{\alpha_l,\alpha_l'\}]_{\mathbb{N}})$ for which
$(\forall j\in[j_l,j_l^*]_{\mathbb{N}})(\exists i=i(j)\in[\max\{f(j),f'(j)\},j_{l+1}-1]_{\mathbb{N}})$
$[f_j][p_{f(j)i}]=[f_j'][p_{f'(j)i}]$,
such that, in addition,
$(\forall l\in[1,m]_{\mathbb{N}})$, $j_l^*\geq j_l'$ and
$(\forall l\in[1,m]_{\mathbb{N}})$ $i=i(j_l')\leq i_*^l$.

Observe that the last condition implies that $i=i(j)\leq i_*^l$, for every l and every $j\in[j_l,j_l']_{\mathbb{N}}$. Further, for the indices i_*^l in this definition and for the indices i^l, i' (for F_n, F_n' respectively),
$(\forall l\in[1,m]_{\mathbb{N}})$ $i_*^l\geq\max\{i^l,i'^l\}$
must hold. A very nontrivial fact is that the homotopy of n-hyperladders is an equivalence relation which is compatible with the category composition. So, finally, we have got the sequence of quotient categories (originally denoted by $\underline{\mathcal{S}}(n)$)

$$\underline{\mathcal{S}}_*(n)=\underline{\mathcal{L}}(n)/(\simeq),\quad n\in\mathbb{N}\cup\{\omega\},$$

on the object class $Ob(tow.HcANR)$ such that

$$\underline{\mathcal{S}}_*(n)(\boldsymbol{X},\boldsymbol{Y})=\{\boldsymbol{F}_n=[F_n]\mid F_n\in\underline{L}_n(\boldsymbol{X},\boldsymbol{Y})\}=\underline{\mathcal{L}}(n)(\boldsymbol{X},\boldsymbol{Y})/(\simeq)$$

and $\boldsymbol{G}_n\boldsymbol{F}_n=[G_n][F_n]=[G_nF_n]$ ([18], Theorem 2.9).

Notice that this sequence of categories yields a "sequential" category $\underline{\mathcal{S}}_*(\mathbb{N})$ on the same object class, where

$$\underline{\mathcal{S}}_*(\mathbb{N})(\boldsymbol{X},\boldsymbol{Y})=\{\boldsymbol{F}\equiv(\boldsymbol{F}_n)\mid\boldsymbol{F}_n=[F_n]\in\underline{\mathcal{S}}_*(n)(\boldsymbol{X},\boldsymbol{Y}),n\in\mathbb{N}\},$$

and $\boldsymbol{GF} = (\boldsymbol{G}_n)(\boldsymbol{F}_n) = (\boldsymbol{G}_n\boldsymbol{F}_n)$.

All the constructed categories are related by appropriate functors, which keep the objects fixed and mutually commute according to the indices ([18], Theorems 2.11, 2.12, and 2.13). Moreover, the isomorphism classification in the category $\underline{\mathcal{S}}_*(\omega)$ coincides with that of *tow-HcANR* ([18], Theorem 3.4 (i)).

In the same way as the shape category of compacta $Sh(cM)$ is defined via *tow-HcANR*, there exists, for each $n \in \mathbb{N} \cup \{\omega\}$, the "$*$-shape" category of compacta $\mathcal{S}_*(n)$ which is defined (realized) via $\underline{\mathcal{S}}_*(n)$, i.e.

$$Ob(\mathcal{S}_*(n)) = Ob(c\mathcal{M}) \quad \text{and} \quad \mathcal{S}_*(n)(X,Y) \approx \underline{\mathcal{S}}_*(n)(\boldsymbol{X},\boldsymbol{Y}).$$

Further, every functor relating a pair of $\underline{\mathcal{S}}_*(n)$, $\underline{\mathcal{S}}_*(n')$, $\underline{\mathcal{S}}_*(\mathbb{N})$, *tow-HcANR* induces the corresponding functor relating the pair of $\mathcal{S}_*(n)$, $\mathcal{S}_*(n')$, $\mathcal{S}_*(\mathbb{N})$, $Sh(cM)$ respectively ([18], Corollary 2) .The isomorphism classification in $S_*(\omega)$ coincides with the shape type classification. The corresponding commutative diagrams are given below.

