The equiform differential geometry of curves in the pseudo-Galilean space*

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Abstract. *In this paper the equiform differential geometry of curves in the pseudo-Galilean space G_3^1 is introduced. Basic invariants and a moving trihedron are described. Frenet formulas are derived and the fundamental theorem of curves in equiform geometry of G_3^1 is proved. The curves of constant curvatures are described.*

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1. Introduction

Theory of curves and the curves of constant curvature in the equiform differential geometry of the isotropic spaces I_3^1 and I_3^2, and the Galilean space G_3 are described in [7] and [8], respectively. In this paper we introduce the equiform differential geometry, prove the fundamental theorem of curves and describe the curves of constant curvature in the equiform differential geometry of G_3^1. Although the equiform geometry has minor importance related to usual one, the curves that appear here in the equiform geometry, can be seen as generalizations of well-known curves from above mentioned geometries and therefore could have been of research interest.

The pseudo-Galilean space is one of the real Cayley-Klein spaces. It has projective signature $(0, 0, +, -)$ according to [6]. The absolute of the pseudo-Galilean space is an ordered triple $\{\omega, f, I\}$ where ω is the ideal (absolute) plane, f a line in ω and I is the fixed hyperbolic involution of the points of f.

The geometry of a pseudo-Galilean space G_3^1 has been explained in details in dissertation [1]. The curves in G_3^1 are described in [2] and [4] and the surfaces in [3] and [5]. The notions and the symbols therein will be used in this paper.

Let us first recall basic notions from pseudo-Galilean geometry.

*This paper is dedicated to the memory of Professor B. Pavković.

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In the inhomogeneous affine coordinates for points and vectors (point pairs) the similarity group H_8 of G_3^1 has the following form

$$\begin{aligned} \bar{x} &= a + b \cdot x, \\ \bar{y} &= c + d \cdot x + r \cdot \cosh\varphi \cdot y + r \cdot \sinh\varphi \cdot z, \\ \bar{z} &= e + f \cdot x + r \cdot \sinh\varphi \cdot y + r \cdot \cosh\varphi \cdot z, \end{aligned} \tag{1.1}$$

where a, b, c, d, e, f, r and φ are real numbers. Particularly, for $b = r = 1$, the group (1.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space G_3^1. The motion group leaves invariant the absolute figure and defines the other invariants of this geometry.

It has the following form

$$\begin{aligned} \bar{x} &= a + x, \\ \bar{y} &= c + d \cdot x + \cosh\varphi \cdot y + \sinh\varphi \cdot z, \\ \bar{z} &= e + f \cdot x + \sinh\varphi \cdot y + \cosh\varphi \cdot z. \end{aligned} \tag{1.2}$$

According to the motion group in the pseudo-Galilean space, there are non-isotropic vectors $\mathbf{x}(x, y, z)$ (for which holds $x \neq 0$) and four types of isotropic vectors: spacelike ($x = 0,\ y^2 - z^2 > 0$), timelike ($x = 0\ y^2 - z^2 < 0$) and two types of lightlike vectors ($x = 0,\ y = \pm z$).

The scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in G_3^1 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} a_1 \cdot b_1, & a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2 \cdot b_2 - a_3 \cdot b_3, & a_1 = 0 \text{ and } b_1 = 0. \end{cases}$$

Let us recall basic facts about curves in pseudo-Galilean space, that were introduced in [2].

A curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ is admissible if it has no inflection points, no isotropic tangents or tangents or normals whose projections on the absolute plane would be light-like vectors.

For an admissible curve $\mathbf{r} : I \to G_3^1$, $I \subseteq \mathbb{R}$ the curvature $\kappa(t)$ and the torsion $\tau(t)$ are defined by

$$\begin{aligned} \kappa(t) &= \frac{\sqrt{|\ddot{y}(t)^2 - \ddot{z}(t)^2|}}{(\dot{x}(t))^2}, \\ \tau(t) &= \frac{\ddot{y}(t)\dddot{z}(t) - \dddot{y}(t)\ddot{z}(t)}{|\dot{x}(t)|^5 \cdot \kappa^2(t)}, \end{aligned} \tag{1.3}$$

expressed in components.

