Uniqueness of meromorphic functions sharing one value with their derivatives

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Abstract. *In the paper we deal with the uniqueness problem of meromorphic functions sharing a finite value with their derivatives. The results in this paper improve those given by Lahiri-Sarkar, Liu-Yang and others. In addition, a recent result of the first present author is complemented in this paper.*

Key words: *meromorphic function, derivative, weighted sharing*

AMS subject classifications: 30D35

Received September 18, 2007 Accepted August 10, 2008

1. Introduction definitions and results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [3]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying

$$S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E).$$

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see [11]).

Throughout this paper, we denote by I any set of $r \in (0,\infty)$ with infinite linear measure.

In 1996 R. Brück, proved the following result.

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Theorem A. *[2] Let f be a nonconstant entire function. If f and $f^{'}$ share the value 1 CM and if $N(r,0;f^{'}) = S(r,f)$ then $\frac{f^{'}-1}{f-1}$ is a nonzero constant.*

For entire functions of finite order, Yang proved the following result which improved *Theorem A.*

Theorem B. *[10] Let f be a nonconstant entire function of finite order and let $a(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

In 1998, Zhang proved the following two results, which extended *Theorem A.*

Theorem C. *[13]Let f be a non-constant meromorphic function. If f and $f^{'}$ share the value 1 CM, and if*

$$\overline{N}(r,\infty;f) + N(r,0;f^{'}) < (\lambda + o(1))T(r,f^{'}) \tag{1.1}$$

for some real constant $\lambda \in (0;\frac{1}{2})$, then $\frac{f^{'}-1}{f-1}$ is a nonzero constant.

Theorem D. *[13] Let f be a non-constant meromorphic function. If f and $f^{(k)}$ share the value 1 CM, and if*

$$2\overline{N}(r,\infty;f) + \overline{N}(r,0;f^{'}) + N(r,0;f^{(k)}) < (\lambda + o(1))T(r,f^{(k)}) \tag{1.2}$$

for some real constant $\lambda \in (0;1)$, then $\frac{f^{(k)}-1}{f-1}$ is a nonzero constant.

We now give the following two examples.

Example 1.1. $f(z) = 1 + tanz$.

Clearly $f(z) - 1 = tanz$ and $f^{'}(z) - 1 = tan^2 z$ share 1 IM and $\overline{N}(r,\infty;f) + N(r,0;f^{'}) = \overline{N}(r,-1;e^{2iz}) + N(r,0;sec^2(z)) \sim 2T(r,e^{iz})$. Again it follows from Mohon'ko's Lemma (see [9]) that $T(r,f^{'}) = 2T(r,sec\, z) + O(1) = 4T(r,e^{iz}) + O(1)$.

Example 1.2. $f(z) = \frac{2}{1-e^{-2z}}$.

Clearly $f^{'}(z) = -\frac{4e^{-2z}}{(1-e^{-2z})^2}$. Here $f - 1 = \frac{1+e^{-2z}}{1-e^{-2z}}$ and $f' - 1 = -\frac{(1+e^{-2z})^2}{(1-e^{-2z})^2}$. Here $\overline{N}(r,\infty;f) + N(r,0;f^{'}) = \overline{N}(r,1;e^{2z}) \sim 2T(r,e^z)$ and from Mohon'ko's Lemma {See [9]} we have $T(r,f^{'}) = 4T(r,e^z) + O(1)$.

So when $\lambda \geq \frac{1}{2}$ the condition (1.1) satisfies but the conclusion of Theorem C ceasess to hold. From the above two examples it is clear that in *Theorem C* when the nature of sharing the value 1 is relaxed from CM to IM the condition (1.1) can not be further weakened.

Throughout this paper we also need the following ten definitions.