$$\begin{array}{ccc} & Sh & \\ T_n \swarrow & & \searrow T_{n'} \\ \mathcal{S}_*(n) & \underset{R_{nn'}}{\leftarrow} & \mathcal{S}_*(n') \end{array}, \quad \begin{array}{ccc} & Sh & \\ T_n \swarrow & & \searrow T \\ \mathcal{S}_*(n) & \underset{P_n}{\leftarrow} & \mathcal{S}_*(\mathbb{N}) \end{array}$$

$$\begin{array}{ccc} & HcM & \\ S_n \swarrow & & \searrow S \\ \mathcal{S}_*(n) & \underset{T_n}{\leftarrow} & Sh \end{array}, \quad \begin{array}{ccc} & HcM & \\ \Sigma \swarrow & & \searrow S \\ \mathcal{S}_*(\mathbb{N}) & \underset{T}{\leftarrow} & Sh \end{array}.$$

The obtained graded family of categories and functors $(\mathcal{S}_*(n); S_n, T_n, R_{nn'})$, $n \leq n' \in \mathbb{N} \cup \{\omega\}$, was called the *subshape spectrum* for compacta.

As the main application of this "$*$-shape" theory we mention the following two facts:

(i) *Two metrizable compacta X and Y are S^*-equivalent if and only if they are isomorphic objects of* $\underline{\mathcal{S}}_*(\mathbb{N})$ ([18], Theorem 4.7);

(ii) *The q^*-equivalence strictly implies S^*-equivalence*, i.e. *the q^*-shape is strictly finer than S^*-shape* ([18], Corollary 5.7).

3.4. The weak shape

In the very recent paper [20], Červar and the author succeeded to generalize the above "subshape" theory to arbitrary topological spaces, and moreover, to any category pair $(\mathcal{C}, \mathcal{D})$, whenever $\mathcal{D} \subseteq \mathcal{C}$ is dense. However, the generalization is made only in the case $n = 1$, because it is proved ([20], Remark 2) that, although the categories $\mathcal{S}^*(n)$ and $\mathcal{S}^*(n')$, $n \neq n'$, are not equivalent, the isomorphism classifications in all the categories $\mathcal{S}^*(n)$ (and $\mathcal{S}_*(\mathbb{N})$ as well) coincide with the S^*-equivalence.

Notice that a generalization of a ladder (and of a hyperladder as well) to inverse systems supposes the same index set. Hence, the first problem was the passage from the index set $\mathbb{N}$ (positive integers) to an arbitrary common index set Λ (directed, ordered, infinite, cofinite, having no maximal element). Further, since a $\mathcal{C}$-object admits many $\mathcal{D}$-expansions, the second problem was the independence of a chosen index set.

The first step was to pull out the *reduced* inv- and pro-categories, which are subcategories of inv- and pro-categories respectively. Given a category $\mathcal{A}$, the category $inv^{\sim}$-$\mathcal{A}$ is defined by requirement (based on [13], Theorem I.1.3)

$$inv^{\sim} - \mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \begin{cases} inv\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}),\ \Lambda = M \\ \varnothing,\ \Lambda \neq M. \end{cases}$$

Clearly, for every Λ, there exists the full subcategory inv^{Λ}-$\mathcal{A} \subseteq inv^{\sim}$-$\mathcal{A}$. Especially, for $\Lambda = \mathbb{N}$, $inv^{\mathbb{N}}$-$\mathcal{A} = \mathcal{A}^{\mathbb{N}}$. The corresponding reduced pro-category is the quotient category, i.e.

$$pro^{\sim} - \mathcal{A} = (inv^{\sim} - \mathcal{A})/(\simeq).$$

For a fixed Λ, there is the full subcategory pro^{Λ}-$\mathcal{A} \subseteq pro^{\sim}$-$\mathcal{A}$, and in the case $\Lambda = \mathbb{N}$, $pro^{\mathbb{N}}$-$\mathcal{A} = tow$-$\mathcal{A}$.