Hence, for an admissible curve $\mathbf{r} : I \to G_3^1$, $I \subseteq \mathbb{R}$ parameterized by the arc length s with differential form $ds = dx$, given by

$$\mathbf{r}(x) = (x, y(x), z(x)), \tag{1.4}$$

the formulas (1.3) have the following form

$$\begin{aligned} \kappa(x) &= \sqrt{|y''(x)^2 - z''(x)^2|}, \\ \tau(x) &= \frac{y''(x)z'''(x) - y'''(x)z''(x)}{\kappa^2(x)}. \end{aligned} \tag{1.5}$$

The associated trihedron is given by

$$\begin{aligned}
\mathbf{t} &= \mathbf{r}'(x) = (1, y'(x), z'(x)),\\
\mathbf{n} &= \frac{1}{\kappa(x)}\mathbf{r}''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)),\\
\mathbf{b} &= \frac{1}{\kappa(x)}(0, \varepsilon z''(x), \varepsilon y''(x)),
\end{aligned} \tag{1.6}$$

where $\varepsilon = +1$ or -1, chosen by criterion $\det(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1$, that means

$$|y''^2(x) - z''^2(x)| = \varepsilon(y''^2(x) - z''^2(x)).$$

The curve $\mathbf{r}$ given by (1.4) is time-like (resp. space-like) if $\mathbf{n}(x)$ is a space-like (resp. time-like) vector. The principal normal vector or simply normal is space-like if $\varepsilon = +1$ and time-like if $\varepsilon = -1$.

For derivatives of the tangent (vector) $\mathbf{t}$, the normal $\mathbf{n}$ and the binormal $\mathbf{b}$, respectively, the following Serret-Frenet formulas hold

$$\mathbf{t}' = \kappa \cdot \mathbf{n}, \qquad \mathbf{n}' = \tau \cdot \mathbf{b}, \qquad \mathbf{b}' = \tau \cdot \mathbf{n}. \tag{1.7}$$

From (1.7), we derive an important relation

$$\mathbf{r}'''(x) = \kappa'(x) \cdot \mathbf{n}(x) + \kappa(x) \cdot \tau(x) \cdot \mathbf{b}(x). \tag{1.8}$$

2. Equiform transformations of the pseudo-Galilean space

Let us now introduce the notion of equiform transformations of the pseudo-Galilean space G_3^1.

Similarity group (1.1) maps a usual (formal) line element ($dx = 0$, dy, dz) in a pseudo-Euclidean plane (i.e. x = const.) into a segment of length proportional to the original with the coefficient of proportionality r. Other line elements (dx, dy, dz), which lie on an isotropic plane ($dx \neq 0$) are mapped into proportional ones with the coefficient b. Therefore, all line segments are mapped into proportional ones with the same coefficient of proportionality if and only if $b = r$.

With this condition we get a subgroup $H_7 \subset H_8$ which preserves length ratio of segments and angles between planes and lines, respectively. Therefore it is justified to call this group the *group of equiform transformations of the pseudo-Galilean space.*

The structure of this group is described by the following theorem.

Theorem 2.1. *Every equiform transformation* $\boldsymbol{e}$ *of the space* G_1^3 *is a semi-direct composition of a homothety* $\boldsymbol{h}$ *and an isometry* $\boldsymbol{i}$*, i.e.* $\boldsymbol{e} = \boldsymbol{i} \circ \boldsymbol{h} = \boldsymbol{h} \circ \boldsymbol{i}$. *Here homothety* $\boldsymbol{h}$ *denotes the mapping* $(x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z})$ *given by*

$$\bar{x} = \lambda \cdot x, \qquad \bar{y} = \lambda \cdot y, \qquad \bar{z} = \lambda \cdot z, \tag{2.1}$$

with given coefficient $\lambda > 0$, *fixing the origin* $(0, 0, 0)$.

Definition 2.2. *Geometry of* G_1^3 *induced by the 7-parameter equiform group* H_7 *is called the* equiform geometry of the space G_1^3.

According to the previous theorem, we can find invariants of the equiform group by finding invariants of the homothety group and those of the isometry group.

3. Frenet formulas in the equiform geometry of the pseudo-Galilean space

Let $\mathbf{r} : I \to G_3^1$ be an admissible curve. We define the equiform parameter of $\mathbf{r}$ by

$$\sigma := \int \frac{ds}{\rho} = \int \kappa ds,$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve $\mathbf{r}$.