Definition 1.1. *[7] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

(i) $N(r,a;f \mid\geq p)$ ($\overline{N}(r,a;f \mid\geq p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f \mid\leq p)$ ($\overline{N}(r,a;f \mid\leq p)$) *denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 1.2. *{6, cf.[12]} For $a \in \mathbb{C}\cup\{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f \mid\geq 2)+\ldots\overline{N}(r,a;f \mid\geq p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.*

Definition 1.3. *[6] Let $a,b \in \mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f \mid\ g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.*

Definition 1.4. *[6] Let $a,b \in \mathbb{C}\ \cup\{\infty\}$. We denote by $N(r,a;f \mid\ g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.*

Definition 1.5. *[14] For a positive integer p and $a \in \mathbb{C}\cup\{\infty\}$ we put*

$$\delta_p(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{N_p(r,a;f)}{T(r,f)}.$$

Clearly $0\leq\delta(a;f)\leq\delta_p(a;f)\leq\delta_{p-1}(a;f)\ldots\leq\delta_2(a;f)\leq\delta_1(a;f)=\Theta(a;f)$.

Definition 1.6. *[1] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$.*

Definition 1.7. *[4, 5] Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 1.8. *Let $a\in\mathbb{C}\ \cup\{\infty\}$ and m, n be two positive integers. We denote by $N(r,a;f \mid\ m\leq f\leq n)$ ($\overline{N}(r,a;f \mid\ m\leq f\leq n)$) the counting function (reduced counting function) of those a-points of f whose multiplicity p satisfies $m\leq p\leq n$.*

Definition 1.9. *Let $a\in\mathbb{C}\ \cup\{\infty\}$ and m be a positive integer. We denote by $\overline{N}(r,a;f \mid\ g\neq a\mid\geq m)$ the reduced counting function of those a-points of f which are not the a points of g whose multiplicities are $\geq m$.*

To state the next results we require the following definition known as weighted sharing of values which measure how close a shared value is to be shared IM or to be shared CM.

Definition 1.10. *[4, 5] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2004, Lahiri and Sarkar proved the following two results in the direction of weighted sharing of values which improved the results in [13].

Theorem E. *[7] Let f be a nonconstant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share $(1, 2)$ and*

$$2\overline{N}(r, \infty; f) + N_2\left(r, 0; f^{(k)}\right) + N_2(r, 0; f^{'}) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \tag{1.3}$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}/\{0\}$.

Theorem F. *[7] Let f be a nonconstant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share $(1, 1)$ and*

$$2\overline{N}(r, \infty; f) + N_2\left(r, 0; f^{(k)}\right) + 2\overline{N}(r, 0; f) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \tag{1.4}$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}/\{0\}$.

Recently the first present author proved the following result, which shows that the conditions (1.3) and (1.4) in *Theorem E* and *Theorem F* can be further weakened if $l \geq k$.

Theorem G. *[1] Let f be a nonconstant meromorphic function and $k(\geq 1)$, $l(\geq 1)$ be integers and a $(\not\equiv 0, \infty)$ be a constant. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l(\geq k)$ and*

$$2\overline{N}(r, \infty; f) + N_2\left(r, 0; f^{(k)}\right) + \overline{N}\left(r, 0; (f/a)^{'}\right) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \tag{1.5}$$

for $r \in I$, where $0 < \lambda < 1$ then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant.

Regarding Theorem G, it is natural to ask the following question.

Question 1.1 What can be said concerning the condition (1.5), provided $l < k$?

Further results in this direction have been obtained by Liu and Yang in the following four theorems.

Theorem H. ([see [8], Theorem 1.2]) *Let f be a nonconstant meromorphic function. If f and $f^{'}$ share $(1, 0)$ and if*

$$\overline{N}(r, \infty; f) + \overline{N}\left(r, 0; f^{'}\right) < (\lambda + o(1))\, T\left(r, f^{(k)}\right) \tag{1.6}$$

for $r \in I$, where $0 < \lambda < \frac{1}{4}$, then $\frac{f^{'}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}/\{0\}$.