A *ladder* of an $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ to a $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, with $M = \Lambda$, over a segment

$$\boldsymbol{\mu} = [\mu_1, \mu_2] = \{\mu \in \Lambda \mid \mu_1 \leq \mu \leq \mu_2\} \subseteq \Lambda,$$

denoted by $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, consists of an increasing (index) function

$$f : \boldsymbol{J} \to \boldsymbol{\lambda} = \mu,$$

where $\boldsymbol{J}$ is an initial subset of $\boldsymbol{\mu}$, and of $\mathcal{A}$-morphisms

$$f_\mu : X_{f(\mu)} \to Y_\mu, \quad \mu \in \boldsymbol{J},$$

such that, for every related pair $\mu \leq \mu'$,

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}.$$

In the case $\boldsymbol{J} = \varnothing$ (i.e. $\mu_1 \notin \boldsymbol{J}$), $f_{\boldsymbol{\mu}}$ is said to be the *empty* ladder. The *identity ladder* on an $\boldsymbol{X}$ over a $\boldsymbol{\lambda}$, denoted by $1_{\boldsymbol{X}\boldsymbol{\lambda}}$, is given by $1_{\boldsymbol{\lambda}}$ and 1_{X_λ}, $\lambda \in \boldsymbol{\lambda}$. An $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ and a $g_{\boldsymbol{\nu}} : \boldsymbol{Y} \to \boldsymbol{Z}$ admit *composition* provided $\boldsymbol{\mu} = \boldsymbol{\nu}$.

We say that two ladders $f_{\boldsymbol{\mu}}, f'_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, over the same $\boldsymbol{\mu} = \boldsymbol{\lambda}$, are *equivalent* (*homotopic*), denoted by $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$, provided they both are empty or there exists an initial subset $\boldsymbol{J}^* \subseteq \boldsymbol{J} \cap \boldsymbol{J}'$ of $\boldsymbol{\mu}$ such that

$$(\forall \mu \in \boldsymbol{J}^*)(\exists \lambda(\mu) \in \boldsymbol{\mu},\ \lambda \geq f(\mu), f(\mu'))\ f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

The homotopy relation $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$ is an equivalence relation on the set of all ladders of $\boldsymbol{X}$ to $\boldsymbol{Y}$.

A *hyperladder* of $\boldsymbol{X}$ to $\boldsymbol{Y}$, denoted by $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, is a family of ladders $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, indexed by all the segments $\boldsymbol{\mu} = [\mu_1, \mu_2]$ in $\Lambda = M$, such that every related pair $\mu_1 \leq \mu'_1$ in Λ admits a $\lambda^1 \in \Lambda$, $\lambda^1 \geq \mu'_1$, such that, for every $\mu_2 \geq \lambda^1$, the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$, assigned to $\boldsymbol{\mu} = [\mu_1, \mu_2]$, fulfills the requirement that $\mu'_1 \in \boldsymbol{J}$ (the domain of f) and $f(\mu'_1) \leq \lambda^1$. Briefly,

$$(\forall \mu_1 \in \Lambda)(\forall \mu'_1 \geq \mu_1)(\exists \lambda^1 \geq \mu'_1)(\forall \mu_2 \geq \lambda^1)$$

the index function $f : \boldsymbol{J} \to \boldsymbol{\lambda} = \boldsymbol{\mu} = [\mu_1, \mu_2]$ of the corresponding $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ fulfills the following two conditions:

$$\mu_1' \in \boldsymbol{J} \quad \text{and} \quad f(\mu_1') \leq \lambda^1.$$

The *identity hyperladder* on an $\boldsymbol{X}$, denoted by $(1_{\boldsymbol{X}\boldsymbol{\lambda}})$, is given by the family of all the identity ladders. A hyperladder $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ and a hyperladder $(g_{\boldsymbol{\nu}}) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, where $\Lambda = M = N$, are *composing* coordinatewise. All the inverse systems in a category $\mathcal{A}$ and all the appropriate hyperladders form a category, denoted by $inv_*^\sim$-$\mathcal{A}$. Clearly, for each fixed Λ, there exists the corresponding full subcategory inv_*^Λ-$\mathcal{A} \subseteq inv_*^\sim$-$\mathcal{A}$.