It follows

$$\frac{d\sigma}{ds} = \frac{1}{\rho} \qquad \text{i.e.} \qquad \frac{ds}{d\sigma} = \rho. \tag{3.1}$$

Let h be a homothety with the center in the origin and the coefficient λ. If we put $\tilde{\mathbf{r}} = h(\mathbf{r})$ then it follows

$$\tilde{s} = \lambda s \quad \text{and} \quad \tilde{\rho} = \lambda \rho, \tag{3.2}$$

where $\tilde{s}$ is the arc length parameter of $\tilde{\mathbf{r}}$ and $\tilde{\rho}$ the radius of curvature of this curve. Therefore, σ is an equiform invariant parameter of $\mathbf{r}$.

Remark 3.1. *Let us note that κ and τ are not invariants of the homothety group, since from (1.3) it follows $\tilde{\kappa} = \frac{1}{\lambda}\kappa$ and $\tilde{\tau} = \frac{1}{\lambda}\tau$.*

The vector

$$\mathbf{T} = \frac{d\mathbf{r}}{d\sigma}$$

is called a tangent vector of the curve $\mathbf{r}$ in the equiform geometry. From (1.6) and (3.1) we get

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{d\sigma} = \rho \cdot \frac{d\mathbf{r}}{ds} = \rho \cdot \mathbf{t}. \tag{3.3}$$

Further, we define the principal normal and the binormal vector, or simply the normal and the binormal by

$$\mathbf{N} = \rho \cdot \mathbf{n}, \qquad \mathbf{B} = \rho \cdot \mathbf{b}. \tag{3.4}$$

It is easy to check that the trihedron $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an equiform invariant trihedron of the curve $\mathbf{r}$.

Let us find now the derivatives of these triples with respect to σ, denoted by prime ($'$). From (1.7), (3.1), (3.3) and (3.4) it follows

$$\mathbf{T}' = \frac{d\mathbf{T}}{d\sigma} = \frac{d}{d\sigma}(\rho\mathbf{t}) = \frac{d}{ds}(\rho\mathbf{t})\frac{ds}{d\sigma} = (\dot{\rho}\mathbf{t} + \rho\dot{\mathbf{t}})\rho = \dot{\rho}\mathbf{T} + \mathbf{N},$$

where the derivative with respect to the arc length s is denoted by a dot. Similarly, we obtain

$$\mathbf{N}' = \frac{d\mathbf{N}}{d\sigma} = \dot{\rho}\mathbf{N} + \rho\tau\mathbf{B},$$

$$\mathbf{B}' = \frac{d\mathbf{B}}{d\sigma} = \rho\tau\mathbf{N} + \dot{\rho}\mathbf{B}.$$

Definition 3.2. *The function $\mathcal{K} : I \to \mathbb{R}$ defined by*

$$\mathcal{K} = \dot{\rho} \tag{3.5}$$

is called the equiform curvature *of the curve* $\mathbf{r}$. Let us prove that $\mathcal{K}$ is a differential invariant of the group of equiform transformations. If we put $\tilde{\mathcal{K}}$ to be equiform curvature of the curve $\tilde{\mathbf{r}}$, then we have

$$\tilde{\mathcal{K}} = \dot{\tilde{\rho}} = \frac{d\tilde{\rho}}{d\tilde{s}} = \frac{d(\lambda\rho)}{ds}\frac{ds}{d\tilde{s}} = \dot{\rho} = \mathcal{K}.$$

Definition 3.3. *The function $\mathcal{T} : I \to \mathbb{R}$ defined by*

$$\mathcal{T} = \rho\tau = \frac{\tau}{\kappa} \tag{3.6}$$

is called the equiform torsion *of the curve* $\mathbf{r}$. *It is a differential invariant of the group of equiform transformations, too.*

Thus the formulas analogous to the Frenet formulas in the equiform geometry of the pseudo-Galilean space have the following form

$$\begin{aligned}
\frac{d\mathbf{T}}{d\sigma} &= \mathcal{K} \cdot \mathbf{T} + \mathbf{N} \\
\frac{d\mathbf{N}}{d\sigma} &= \mathcal{K} \cdot \mathbf{N} + \mathcal{T} \cdot \mathbf{B} \\
\frac{d\mathbf{B}}{d\sigma} &= \mathcal{T} \cdot \mathbf{N} + \mathcal{K} \cdot \mathbf{B}.
\end{aligned}$$

Remark 3.4. *The equiform parameter $\sigma = \int \kappa(s)ds$ for closed curves is called the* total curvature, *and it plays an important role in global differential geometry of the Euclidean space. Also, the function $\frac{\tau}{\kappa}$ has been already known as a* conical curvature *and it also has interesting geometric interpretation.*

4. The fundamental theorem of curves in equiform geometry of the pseudo-Galilean space

First of all, we recall the fundamental theorem of the pseudo-Galilean theory of curves, proved in [2]. This theorem differs crucially from the analogous theorem in Euclidian, isotropic or Galilean space. The uniqueness in this theorem is not fulfilled and the reason for this is the existence of pseudo-Euclidean planes in pseudo-Galilean space. As it is well known in pseudo-Euclidean plane geometry, the uniqueness in the fundamental theorem of plane curves does not hold. Here is the theorem.