Theorem I. ([see [8], Theorem 1.4]) *Let f be a nonconstant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share $(1,0)$ and*

$$(3k+6)\overline{N}(r,\infty;f)+5N(r,0;f)<(\lambda+o(1))\,T\left(r,f^{(k)}\right) \tag{1.7}$$

for $r\in I$, where $0<\lambda<1$, then $\frac{f^{(k)}-1}{f-1}\equiv c$ for some constant $c\in\mathbb{C}/\{0\}$.

Theorem J. ([see [8], Theorem 1.6]) *Let f be a nonconstant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (a,∞), where $a\neq 0,\infty$ is a constant and satisfy one of the following conditions,*

(i) $\delta(0;f)+\Theta(\infty;f)>\frac{4k}{2k+1}$,

(ii) $\overline{N}(r,\infty;f)+N(r,0;f)<(\lambda+o(1))\,T(r,f),\quad \left(0<\lambda<\frac{2}{2k+1}\right)$,

(iii) $\left(k+\frac{1}{2}\right)\overline{N}(r,\infty;f)+\frac{3}{2}N(r,0;f)<(\lambda+o(1))\,T(r,f),\quad (0<\lambda<1)$

then $f\equiv f^{(k)}$.

Theorem K. ([see [8], Theorem 1.7]) *Let f be a nonconstant meromorphic function. If f and f' share $(a,0)$, where $a\neq 0,\infty$ is a constant and if*

$$\overline{N}(r,\infty;f)+N(r,0;f)<(\lambda+o(1))\,T(r,f),\quad \left(0<\lambda<\frac{2}{3}\right), \tag{1.8}$$

then $f\equiv f'$.

In this paper we will establish the following three theorems of which *Theorem 1.1* and *Theorem 1.3* improve *Theorem E*, *Theorem F* and *Theorem I* and deal with *Question 1.1*, *Theorem 1.2* improves *Theorem J* and *Theorem K*. Following theorems are the main results of the paper.

Theorem 1.1. *Let f be a nonconstant meromorphic function, $k(>1)$, $l(\geq 0)$ integers and a $(\not\equiv 0,\infty)$ be a constant. Suppose that $f-a$ and $f^{(k)}-a$ share $(0,l)$. If $k>l>0$ and*

$$\begin{aligned}&2\overline{N}(r,\infty;f)+N_2\left(r,0;f^{(k)}\right)+\overline{N}(r,0;f\mid\geq 2)+\overline{N}\left(r,0;f'\mid f\neq 0\mid\geq l\right)\\&<(\lambda+o(1))\,T\left(r,f^{(k)}\right)\end{aligned} \tag{1.9}$$

or $l=0$, $k>1$ and

$$\begin{aligned}&(3k+6)\overline{N}(r,\infty;f)+N_{k+2}(r,0;f)+2N_{k+1}(r,0;f)+N_2(r,0;f)+\overline{N}(r,0;f)\\&<(\lambda+o(1))\,T\left(r,f^{(k)}\right)\end{aligned} \tag{1.10}$$

for $r\in I$, where $0<\lambda<1$, then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant.

Theorem 1.2. *Let f be a nonconstant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ two integers. If f and $f^{(k)}$ share (a,l), where $a \neq 0, \infty$ is a constant, $l \geq k-1$ and satisfy one of the following conditions,*

(i) $\delta(0;f) + 2\delta_k(0;f) + (2k+1)\Theta(\infty;f) > 2k+2$,

(ii) $\overline{N}(r,\infty;f) + \frac{1}{2k+1}N(r,0;f) + \frac{2}{2k+1}N_k(r,0;f) < (\lambda + o(1))\,T(r,f)$, $\left(0 < \lambda < \frac{2}{2k+1}\right)$,

(iii) $\left(k+\frac{1}{2}\right)\overline{N}(r,\infty;f) + \frac{1}{2}N(r,0;f) + N_k(r,0;f) < (\lambda + o(1))\,T(r,f)$, $(0 < \lambda < 1)$

then $f \equiv f^{(k)}$.

