Integral identities for sums

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Abstract. *We consider some finite binomial sums involving the derivatives of the binomial coefficient and develop some integral identities. In particular cases it is possible to express the sums in closed form.*

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1. Introduction

The Beta function

$$B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha,\beta>0$$

and the Gamma function

$$\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$$

are useful tools that are employed in the summation of series. In this paper we will develop integral identities for sums of the form

$$\sum_{n=1}^{p} n^k t^n \binom{p}{n} \frac{d^q}{dj^q} Q(a,j)$$

where

$$Q(a,j)=\binom{an+j}{j}^{-1}$$

is the binomial coefficient and let $Q^{(q)}(a,j)=\frac{d^q}{dj^q}(Q(a,j))$ be the q^{th} derivative of the reciprocal binomial coefficient. There has recently been renewed interest in the

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study of series involving binomial coefficients and a number of authors have obtained either closed form representation or integral representation for some of these series. The interested reader is referred to [1, 2, 3, 4, 5, 9, 10, 11, 12] and references therein. The following Lemma and Theorem are the main results presented in this paper.

2. Integral identity

The following Lemma deals with the derivatives of binomial coefficients.

First we define $\mathbb{N} = \{0, 1, 2, 3, ...\}$ and note the polygamma functions $\psi^{(k)}(z)$, $k \in \mathbb{N}$ are defined by

$$\psi^{(k)}(z) := \frac{d^{k+1}}{dz^{k+1}} \log \Gamma(z) = \frac{d^k}{dz^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)$$
$$= -\int_0^1 \frac{[\log(t)]^k \; t^{z-1}}{1-t} dt,$$

and $\psi^{(0)}(z) = \psi(z)$, denotes the digamma function, defined by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We also recall [7] the series representation for $\psi(z)$

$$\psi(z) = \sum_{r=0}^{\infty} \left(\frac{1}{r+1} - \frac{1}{r+z}\right) - \gamma,$$

where $\gamma = \lim_{n \to \infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log(n)\right) = -\psi(1) \approx 0.5772156649015328606065$ is the Euler-Mascheroni constant.

$$\zeta(q, b) = \sum_{k=0}^{\infty} \frac{1}{(k+b)^q}$$

is the generalised Zeta function where any term with $k + b = 0$ is excluded. We also define the generalised Harmonic number in power k as

$$H_n^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k}$$

and for convenience we put for $i = 1, 2, 3, 4, ...$

$$P^{(i)}(a, j) = \frac{d^i P}{dj^i} = \frac{d^i}{dj^i}\left(\sum_{r=1}^{an} \frac{1}{r+j}\right) \tag{1}$$
$$= (-1)^i i! \sum_{r=1}^{an} \frac{1}{(r+j)^{i+1}}$$
$$= (-1)^i i! \left[\zeta(i+1, j+1) - \zeta(i+1, j+1+an)\right].$$

Lemma 1. *Let a be a positive integer with $j \geq 0$, n a positive integer and* $Q(a,j) = Q^{(0)}(a,j) = \binom{an+j}{j}^{-1}$. *Then,*

$$Q^{(1)}(a,j) = \frac{dQ}{dj} = \begin{cases} -Q(a,j)P(a,j), \quad \textit{where} \\ P(a,j) = P^{(0)}(a,j) = \sum_{r=1}^{an} \frac{1}{r+j} \quad \textit{for } j > 0 \ , \\ = -Q(a,j)\left[\psi\left(j+1+an\right) - \psi(j+1)\right] \end{cases} \tag{2}$$

and for $\lambda \geq 2$

$$\begin{aligned} Q^{(\lambda)}(a,j) &= \frac{d^{\lambda}Q}{dj^{\lambda}} \\ &= -\sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q^{(\rho)}(a,j) P^{(\lambda-1-\rho)}(a,j). \end{aligned} \tag{3}$$

Proof. Let

$$Q(a,j) = \binom{an+j}{j}^{-1} = \frac{\Gamma\left(an+1\right)\Gamma\left(j+1\right)}{\Gamma\left(an+j+1\right)} = \frac{\Gamma\left(an+1\right)}{\prod_{r=1}^{an}\left(r+j\right)}, \tag{4}$$

taking logs of both sides and differentiating with respect to j we obtain the result (2).