Theorem 4.1 [[2]]. *Let $\kappa = \kappa(x)$ and $\tau = \tau(x)$ be given functions so that $0 < \kappa(x) \in C^1$, $0 \neq \tau(x) \in C$. There are two admissible curves (one timelike and one spacelike) so that the following statements are true:*

(1) c passes through a given point;

(2) at this point the Frenet trihedron of c concides with a given orthonormal positively oriented trihedron;

(3) c can be represented as a vector function $\mathbf{r}(x) \in C^3$ *with arc length parameter* x*;*

(4) $\kappa(x)$ *and* $\tau(x)$ *are the curvature and torsion of c, respectively.*

Here $C^0 = C$, C^1, C^2, C^3 *denote the corresponding continuous differentiability classes of functions, respectively.* This theorem implies the fundamental theorem of curves of the equiform geometry of the pseudo-Galilean space.

Theorem 4.2. *Let* $\mathcal{K} = \mathcal{K}(x)$ *and* $\mathcal{T} = \mathcal{T}(x)$ *be given functions so that* $0 < \mathcal{K}(x) \in C^1$ and $0 \neq \mathcal{T}(x) \in C$. *There are two admissible curves (one timelike and one spacelike) so that the following statements are true:*

(1) c passes through a given point;

(2) at this point the equiform Frenet trihedron of c concides with a given positively oriented equiform trihedron;

(3) c can be represented as a vector function $\mathbf{r}(x) \in C^3$ *with arc length parameter* x*;*

(4) $\mathcal{K}(x)$ *and* $\mathcal{T}(x)$ *are the equiform curvature and equiform torsion of c, respectively.*

Proof. From $\mathcal{K}(x) > 0$ it follows

$$\rho(x) := \int_0^x \mathcal{K}(t)dt > 0. \tag{4.1}$$

Hence,

$$\kappa(x) := \frac{1}{\rho(x)} > 0 \qquad \text{and} \qquad \tau(x) := \mathcal{T}(x) \cdot \kappa(x) \neq 0. \tag{4.2}$$

Furthermore, by Theorem 4.1 it follows that there are two admissible curves that pass through a given point, that the Frenet trihedron coincides with a given orthonormal positively oriented trihedron $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, the curves can be represented as a vector functions $\mathbf{r}(x)$ and $\kappa(x)$ and $\tau(x)$ are the curvature and torsion of these curves, respectively. Finally, we only need check the statement (2) and (4) of the Theorem 4.2, but they follow from (3.3), (3.4), (4.1) and (4.2). □

5. Curves of the constant equiform curvature and torsion

In this section we consider four different cases for curves of constant equiform curvature and torsion, which fulfill the conditions of Theorem 4.2.

A) $\mathcal{K} = const. \neq 0$, $\mathcal{T} = const. \neq 0$

According to (3.5) and (3.6) the curve is characterized by

$$\dot{\rho} = a, \qquad \frac{\tau}{\kappa} = b,$$

where a and b are non-zero real constants. It follows

$$\kappa = \frac{1}{as+c}, \qquad \tau = \frac{b}{as+c}.$$

We can assume without loss of generality that $c = 0$, thus the natural equations of the curve have the form

$$\kappa = \frac{1}{as}, \qquad \tau = \frac{b}{as}. \tag{5.1}$$

Further, the coordinate functions of the curve are:

$$x = s, \qquad y = y(s), \qquad z = z(s).$$

The relation (1.8), using (1.6), can be written into the coordinates, in the following way

$$(0, y''', z''') = \frac{\kappa'}{\kappa} \cdot (0, y'', z'') + \tau \cdot (0, \varepsilon z'', \varepsilon y'').$$