Theorem 1.3. *Let f be a nonconstant meromorphic function and $k(\geq 2)$, $l(\geq 0)$ integers. If f and $f^{(k)}$ share (a,l), where $a \neq 0, \infty$ is a constant $l < k-1$ and satisfy the following condition*
$\left(\frac{2k^2+l+1}{2(l+1)}\right)\overline{N}(r,\infty;f) + \frac{1}{2}N(r,0;f) + \frac{k}{l+1}N_k(r,0;f) < (\lambda + o(1))\,T(r,f), (0 < \lambda < 1)$
then $f \equiv f^{(k)}$.

Remark 1.1. *Putting $l = 0$ and $k = 1$ in* Theorem 1.2 (ii) *we obtain the conclusion of* Theorem K *under a weaker condition than (1.8).*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f be a nonconstant meromorphic function. Henceforth we shall denote by H the following function.

$$H = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{f^{(2+k)}}{f^{(1+k)}} - \frac{2f^{(1+k)}}{f^{(k)}-1}\right). \tag{2.1}$$

Lemma 2.1. *[3] Let f be a nonconstant meromorphic function. Then*

$$T\left(r, f^{(k)}\right) \leq (1+k)T(r,f) + S(r,f).$$

Lemma 2.2. *If for two positive integers p, and k, $N_p\left(r,0;f^{(k)} \mid f \neq 0\right)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ with multiplicity m is counted m times if $m \leq p$ and p times if $m > p$ then*

$$N_p\left(r,0;f^{(k)} \mid f \neq 0\right) \leq N_k(r,0;f) + k\overline{N}(r,\infty;f) - \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right) + S(r,f).$$

Proof. By the first fundamental theorem and Milloux theorem ([see [3], Theorem 3.1]) we get

$$\begin{aligned}
N\left(r,0;f^{(k)}\mid f\neq 0\right) &\leq N\left(r,0;\frac{f^{(k)}}{f}\right)\\
&\leq N\left(r,\infty;\frac{f^{(k)}}{f}\right)+m(r,\frac{f^{(k)}}{f})+O(1)\\
&\leq N(r,0;f\mid< k)+k\overline{N}(r,0;f\mid\geq k)+k\overline{N}(r,\infty;f)+S(r,f)\\
&= N_k(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}$$

Now

$$\begin{aligned}
&N_p\left(r,0;\frac{f^{(k)}}{f}\right)+\sum_{m=p+1}^{\infty}\overline{N}\left(r,0;\frac{f^{(k)}}{f}\mid\geq m\right)=N\left(r,0;\frac{f^{(k)}}{f}\right)\\
&\leq N_k(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}$$

Since $N_p\left(r,0;f^{(k)}\mid f\neq 0\right)\leq N_p\left(r,0;\frac{f^{(k)}}{f}\right)$, the lemma follows from above. □

Lemma 2.3. *[14] For two positive integers p and k*

$$N_p\left(r,0;f^{(k)}\right)\leq N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.4. *[9] Let f be a nonconstant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\neq 0$ and $b_m\neq 0$ Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *[see [3],p.68] Suppose that f is meromorphic and transcendental in the plane and that*

$$f^nP=Q$$

where P and Q are differential polynomials in f and the degree of Q is at most n. Then

$$m(r,P)=S(r,f)\ \ as\ \ \ r\longrightarrow+\infty$$

3. Proofs of the theorems

Proof of Theorem 1.1. Without loss of generality we assume that $a = 1$, since otherwise we can start the proof with $\frac{f}{a}$ and $\frac{f^{(k)}}{a}$.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1: $l \geq 1$. From (2.1) we get

$$N(r,\infty;H) \leq \overline{N}(r,\infty;f) + \overline{N}_*\left(r,1;f,f^{(k)}\right) + \overline{N}\left(r,0;f^{(k)} \mid\geq 2\right) + \overline{N}\left(r,0;f^{'}\right) - \overline{N}(r,1;f \mid\geq 2) + \overline{N}_0\left(r,0;f^{(1+k)}\right), \tag{3.1}$$

where $\overline{N}_0\left(r,0;f^{(1+k)}\right)$ is the reduced counting function of those zeros of $f^{(1+k)}$ which are not the zeros of $f^{'}(f^{(k)}-1)f^{(k)}$.