Now from (2) and for $\lambda \geq 2$

$$Q^{(\lambda)}(a,j) = \frac{d^{\lambda}Q}{dj^{\lambda}} = Q^{(\lambda)}(a,j) = \frac{d^{\lambda-1}}{dj^{\lambda-1}}\left(-QP\right) = -\sum_{\rho=0}^{\lambda-1}\binom{\lambda-1}{\rho} Q^{(\rho)} P^{(\lambda-1-\rho)}$$

and $P^{(\lambda-1-\rho)}\left(a,j\right)$ is given by (1). □

We list the following

$$Q^{(1)}(a,j) = -\binom{an+j}{j}^{-1} \sum_{r=1}^{an} \frac{1}{r+j},$$

$$Q^{(2)}(a,j) = \binom{an+j}{j}^{-1} \left[\left(\sum_{r=1}^{an} \frac{1}{r+j}\right)^2 + \sum_{r=1}^{an} \frac{1}{(r+j)^2}\right]$$

$$Q^{(3)}(a,j) = -\binom{an+j}{j}^{-1} \left[\begin{array}{c} \left(\sum_{r=1}^{an} \frac{1}{r+j}\right)^3 + 2\sum_{r=1}^{an} \frac{1}{(r+j)^3} \\ +3\sum_{r=1}^{an} \frac{1}{(r+j)^2} \sum_{r=1}^{an} \frac{1}{r+j} \end{array}\right]$$

and

$$Q^{(4)}(a,j) = \binom{an+j}{j}^{-1} \left[\begin{array}{c} 6\sum_{r=1}^{an} \frac{1}{(r+j)^2} \left(\sum_{r=1}^{an} \frac{1}{r+j}\right)^2 \\ +8\sum_{r=1}^{an} \frac{1}{(r+j)^3} \sum_{r=1}^{an} \frac{1}{r+j} + 3\left(\sum_{r=1}^{an} \frac{1}{(r+j)^2}\right)^2 \\ +\left(\sum_{r=1}^{an} \frac{1}{r+j}\right)^4 + 6\sum_{r=1}^{an} \frac{1}{(r+j)^4} \end{array}\right]. \tag{5}$$

In the special case when $a = 1$ and $j = 0$ we may write

$$Q^{(1)}(1,0) = -H_n^{(1)}, \tag{6a}$$

$$Q^{(2)}(1,0) = \left(H_n^{(1)}\right)^2 + H_n^{(2)}, \tag{6b}$$

$$Q^{(3)}(1,0) = \left(H_n^{(1)}\right)^3 + 3H_n^{(1)}H_n^{(2)} + 2H_n^{(3)} \tag{6c}$$

and

$$Q^{(4)}(1,0) = \left(H_n^{(1)}\right)^4 + 6\left(H_n^{(1)}\right)^2 H_n^{(2)} + 8H_n^{(1)}H_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}, \tag{7}$$

where $H_n^{(k)}$ are the generalised Harmonic numbers.

We now state the following theorem.

Theorem 1. *Let a be a positive real number, $t \in \mathbb{R}$, $p = 1,2,3,...,k = 1,2,3,...,$ $q = 1,2,3...$ and $j \geq 0$, then*

$$V_p\left(a,j,k,q,t\right) = \sum_{n=1}^{p} n^k t^n \binom{p}{n} Q^{(q)}(a,j) \tag{8}$$

$$= q\sum_{r=1}^{k} t^r \binom{p}{r} r! \left\{ {k \atop r} \right\} \int_0^1 x^{ar}\left(1-x\right)^{j-1}\left(1+tx^a\right)^{p-r}\left(\log(1-x)\right)^{q-1} dx$$

$$+j\sum_{r=1}^{k} t^r \binom{p}{r} r! \left\{ {k \atop r} \right\} \int_0^1 x^{ar}\left(1-x\right)^{j-1}\left(1+tx^a\right)^{p-r}\left(\log(1-x)\right)^{q} dx$$

where $Q^{(q)}(a,j)$ is defined by (3), and

$$\left\{ {k \atop r} \right\} = \frac{1}{r!}\sum_{\mu=0}^{r}(-1)^\mu \binom{r}{\mu}(r-\mu)^k$$

are Stirling numbers of the second kind.

Proof. Consider

$$\sum_{n=0}^{p} \frac{t^n \binom{p}{n}}{\binom{an+j}{j}} = j\sum_{n=0}^{p}\binom{p}{n}\frac{t^n\,\Gamma(j)\,\Gamma(an+1)}{\Gamma(an+j+1)}$$

$$= j\sum_{n=0}^{p}\binom{p}{n} t^n\, B\left(an+1,j\right)$$

$$= j\sum_{n=0}^{p}\binom{p}{n} t^n \int_0^1 x^{an}\left(1-x\right)^{j-1} dx$$

where $B(\cdot,\cdot)$ is the classical Beta function and $\Gamma(\cdot)$ is the Gamma function. By an allowable change of sum and integral we have

$$\sum_{n=0}^{p} \frac{t^n \binom{p}{n}}{\binom{an+j}{j}} = j\int_0^1 \left(1-x\right)^{j-1}\left(1+tx^a\right)^p dx.$$