Let $(f_{\boldsymbol{\mu}}), (f'_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a pair of hyperladders. Then $(f_{\boldsymbol{\mu}})$ is said to be *equivalent* (*homotopic*) to $(f'_{\boldsymbol{\mu}})$, denoted by $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$, provided

$$(\forall \mu_1 \in \Lambda)(\forall \mu_1' \geq \mu_1)(\exists \lambda_*^1 \geq \mu_1')(\forall \mu_2 \geq \lambda_*^1)$$

the corresponding $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ and $f'_{\boldsymbol{\mu}} \in (f'_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} = [\mu_1, \mu_2]$, are homotopic, $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$, such that, in addition, $\mu_1' \in \boldsymbol{J}^* \subseteq \boldsymbol{J} \cap \boldsymbol{J}'$ and $\lambda(\mu_1') \leq \lambda_*^1$.
The homotopy relation $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$ is an equivalence relation on each set $inv_*^\sim$-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$, and it is compatible with the category composition. The homotopy class $[(f_{\boldsymbol{\mu}})]$ of an $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ is denoted by $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$, and these classes are composing by the rule $\boldsymbol{g}_* \boldsymbol{f}_* = [(g_{\boldsymbol{\nu}})][(f_{\boldsymbol{\mu}})] = [(g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}})]$. Therefore, the resume may be as follows ([20], Theorem 1):

For every category $\mathcal{A}$, *there exists a quotient category* ("$*$-reduced pro-category") $pro_*^\sim$-$\mathcal{A} \equiv (inv_*^\sim$-$\mathcal{A})/(\simeq)$.

Further, *for each fixed* Λ, *there exists the corresponding quotient category*

$$pro_*^\Lambda - \mathcal{A} \equiv (inv_*^\Lambda - \mathcal{A})/(\simeq),$$

which is a full subcategory of $pro_*^\sim$-$\mathcal{A}$.
Especially, in the case $\mathcal{A} = HcANR$ and $\Lambda = \mathbb{N}$, the category $pro_*^{\mathbb{N}}$-$HcANR \equiv tow_*$-$HcANR$ is equal to the before constructed category $\mathcal{S}_*(1)$. In this way, the first problem is solved: For every admissible index set Λ and the subcategory $\mathcal{C}_\Lambda \subseteq \mathcal{C}$ determined by all the $\mathcal{C}$-objects admitting a $\mathcal{D}$-expansion over Λ, there exists so called (abstract) Λ-*weak shape category* $Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$, defined via pro_*^Λ-$\mathcal{D}$ in the usual manner, i.e.

$$\begin{aligned} &Ob(Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}) = Ob\mathcal{C}_\Lambda, \\ &Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}(X, Y) \approx pro_*^\Lambda - \mathcal{D}(\boldsymbol{X}, \boldsymbol{Y}), \end{aligned}$$

whenever $\boldsymbol{X}, \boldsymbol{Y}$ are $\mathcal{D}$-expansions over Λ of X and Y respectively. There also exists the (abstract) Λ-*weak shape functor* $S_*^\Lambda : \mathcal{C}_\Lambda \to Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}$ such that $T^\Lambda S^\Lambda = S_*^\Lambda$, i.e.

$$\begin{array}{ccc} & \mathcal{C}_\Lambda & \\ S^\Lambda \swarrow & & \searrow S_*^\Lambda \\ Sh^\Lambda_{(\mathcal{C},\mathcal{D})} & \underset{T^\Lambda}{\rightarrow} & Sh^\Lambda_{*(\mathcal{C},\mathcal{D})}, \end{array}$$

where S^Λ is the corresponding (abstract) shape functor, and T^Λ is a faithful functor which keeps the objects fixed ([20], Section 5).

The index set changing problem is solved in [20], Section 6. Briefly, for every pair of admissible index sets Λ, Λ', there exists a functor

$$H^{\Lambda,\Lambda'} : Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})} \to Sh^{\Lambda'}_{*(\mathcal{C},\mathcal{D})},$$

which is a category isomorphism keeping the objects fixed. As a resume:

There exists a category $Sh_{*(\mathcal{C},\mathcal{D})}$, called the (abstract) weak shape category, *such that*

$$\begin{aligned}&Ob(Sh_{*(\mathcal{C},\mathcal{D})}) = Ob\mathcal{C},\\ &Sh_{*(\mathcal{C},\mathcal{D})}(X,Y) \approx Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}(X,Y) \approx pro^{\Lambda}_{*} - \mathcal{D}(\boldsymbol{X},\boldsymbol{Y}),\end{aligned}$$

whenever $\boldsymbol{X}$, $\boldsymbol{Y}$ are $\mathcal{D}$-expansions over the same Λ of X and Y respectively. There also exists the (abstract) weak shape functor $S_* : \mathcal{C} \to Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$ *such that* $TS = S_*$, i.e.

$$\begin{array}{ccc} & \mathcal{C} & \\ S \swarrow & & \searrow S_* \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{T}{\to} & Sh_{*(\mathcal{C},\mathcal{D})} \end{array},$$

where S is the corresponding (abstract) shape functor, and T is a faithful functor which keeps the objects fixed.

