A fixed fuzzy point for fuzzy mappings in complete metric spaces

SHABAN SEDGHI*, NABI SHOBE† AND ISHAK ALTUN‡

Abstract. *In this paper we prove a fixed point theorem for fuzzy mappings over a complete metric space.*

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1. Introduction and preliminaries

After the introduction of the concept of a fuzzy set by Zadeh [11], several researches were conducted on the generalizations of the concept of a fuzzy set. The idea of an intuitionistic fuzzy set is due to Atanassov [1, 2, 3] and Çöker [5] has defined the concept of fuzzy topological spaces induced by Chang [4]. Heilpern [7], introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [8]. Estruch and Vidal [6] give a fixed point theorem for fuzzy contraction mappings over a complete metric spaces which is a generalization of the given Heilpern's fixed point theorem. Recently, Türkoğlu and Rhoades [9] give an extended version of their main theorem. In this paper we give a common fixed point theorem for two fuzzy mappings over a complete metric space which is a generalization of fixed point theorems given by Estruch and Vidal, and Türkoğlu and Rhoades. We give a common fixed point theorem under the condition $D_\alpha(F(x), G(y)) \leq K(M(x, y))$ for each $x, y \in X$, where D_α, F, G, X, K and M are defined in the following section.

*Department of Mathematics, Islamic Azad University-Ghaemshar Branch, Iran, e-mail: sedghi_gh@yahoo.com

†Department of Mathematics, Islamic Azad University-Babol Branch, Iran, e-mail: nabi_shobe@yahoo.com

‡Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71 450 Yahsihan, Kirikkale, Turkey, e-mail: ishakaltun@yahoo.com

2. The main result

Let X be a nonempty set and $I = [0,1]$. A fuzzy set of X is an element of I^X. For $A, B \in I^X$ we denote $A \subseteq B$ if and only if $A(x) \leq B(x)$ for each $x \in X$. For $\alpha \in (0,1]$ the fuzzy point x_α of X is the fuzzy set of X given by $x_\alpha(y) = \alpha$ if $y = x$ and $x_\alpha(y) = 0$ else [10]. Let (X, d) be a metric linear space (i.e., a complex or real vector space).

The α-level set of A, denote by A_α, is defined by

$$A_\alpha = \{x \in X : A(x) \geq \alpha\} \tag{1}$$

for each $\alpha \in (0,1]$ and

$$A_0 = \overline{\{x \in X : A(x) > 0\}} \tag{2}$$

where $\overline{B}$ denotes the closure of the (non fuzzy) set B. Heilpern [7] called a fuzzy mapping a mapping from the set of X into a family $W(X) \subset I^X$ defined as follows: $A \in W(X)$ if and only if A_α is compact and convex in X for each $\alpha \in [0,1]$ and $\sup\{A(x) : x \in X\} = 1$. In this context we give the following definitions.

Definition 2.1 [see [7]]. *Let $A, B \in W(X)$ and $\alpha \in [0,1]$. Define*

$$\begin{aligned} p_\alpha(A,B) &= \inf\{d(a,b) : a \in A_\alpha, b \in B_\alpha\}, \\ D_\alpha(A,B) &= H(A_\alpha, B_\alpha), \\ D(A,B) &= \sup_\alpha D_\alpha(A,B), \end{aligned}$$

where H is the Hausdorff distance. For $x \in X$ we write $p_\alpha(x,B)$ instead of $p_\alpha(\{x\},B)$.

Definition 2.2 [see [6]]. *Let X be a metric space and $\alpha \in [0,1]$. Consider the following family $W_\alpha(X)$:*

$$W_\alpha(X) = \{A \in I^X : A_\alpha \ \ \textit{is nonempty, compact and convex}\} \tag{3}$$

The following lemmas are of great use for our further discussion. Let (X, d) be metric space.

Lemma 2.3 [see [7]]. *Let $x \in X, A \in W(X)$. Then $x_\alpha \subset A$ if and only if $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$, where x_α is a fuzzy point.*

Lemma 2.4 [see [7]]. *$p_\alpha(x,A) \leq d(x,y) + p_\alpha(y,A)$ for $x, y \in X$, $A \in W(X)$ and every $\alpha \in [0,1]$.*

Lemma 2.5 [see [7]]. *If $x_\alpha \subset A$, then $p_\alpha(x,B) \leq D_\alpha(A,B)$, for each $A, B \in W(X)$.*

Definition 2.6 [see [6]]. *Let x_α be a fuzzy point of X. We will say that x_α is a fixed fuzzy point of the fuzzy mapping F over X if $x_\alpha \subset F(x)$ (i.e., the fixed degree of x is at least α). In particular, and according to [7], if $\{x\} \subset F(x)$, we say that x is a fixed point of F.*

Using these lemmas, our main result is:

Theorem 2.7. *Let $\alpha \in (0,1]$ and (X,d) be a complete metric space. Let F and G be two fuzzy mappings from X into $W_\alpha(X)$ satisfying the following condition:*