Lemma 2.1 implies that $S\left(r,f^{(k)}\right)$ can be replaced by $S(r,f)$. Let z_0 be a simple zero of $f-1$. Then z_0 must be a simple zero of $f^{(k)}-1$ and a zero of H. So

$$N(r,1;f \mid= 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,f) \tag{3.2}$$

Hence

$$\begin{aligned}\overline{N}\left(r,1;f^{(k)}\right) &= \overline{N}(r,1;f) \\ &= N(r,1;f \mid= 1) + \overline{N}(r,1;f \mid\geq 2) \\ &\leq N(r,H) + \overline{N}(r,1;f \mid\geq 2) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}\left(r.0;f^{(k)} \mid\geq 2\right) + \overline{N}(r,1;f \mid\geq l+1) \\ &\quad + \overline{N}\left(r,0;f^{'}\right) + \overline{N}_0\left(r,0;f^{(1+k)}\right) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}\left(r.0;f^{(k)} \mid\geq 2\right) + \overline{N}\left(r,0;f^{'}\right) \\ &\quad + N\left(r,0;f^{'} \mid f \neq 0 \mid\geq l\right) + \overline{N}_0\left(r,0;f^{(1+k)}\right) + S(r,f)\end{aligned} \tag{3.3}$$

Suppose $\overline{N}_{\otimes}(r,0;f^{(1+k)})$ is the reduced counting function of those zeros of $f^{(1+k)}$ which are not the zeros of $(f^{(k)}-1)f^{(k)}$.

Since

$$\begin{aligned}&\overline{N}\left(r,0;f^{'}\right) + N\left(r,0;f^{'} \mid f \neq 0 \mid\geq l\right) \\ &= \overline{N}(r,0;f \mid\geq 2) + \overline{N}\left(r,0;f^{'} \mid f \neq 0\right) + N\left(r,0;f^{'} \mid f \neq 0 \mid\geq l\right),\end{aligned}$$

by the second fundamental theorem, (3.3) and the above explanation we get

$$\begin{aligned}T(r,f^{(k)}) &\leq \overline{N}\left(r,\infty;f^{(k)}\right) + \overline{N}\left(r,0;f^{(k)}\right) + \overline{N}\left(r,1;f^{(k)}\right) \\ &\quad - \overline{N}_{\otimes}\left(r,0;f^{(1+k)}\right) + S\left(r,f^{(k)}\right) \\ &\leq 2\overline{N}(r,\infty;f) + N_2\left(r,0;f^{(k)}\right) + \overline{N}(r,0;f \mid\geq 2) \\ &\quad + \overline{N}\left(r,0;f^{'} \mid f \neq 0\right) + N\left(r,0;f^{'} \mid f \neq 0 \mid\geq l\right) + S(r,f),\end{aligned}$$

which contradicts (1.9).

Subcase 1.2: $l=0$. In this case (3.2) reduces to

$$N_E^{1)}(r,1;f) \le N(r,\infty;H) + S(r,f) \tag{3.4}$$

Since $k \ge 2$ and f and $f^{(k)}$ share $(1,0)$ it follows that $f-1$ may have multiple zeros.

So in view of *Definition 1.7* we get from (3.1) and (3.4) that

$$\begin{aligned}
\overline{N}\left(r,1;f^{(k)}\right) &= \overline{N}(r,1;f) \qquad (3.5)\\
&= N_E^{1)}(r,1;f) + \overline{N}_E^{(2}(r,1;f) + \overline{N}_L(r,1;f) + \overline{N}_L\left(r,1;f^{(1+k)}\right)\\
&\le N(r,H) + \overline{N}(r,1;f \mid\ge 2) + \overline{N}_L\left(r,1;f^{(1+k)}\right) + S(r,f)\\
&\le \overline{N}(r,\infty;f) + \overline{N}\left(r.0;f^{(k)} \mid\ge 2\right) + \overline{N}_L(r,1;f)\\
&\quad +2\overline{N}_L\left(r,1;f^{(k)}\right) + \overline{N}\left(r,0;f^{'}\right) + \overline{N}_0\left(r,0;f^{(1+k)}\right) + S(r,f).
\end{aligned}$$