The most interesting is the standard case $\mathcal{C} = HTop$ and $\mathcal{D} = HANR$ (or $HPol$). Then the notation is simplified to Sh_*. Further, in the special case $\mathcal{C} = HcM$ and $\mathcal{D} = HcANR$ (or $HcPol$), the index set $\mathbb{N}$ suffices. Therefore, the weak shape category for compacta $Sh_*(cM)$ can be realized via tow_*-$HcANR$ or tow_*-$HcPol$.

3.5. Weak shape versus coarse shape

Let us briefly recall only the main results of [20], Section 7, where the weak and coarse shape are compared:

(1) *There exists a functor $W : Sh^*_{(\mathcal{C},\mathcal{D})} \to Sh_{*(\mathcal{C},\mathcal{D})}$, which keeps the objects fixed, and the following diagram commutes:*

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow S & \downarrow S^* & \searrow S_* & \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{J}{\to} & Sh^*_{(\mathcal{C},\mathcal{D})} & \underset{W}{\to} & Sh_{*(\mathcal{C},\mathcal{D})} \end{array}, \quad WJ = T.$$

(2) *If every $\mathcal{C}$-object admits a* countable *$\mathcal{D}$-expansion, then the functor W is* faithful; *especially, the functor $W : Sh^*(cM) \to Sh_*(cM)$ is faithful. Furthermore, in such a case there exists a converse functor $U : Sh_{*(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})}$, which is* not *faithful in general.*

(3) *For every pair of metrizable compacta X, Y, the following are equivalent:*

(*i*) $Sh_*(X) = Sh_*(Y)$;

(*ii*) $Sh^*(X) = Sh^*(Y)$;

(*iii*) $S^*(X) = S^*(Y)$.

There exists a metric continuum X such that its shape type is strictly finer than the above three (coinciding) types of X.
(4) *Let X be a topological space which does not admit any countable $HPol$-expansion, and let Q be a polyhedron consisting of two points. Then*
(*i*) $card(Sh(X,Q) = 2$;
(*ii*) $card(Sh^*(X,Q)) = 2^{\aleph_0}$;
(*iii*) $card(Sh_*(X,Q)) > 2^{\aleph_0}$.
Consequently, in order to provide a pair of spaces belonging to different coarse shape types and to the same weak shape type, one has to consider a class of spaces which do not admit any countable polyhedral (or ANR) expansion.

Let us finally mention some important shape invariants which are invariants of the weak and coarse shape as well ([20], Section 9 ; [8]; [17]). Those are as follows:
(1) In all (abstract) cases: *movability, $\mathcal{D}_0$-movability ($\mathcal{D}_0 \subseteq \mathcal{D}$), semi-stability, strong movability, stability.*
(2) Moreover, for topological spaces: *connectedness, triviality of shape, shape dimension $\leq n$, n-shape connectedness, n-movability.*
(3) For pointed pro-sets and pro-groups: *the Mittag-Leffler property.*

4. The S_n-equivalence

Hereby we demonstrate how the S-equivalence is "decomposed" into a sequence of so called S_n- and S_n^+-equivalences, $n \in \{0\} \cup \mathbb{N}$, providing also the category descriptions of these equivalences as well as a category characterization of the S-equivalence, which is not a full one, [19] and [4]. In [14], Remark 1, the authors had noticed that it makes sense and it could be useful to "decompose" the S-equivalence into "finite parts", called the S_n-equivalences, $n \in \mathbb{N}$. Following this idea, Červar and the author first, in [19], defined and studied those equivalence relations. Afterwards, in [4], they provide a category description for each S_n-equivalence as well as a category characterization of the S-equivalence. Let us briefly recall the definition (Definitions 2.1 and 2.2 of [19] are slightly refined by Definition 1 of [4]).