There exists $K : [0,\infty) \longrightarrow [0,\infty)$, $K(0) = 0, K(t) < t$ *for all* $t \in (0,\infty)$ *and* K *is non-decreasing such that for every* $x \in X$, $(F(x))_\alpha, (G(x))_\alpha$ *are closed, bounded and nonempty subsets of* X. *Moreover*

$$D_\alpha(Fx, Gy) \leq K(M(x,y)), \tag{4}$$

for all $x, y \in X$, *where*

$$M(x,y) = \phi(d(x,y), p_\alpha(x, F(x)), p_\alpha(y, G(y)), p_\alpha(x, G(y)), p_\alpha(y, F(x))), \tag{5}$$

where $\phi : [0,\infty)^5 \longrightarrow [0,\infty)$, *is continuous , increasing in each co-ordinate variable and* $\phi(t,t,t,at,bt) \leq t$ *for every* $t \in [0,\infty)$, *where* $a + b = 2$. *Then there exists* $x \in X$, *such that* x_α *is common fixed fuzzy point of* F, G *if and only if there exists* $x_0, x_1 \in X$ *such that* $x_1 \in (F(x_0))_\alpha$ *with* $\sum_{n=1}^{\infty} K^n(d(x_0,x_1)) < \infty$. *In particular, if* $\alpha = 1$, *then* x *is a common fixed point of* F, G.

Proof. If there exists $x \in X$ such that x_α is a common fixed fuzzy point of F and G, then $x_\alpha \subset F(x)$ also $x_\alpha \subset G(x)$ and $d(x,x) = 0, 0 = K(0) = K^2(0) = \cdots = K^n(0) = \cdots$ and $\sum_{n=1}^{\infty} K^n(d(x,x)) = 0$. Let $x_0 \in X$. Since $(F(x_0))_\alpha$ is nonempty subset of X, then $\exists x_1 \in (F(x_0))_\alpha$, also since $(G(x_1))_\alpha$ is nonempty subset of X, $\exists x_2 \in (G(x_1))_\alpha$ such that

$$d(x_1,x_2) = p_\alpha(x_1, G(x_1)) \leq D_\alpha(F(x_0), G(x_1)) \leq K(M(x_0,x_1)), \tag{6}$$

where

$$\begin{aligned} M(x_0,x_1) &= \phi(d(x_0,x_1), p_\alpha(x_0, F(x_0)), p_\alpha(x_1, G(x_1)), p_\alpha(x_0, G(x_1)), p_\alpha(x_1, F(x_0))) \\ &\leq \phi(d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), d(x_0,x_2), d(x_1,x_1)) \\ &\leq \phi(d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), d(x_0,x_1) + d(x_1,x_2), 0). \end{aligned}$$

We prove that $d(x_1,x_2) \leq d(x_0,x_1)$. If $d(x_1,x_2) > d(x_0,x_1)$, by above inequality we have

$$\begin{aligned} M(x_0,x_1) &\leq \phi(d(x_1,x_2), d(x_1,x_2), d(x_1,x_2), d(x_1,x_2) + d(x_1,x_2), 0) \\ &\leq d(x_1,x_2). \end{aligned}$$

Hence

$$d(x_1,x_2) \leq K(d(x_1,x_2)) < d(x_1,x_2) \tag{7}$$

which is a contradiction. Therefore, we get $d(x_1,x_2) \leq d(x_0,x_1)$, thus by above inequality we have $d(x_1,x_2) \leq K(d(x_0,x_1))$.

By induction we construct a sequence $\{x_n\}$ in X such that $x_{2n+1} \in (F(x_{2n}))_\alpha$, $x_{2n+2} \in (G(x_{2n+1}))_\alpha$ and

$$d(x_{2n+1}, x_{2n+2}) \leq D_\alpha(F(x_{2n}), G(x_{2n+1})) \leq K(M(x_{2n}, x_{2n+1})), \tag{8}$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &= \phi(d(x_{2n}, x_{2n+1}), p_\alpha(x_{2n}, F(x_{2n})), p_\alpha(x_{2n+1}, G(x_{2n+1})), p_\alpha(x_{2n}, G(x_{2n+1})), \\ &\qquad\qquad p_\alpha(x_{2n+1}, F(x_{2n}))) \\ &\leq \phi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})) \\ &\leq \phi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad\qquad d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0). \end{aligned}$$

Similarly, we prove that $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$, for every $n \in \mathbb{N}$. Suppose $d(x_{2n+1}, x_{2n+2}) > d(x_{2n}, x_{2n+1})$ for some $n \in \mathbb{N}$. Then from above inequality and $K(t) < t$ for all $t \in (0, \infty)$, we have

$$d(x_{2n+1}, x_{2n+2}) \leq K(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}) \tag{9}$$

which is a contradiction. Therefore, we have

$$d(x_{2n+1}, x_{2n+2}) \leq K(d(x_{2n}, x_{2n+1})), \tag{10}$$

similarly we get

$$d(x_{2n}, x_{2n+1}) \leq K(d(x_{2n-1}, x_{2n})). \tag{11}$$

Thus,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq K(d(x_{2n}, x_{2n+1})) \\ &\leq K^2(d(x_{2n-1}, x_{2n})) \\ &\vdots \\ &\leq K^n(d(x_0, x_1)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} K^n(d(x_0, x_1)) < \infty$, hence it is convergent. That is for every $\epsilon > 0$ there exists n_0 such that for every $n, m \geq n_0$ we have $\sum_{k=n}^{n+m-1} K^k(d(x_0, x_1)) < \epsilon$. Hence we obtain