Hence by the second fundamental theorem we get in view of *Lemmas 2.2*, *2.3* and (3.5) that

$$\begin{aligned}
T\left(r,f^{(k)}\right) &\le \overline{N}\left(r,\infty;f^{(k)}\right) + \overline{N}\left(r,0;f^{(k)}\right) + \overline{N}\left(r,1;f^{(k)}\right)\\
&\quad -\overline{N}_{\otimes}\left(r,0;f^{(1+k)}\right) + S(r,f)\\
&\le 2\overline{N}(r,\infty;f) + N_2\left(r,0;f^{(k)}\right) + \overline{N}\left(r,0;f^{'} \mid f \ne 0\right) + \overline{N}\left(r,0;f^{'}\right)\\
&\quad +2\overline{N}\left(r,0;\left(f^{(k)}\right)^{'} \mid f^{(k)} \ne 0\right) + S(r,f)\\
&\le (4+k)\overline{N}(r,\infty;f) + N_{2+k}(r,0;f) + \overline{N}(r,0;f) + N_2(r,0;f)\\
&\quad +2\left(N_{1+k}(r,0;f) + (1+k)\overline{N}(r,\infty;f)\right) + S(r,f),
\end{aligned}$$

which contradicts (1.10).

Case 2 Let $H \equiv 0$. On integration we get from (2.1)

$$\frac{1}{f-1} \equiv \frac{C}{f^{(k)}-1} + D, \tag{3.6}$$

where C, D are constants and $C \ne 0$. If z_0 be a pole of f with multiplicity p then it is a pole of $f^{(k)}$ with multiplicity $p+k$. This contradicts (3.6). It follows that f has no pole and so f is entire function here. Let $D \ne 0$. Then from (3.6) we get

$$f^{(k)} = \frac{(C-D)\,f + D + 1 - C}{-D\,f + D + 1} \tag{3.7}$$

Therefore

$$-Dff^{(k)} = (C-D)f + D + 1 - C - (D+1)f^{(k)} \tag{3.8}$$

Hence by *Lemma 2.5* we obtain

$$m(r,f^{(k)}) = T(r,f^{(k)}) = S(r,f). \tag{3.9}$$

So using *Lemma 2.4* from (3.6) we get $T(r,f) = T(r,f^{(k)}) + S(r,f) = S(r,f)$, which is absurd. Hence $D = 0$ and so $\frac{f^{(k)}-1}{f-1} = C$. This proves the theorem. □

Proof of Theorem 1.2. Suppose $f \not\equiv f^{(k)}$. Let

$$F = \frac{f}{f^{(k)}}$$

Then by the first fundamental theorem we have

$$T(r,F) = m\left(r,\frac{1}{F}\right) + N(r,0;F) = N\left(r,\infty;\frac{f^{(k)}}{f}\right) + S(r,f). \tag{3.10}$$

Since $l \geq k-1$. We first note that $f-a$ has no zero of multiplicity $> k$ since otherwise that will be a zero of $f^{(k)}$, which is impossible. If there exists a zero of $f-a$ of multiplicity p then that will be zero of $F-1$ of multiplicity at least $l+1 \geq k$ when $p > l$ and at least p when $p \leq l$. When $l \geq k$, clearly $f-a$ and $f^{(k)}-a$ share $(0,\infty)$. Also when $l = k-1$ the zeros of $f-a$ whose multiplicities are different from that of $f^{(k)}-a$ is exactly of order $l+1 = k$. It follows that

$$N(r,a;f) \leq N\left(r,\infty;\frac{f^{(k)}}{f-f^{(k)}}\right) \leq T(r,F) + O(1). \tag{3.11}$$