For every $n \in \mathbb{N}$, the condition relating $\boldsymbol{Y}$ to $\boldsymbol{X}$ given by diagram $(*)$ (of our Section 1) is denoted by (D_{2n-1}). Further, by (D_{2n}) is denoted the extension of (D_{2n-1}) by adding one rectangle (with a mapping g_n) preserving commutativity up to homotopy.

Given any $\boldsymbol{X}, \boldsymbol{Y} \in Ob(tow\text{-}HcANR)$ and $n \in \{0\} \cup \mathbb{N}$, let $S_n(\boldsymbol{X}, \boldsymbol{Y})$ denote condition (D_{2n+1}) relating $\boldsymbol{Y}$ to $\boldsymbol{X}$. Further, let $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ denote condition (D_{2n+2}) relating $\boldsymbol{Y}$ to $\boldsymbol{X}$. Then $\boldsymbol{Y}$ is said to be *S_n-dominated by* $\boldsymbol{X}$, denoted by $S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X})$, provided condition $S_n(\boldsymbol{Y}, \boldsymbol{X})$ holds; $\boldsymbol{Y}$ is said to be *S_n-equivalent to* $\boldsymbol{X}$, denoted by $S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$, provided the both conditions $S_n(\boldsymbol{Y}, \boldsymbol{X})$ and $S_n(\boldsymbol{X}, \boldsymbol{Y})$ are fulfilled. Similarly and dually, $\boldsymbol{Y}$ is said to be *S_n^+-dominated by* $\boldsymbol{X}$, denoted by $S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})$, provided condition $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ holds; $\boldsymbol{Y}$ is said to be *S_n^+-equivalent to* $\boldsymbol{X}$, denoted by $S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})$, provided the both conditions $S_n^+(\boldsymbol{X}, \boldsymbol{Y})$ and $S_n^+(\boldsymbol{Y}, \boldsymbol{X})$ are fulfilled.

If X and Y are compacta, then we define $S_n(Y) \leq S_n(X)$ and $S_n(Y) = S_n(X)$ ($S_n^+(Y) \leq S_n^+(X)$ and $S_n^+(Y) = S_n^+(X)$) provided $S_n(\boldsymbol{Y}) \leq S_n(\boldsymbol{X})$ and $S_n(\boldsymbol{Y}) =$

$S_n(\boldsymbol{X})$ ($S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})$ and $S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})$) respectively, for some (equivalently, any) compact ANR inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ associated with X, Y respectively.

It is obviously, by definition, that

$(S_{n+1}(\boldsymbol{Y}) \leq S_{n+1}(\boldsymbol{X})) \Rightarrow (S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X}))$,

$(S_n^+(\boldsymbol{Y}) \leq S_n^+(\boldsymbol{X})) \Rightarrow (S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X}))$,

$(S(\boldsymbol{Y}) = S(\boldsymbol{X})) \Leftrightarrow ((\forall n \in \{0\} \cup \mathbb{N}), S_n(\boldsymbol{Y}) = S_n(\boldsymbol{X})$ (or, equivalently, $S_n^+(\boldsymbol{Y}) = S_n^+(\boldsymbol{X})$).

Analogous statements hold for compacta as well. Consequently, the following sequence of implications (of equivalences on compacta strictly coarser than the shape type classification) is established:

$(**)\quad S_0 \Leftarrow S_0^+ \Leftarrow S_1 \Leftarrow \cdots \Leftarrow S_n \Leftarrow S_n^+ \Leftarrow S_{n+1} \Leftarrow \cdots \Leftarrow S \Leftarrow S^*.$

Here are the main facts obtained in [19]:

- *the S_0-equivalence is the trivial equivalence relation (all nonempty compacta are mutually S_0-equivalent);*

- *the S_0^+-equivalence is* not *trivial ($S_0^+(\{*\}) \leq S_0^+(\{*\} \sqcup \{*\})$ and $S_0^+(\{*\} \sqcup \{*\}) \nleq S_0^+(\{*\})$);*

- *the implications $S_0 \Leftarrow S_0^+ \Leftarrow S_1 \Leftarrow S_1^+$ and $S_1 \Leftarrow S$ are* strict*;*

- *if the S^*-equivalence is strictly finer than S-equivalence, then the sequence $(**)$ admits a* strict *subsequence and, moreover, it can be realized by a single compactum*