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq K^n(d(x_0, x_1)) + \cdots + K^{n+m-1}(d(x_0, x_1)) \\ &= \sum_{k=n}^{n+m-1} K^k(d(x_0, x_1)) < \epsilon. \end{aligned}$$

Therefore the sequence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, then $\{x_n\}$ converges to a point $x \in X$. Suppose that $p_\alpha(x, G(x)) > 0$, then by Lemmas 2.4 and 2.5 we have

$$\begin{aligned} p_\alpha(x, G(x)) &\leq d(x, x_{2n+1}) + p_\alpha(x_{2n+1}, G(x)) \\ &\leq d(x, x_{2n+1}) + D_\alpha(F(x_{2n}), G(x)) \\ &\leq d(x, x_{2n+1}) + K(M(x_{2n}, x)), \end{aligned}$$

where

$$\begin{aligned} M(x_{2n}, x) &= \phi(d(x_{2n}, x), p_\alpha(x_{2n}, F(x_{2n})), p_\alpha(x, G(x)), p_\alpha(x_{2n}, G(x)), p_\alpha(x, F(x_{2n}))) \\ &\leq \phi(d(x_{2n}, x), d(x_{2n}, x_{2n+1}), p_\alpha(x, G(x)), p_\alpha(x_{2n}, G(x)), d(x, x_{2n+1})). \end{aligned}$$

On making $n \longrightarrow \infty$, we get

$$\begin{aligned} \lim_{n \longrightarrow \infty} M(x_{2n}, x) &\leq \phi(0, 0, p_\alpha(x, G(x)), p_\alpha(x, G(x)), 0) \\ &\leq \phi(p_\alpha(x, G(x)), p_\alpha(x, G(x)), p_\alpha(x, G(x)), p_\alpha(x, G(x)), p_\alpha(x, G(x))) \\ &\leq p_\alpha(x, G(x)), \end{aligned}$$

hence we have

$$p_\alpha(x, G(x)) \leq 0 + K(p_\alpha(x, G(x))) < p_\alpha(x, G(x)), \tag{12}$$

which is contradiction. Consequently, $p_\alpha(x, G(x)) = 0$ and by Lemma 2.3, $x_\alpha \subset G(x)$. Similarly, suppose that $p_\alpha(x, F(x)) > 0$, then we have

$$\begin{aligned} p_\alpha(x, F(x)) &\leq d(x, x_{2n+2}) + p_\alpha(x_{2n+2}, F(x)) \\ &\leq d(x, x_{2n+2}) + D_\alpha(F(x), G(x_{2n+1})) \\ &\leq d(x, x_{2n+2}) + K(M(x_{2n+1}, x)), \end{aligned}$$

where

$$\begin{aligned} M(x_{2n+1}, x) &= M(x, x_{2n+1}) \\ &= \phi(d(x, x_{2n+1}), p_\alpha(x, F(x)), p_\alpha(x_{2n+1}, G(x_{2n+1})), p_\alpha(x, G(x_{2n+1})), \\ &\qquad p_\alpha(x_{2n+1}, F(x))) \\ &\leq \phi(d(x, x_{2n+1}), p_\alpha(x, F(x)), d(x_{2n+1}, x_{2n+2}), d(x, x_{2n+2}), \\ &\qquad p_\alpha(x_{2n+1}, F(x)). \end{aligned}$$

On making $n \longrightarrow \infty$, we get

$$\begin{aligned} \lim_{n \longrightarrow \infty} M(x_{2n+1}, x) &\leq \phi(0, p_\alpha(x, F(x)), 0, 0, p_\alpha(x, F(x))) \\ &\leq \phi(p_\alpha(x, F(x)), p_\alpha(x, F(x)), p_\alpha(x, F(x)), p_\alpha(x, F(x)), \\ &\qquad p_\alpha(x, F(x))) \\ &\leq p_\alpha(x, F(x)), \end{aligned}$$

we have

$$p_\alpha(x, F(x)) \leq K(p_\alpha(x, F(x))) < p_\alpha(x, F(x)), \tag{13}$$

which is contradiction. Consequently, $p_\alpha(x, F(x)) = 0$ and by Lemma 2.3, $x_\alpha \subset F(x)$. □

Remark 2.8. *If we give $F = G$ and $\phi(t_1, \cdots, t_5) = qt_1$ in Theorem 2.7, we have main Theorem of [6].*

Remark 2.9. *If we give $F = G$ and $\phi(t_1, \cdots, t_5) = max\{t_1, \cdots, t_5\}$ in Theorem 2.7, we have Theorem 1 of [9].*

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