Again from the second fundamental theorem and Milloux theorem we obtain

$$\begin{aligned} & m\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f-a}\right) < m\left(r,\frac{1}{f^{(k)}}\right) + S(r,f) \qquad (3.12)\\ &\leq T\left(r,f^{(k)}\right) - N\left(r,0;f^{(k)}\right) + S(r,f)\\ &\leq \overline{N}\left(r,0;f^{(k)}\right) + \overline{N}\left(r,\infty;f^{(k)}\right) + \overline{N}\left(r,a;f^{(k)}\right) - N\left(r,0;f^{(k)}\right) + S(r,f)\\ &\leq \overline{N}(r,\infty;f) + \overline{N}\left(r,a;f^{(k)}\right) + S(r,f). \end{aligned}$$

Using the first fundamental theorem, *Lemma 2.2* and (3.10) we have from (3.12)

$$\begin{aligned} 2T(r,f) &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,a;f) + \overline{N}(r,a;f) + S(r,f) \qquad (3.13)\\ &\leq \overline{N}(r,\infty;f) + N(r,0;f) + 2\left(N_k(r,0;f) + k\overline{N}(r,\infty;f)\right) + S(r,f)\\ &= (2k+1)\overline{N}(r,\infty;f) + N(r,0;f) + 2N_k(r,0;f) + S(r,f), \end{aligned}$$

which contradicts the conditions (i) and (ii) of the theorem. Hence $f = f^{(k)}$. Similarly proceeding in the same way as done in (3.13) we can obtain

$$T(r,f) \leq (k+\frac{1}{2})\overline{N}(r,\infty;f) + \frac{1}{2}N(r,0;f) + N_k(r,0;f) + S(r,f), \tag{3.14}$$

which contradicts (iii) of the theorem. Hence $f \equiv f^{(k)}$. □

Proof of Theorem 1.3. Suppose $f \not\equiv f^{(k)}$. Let

$$F = \frac{f}{f^{(k)}}$$

Here $l \leq k-2$. Let z_0 be a zero of $f-a$ with multiplicity p satisfying $l+2 \leq p \leq k$. Then z_0 may be a zero of $f^{(k)}-a$ of multiplicity $l+1$ and so it is counted in the counting function of $F-1$ at most $\frac{k}{l+1}p$ times. Hence in view of the first fundamental theorem, *Lemma 2.2* we observe that (3.11) changes to

$$\begin{aligned} N(r,a;f) &\leq \frac{k}{l+1} N\left(r,\infty;\frac{f^{(k)}}{f-f^{(k)}}\right) \\ &\leq \frac{k}{l+1} T(r,F) + O(1) \\ &\leq \frac{k}{l+1} N\left(r,\infty;\frac{f^{(k)}}{f}\right) + S(r,f) \\ &\leq \frac{k}{l+1} N\left(r,0;\frac{f^{(k)}}{f}\right) + S(r,f) \\ &\leq \frac{k}{l+1}\left[N_k(r,0;f) + k\overline{N}(r,\infty;f)\right] + S(r,f). \end{aligned} \tag{3.15}$$

Using (3.15) we have from (3.12)

$$\begin{aligned} 2T(r,f) &\leq \overline{N}(r,\infty;f) + N(r,0;f) + N(r,a;f) + \overline{N}(r,a;f) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + N(r,0;f) + 2N(r,a;f) + S(r,f) \\ &\leq \frac{2k^2+l+1}{l+1}\overline{N}(r,\infty;f) + N(r,0;f) + \frac{2k}{l+1}\, N_k(r,0;f) + S(r,f). \end{aligned}$$

That is

$$T(r,f) \leq \frac{2k^2+l+1}{2(l+1)}\,\overline{N}(r,\infty;f) + \frac{1}{2}\, N(r,0;f) + \frac{k}{l+1}\, N_k(r,0;f) + S(r,f)$$

which contradicts the given inequality in the Theorem. So $f = f^{(k)}$. □

Acknowledgement

The authors are grateful to the referee for his/her valuable comments and suggestions towards the improvement of the paper.

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