In [4] is followed the same basic idea of [18]. However, in this setting the strong "uniformity" conditions for the hyperladders and their homotopy relation had to be abandoned. Only a slight control over the index functions is possible. A morphism set, denoted by $\underline{\boldsymbol{L}}_n(\boldsymbol{X}, \boldsymbol{Y})$, of each constructed category $\underline{\mathcal{A}}_n$ consists of the the corresponding, so called, *free n-hyperladders*. There exists a certain equivalence ("homotopy") relation on $\underline{\boldsymbol{L}}_n(\boldsymbol{X}, \boldsymbol{Y})$. Unfortunately, there is not any quotient category because the equivalence relation is not compatible with the composition. These categories are related by the "restriction" functors (which are not unique) $\underline{R}_{nnn'} : \underline{\mathcal{A}}_{n'} \to \underline{\mathcal{A}}_n$, for all pairs $n \leq n'$, such that $\underline{R}_{nn'}\underline{R}_{n'n''} = \underline{R}_{nn''}$. The following are main results of [4] (X, Y denote compact metrizable spaces, while $\boldsymbol{X}$, $\boldsymbol{Y}$ are any with them associated inverse sequence of *tow-HcANR* respectively):

- *for each $n \in \{0\} \cup \mathbb{N}$, if $S_{3n+1}(Y) \leq S_{3n+1}(X)$ (or $S_{3n+1}(X) \leq S_{3n+1}(Y)$), then there exist an $F \in \underline{\boldsymbol{L}}_{2n+1}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G \in \underline{\boldsymbol{L}}_{2n+1}(\boldsymbol{Y}, \boldsymbol{X})$ such that $FG \simeq 1_{\boldsymbol{Y}}$ and $GF \simeq 1_{\boldsymbol{X}}$ in $\underline{\mathcal{A}}_{2n+1}$;*

- *for each $n \in \{0\} \cup \mathbb{N}$, if $S_{3n+2}^+(Y) \leq S_{3n+2}^+(X)$ (or $S_{3n+2}^+(X) \leq S_{3n+2}^+(Y)$), then there exist an $F \in \underline{\boldsymbol{L}}_{2n+2}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G \in \underline{\boldsymbol{L}}_{2n+2}(\boldsymbol{Y}, \boldsymbol{X})$ such that $FG \simeq 1_{\boldsymbol{Y}}$ and $GF \simeq 1_{\boldsymbol{X}}$ in $\underline{\mathcal{A}}_{2n+2}$;*

- *for each $n \in \{0\} \cup \mathbb{N}$, if there exist an $F \in \underline{\boldsymbol{L}}_{2n+1}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G \in \underline{\boldsymbol{L}}_{2n+1}(\boldsymbol{Y}, \boldsymbol{X})$ such that $FG \simeq 1_{\boldsymbol{Y}}$ and $GF \simeq 1_{\boldsymbol{X}}$ in $\underline{\mathcal{A}}_{2n+1}$, then $S_n(Y) = S_n(X)$;*

- *for each $n \in \mathbb{N}$, if there exist an $F \in \underline{\boldsymbol{L}}_{2n}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G \in \underline{\boldsymbol{L}}_{2n}(\boldsymbol{Y}, \boldsymbol{X})$ such that $FG \simeq 1_{\boldsymbol{Y}}$ and $GF \simeq 1_{\boldsymbol{X}}$ in $\underline{\mathcal{A}}_{2n}$, then $S_{n-1}^+(Y) = S_{n-1}^+(X)$.*

Finally, let $\underline{\mathcal{A}}$ be the sequential category determined by all the $\underline{\mathcal{A}}_n$, i.e. the morphisms of $\underline{\mathcal{A}}$ are sequences $F = (F_n)$ of morphisms F_n of $\underline{\mathcal{A}}_n$, while the composition and the homotopy relation are defined coordinatewise. Then the following characterization holds ([4], Theorem 7):

$S(X) = S(Y)$ if and only if there exist an $F \in \underline{A}(\boldsymbol{X}, \boldsymbol{Y})$ and a $G \in \underline{A}(\boldsymbol{Y}, \boldsymbol{X})$ such that $GF \simeq 1_{\boldsymbol{X}}$ and $FG \simeq 1_{\boldsymbol{Y}}$.

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