Now apply consecutively k- times the operator $t\frac{d}{dt}(\cdot)$, so that

$$\sum_{n=1}^{p} \frac{nt^n \binom{p}{n}}{\binom{an+j}{j}} = jpt \int_0^1 x^a (1-x)^{j-1} (1+tx^a)^{p-1}\, dx,$$

$$\sum_{n=1}^{p} \frac{n^2\ t^n \binom{p}{n}}{\binom{an+j}{j}} = jpt \int_0^1 x^a (1-x)^{j-1} (1+tx^a)^{p-1}\, dx$$

$$+jp(p-1)\,t^2 \int_0^1 x^{2a} (1-x)^{j-1} (1+tx^a)^{p-2}\, dx,$$

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$$\sum_{n=1}^{p} \frac{n^k\ t^n \binom{p}{n}}{\binom{an+j}{j}} = j\sum_{r=1}^{k} t^r \binom{p}{r} r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\} \int_0^1 x^{ar} (1-x)^{j-1} (1+tx^a)^{p-r}\, dx.$$

Applying the operator $Q^{(q)}(a,j) = \frac{d^q}{dj^q}(Q(a,j))$ as defined in Lemma 1 we have that

$$\sum_{n=1}^{p} n^k\ t^n \binom{p}{n} Q^{(q)}(a,j)$$

$$= q\sum_{r=1}^{k} t^r \binom{p}{r} r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\} \int_0^1 x^{ar} (1-x)^{j-1} (1+tx^a)^{p-r} (\log(1-x))^{q-1}\, dx$$

$$+j\sum_{r=1}^{k} t^r \binom{p}{r} r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\} \int_0^1 x^{ar} (1-x)^{j-1} (1+tx^a)^{p-r} (\log(1-x))^{q}\, dx$$

where $\left\{ \begin{matrix} k \\ r \end{matrix} \right\}$ are Stirling numbers of the second kind. □

Some interesting results follow for particular parameter values, which involve Harmonic numbers. Consider the following corollary.

Corollary 1. *Let* $a = 1,\ t = -1$. *Then*

$$\sum_{n=1}^{p} n^k\ (-1)^n \binom{p}{n} \frac{d^q}{dj^q}(Q(1,j))$$

$$= q!\,(-1)^q \sum_{r=1}^{k} (-1)^r \binom{p}{r} r! \left\{ \begin{matrix} k \\ r \end{matrix} \right\} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left[\frac{s-p}{(p+j-s)^{q+1}} \right] \tag{9}$$

Proof. From Theorem 1

$$\sum_{n=1}^{p} n^k \ (-1)^n \ \binom{p}{n} \frac{d^q}{dj^q}(Q(1,j))$$
$$= q\sum_{r=1}^{k}(-1)^r \binom{p}{r} r! \left\{ {k \atop r} \right\} \int_0^1 x^r (1-x)^{p+j-r-1} (\log(1-x))^{q-1} dx$$
$$+j\sum_{r=1}^{k}(-1)^r \binom{p}{r} r! \left\{ {k \atop r} \right\} \int_0^1 x^r (1-x)^{p+j-r-1} (\log(1-x))^{q} dx,$$

evaluating the integrals we have

$$= q!(-1)^{q-1}\sum_{r=1}^{k}(-1)^r \binom{p}{r} r! \left\{ {k \atop r} \right\} \sum_{s=0}^{r}(-1)^s \binom{r}{s} \frac{1}{(p+j-r+s)^q}$$
$$+jq!(-1)^{q}\sum_{r=1}^{k}(-1)^r \binom{p}{r} r! \left\{ {k \atop r} \right\} \sum_{s=0}^{r}(-1)^s \binom{r}{s} \frac{1}{(p+j-r+s)^{q+1}},$$

collecting like terms and renaming the counter s we obtain (9). □

Remark 1. *For $q=4,\ j=0\ ,\ k=3$ and using (7)*

$$\sum_{n=1}^{p} n^3 \ (-1)^n \ \binom{p}{n} \left[\begin{array}{c} \left(H_n^{(1)}\right)^4 + 6\left(H_n^{(1)}\right)^2 H_n^{(2)} + 8H_n^{(1)}H_n^{(3)} \\ +3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)} \end{array} \right]$$
$$= 24\left(\frac{\begin{array}{c} -1296 + 9504p - 30235p^2 + 54654p^3 - 61280p^4 \\ +44016p^5 - 20255p^6 + 5770p^7 - 926p^8 + 64p^9 \end{array}}{p(p-1)^4(p-2)^4(p-3)^4} \right).$$

Moreover,

$$\sum_{p=4}^{\infty}\sum_{n=1}^{p} n^3 \ (-1)^n \ \binom{p}{n} \left[\begin{array}{c} \left(H_n^{(1)}\right)^4 + 6\left(H_n^{(1)}\right)^2 H_n^{(2)} + 8H_n^{(1)}H_n^{(3)} \\ +3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)} \end{array} \right]$$
$$= 24\zeta(4) + \frac{973}{2}.$$

3. Conclusion

We have applied the method of integral representation for binomial sums that also involve Harmonic numbers and in some cases expressed them in closed form.

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