Thus the computation of the coordinate functions y and z reduces to solving the following symmetric system of ordinary differential equations

$$\begin{aligned} y''' &= \frac{\kappa'}{\kappa} \cdot y'' + \tau \cdot \varepsilon z'', \\ z''' &= \frac{\kappa'}{\kappa} \cdot z'' + \tau \cdot \varepsilon y''. \end{aligned} \tag{5.2}$$

Using formulas (5.1), this system takes the form

$$\begin{aligned} y''' &= -\frac{1}{s} \cdot y'' + \varepsilon \frac{b}{as} \cdot z'', \\ z''' &= -\frac{1}{s} \cdot z'' + \varepsilon \frac{b}{as} \cdot y''. \end{aligned}$$

If we introduce the substitution $as = t$ we get

$$\begin{aligned} t \cdot \frac{d^3 y}{dt^3} &= -\frac{d^2 y}{dt^2} + \varepsilon \frac{b}{a} \cdot \frac{d^2 z}{dt^2}, \\ t \cdot \frac{d^3 z}{dt^3} &= -\frac{d^2 z}{dt^2} + \varepsilon \frac{b}{a} \cdot \frac{d^2 y}{dt^2}. \end{aligned}$$

Reducing the order ($u = \frac{d^2 y}{dt^2}, v = \frac{d^2 z}{dt^2}$) we obtain the system

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{t} \cdot u + \varepsilon \frac{b}{at} \cdot v, \\ \frac{dv}{dt} &= -\frac{1}{t} \cdot v + \varepsilon \frac{b}{at} \cdot u. \end{aligned} \tag{5.3}$$

Eliminating v and $\frac{dv}{dt}$ from this system, for $a \neq b$, we get homogenous Euler's equation

$$a^2 t^2 \cdot \frac{d^2 u}{dt^2} + 3a^2 t \cdot \frac{du}{dt} + (a^2 - b^2) \cdot u = 0. \tag{5.4}$$

The general solution of this equation is given by

$$u(t) = \frac{1}{t}\Big[C_1 \cosh(\frac{b}{a}\ln t) + C_2 \sinh(\frac{b}{a}\ln t)\Big].$$

Further, because of $u = \frac{d^2y}{dt^2} = \frac{1}{a^2}\frac{d^2y}{dx^2}$, we have

$$y''(x) = \frac{a}{x}\Big[C_1 \cosh(\frac{b}{a}\ln ax) + C_2 \sinh(\frac{b}{a}\ln ax)\Big]. \tag{5.5}$$

For $C_1 = 1, C_2 = 0$, after integration, we get the particular solution

$$y'(x) = \frac{a^2}{b}\sinh(\frac{b}{a}\ln ax),$$

or by putting $t = \frac{b}{a}\ln ax$ we obtain

$$y'(t) = \frac{a^2}{b^2}\exp^{\frac{a}{b}t}\sinh t.$$

Finally, after partial integration we have

$$y_1(t) = \frac{a^2}{b(b^2-a^2)}\exp^{\frac{a}{b}t}\big(b\cosh t - a\sinh t\big).$$

The second particular solution of Euler's equation (5.4) we get if in (5.5) we put $C_1 = 0, C_2 = 1$. After integration we obtain function

$$y_2(t) = \frac{a^2}{b(b^2-a^2)}\exp^{\frac{a}{b}t}\big(b\sinh t - a\cosh t\big).$$

In the similar way as described for the first particular solution, eliminating u and $\frac{du}{dt}$ from (5.3), we get the following homogenous Euler's equation for v

$$a^2t^2\cdot\frac{d^2v}{dt^2} + 3a^2t\cdot\frac{dv}{dt} + (a^2-b^2)\cdot v = 0. \tag{5.6}$$

Solutions of this Euler's equation, corresponding to solutions $y_1(t)$ and $y_2(t)$ are

$$z_1(t) = \frac{a^2}{b(b^2-a^2)}\exp^{\frac{a}{b}t}\big(b\sinh t - a\cosh t\big),$$

$$z_2(t) = \frac{a^2}{b(b^2-a^2)}\exp^{\frac{a}{b}t}\big(b\cosh t - a\sinh t\big).$$

According to the Theorem 4.1 we have obtained two curves satisfying conditions (5.1). The parametric representations of the first curve is given by

$$x(t) = \frac{1}{a}\exp^{\frac{a}{b}t}, \qquad y(t) = \frac{a^2}{b(b^2-a^2)}\exp^{\frac{a}{b}t}\big(b\cosh t - a\sinh t\big),$$

$$z(t) = \frac{a^2}{b(b^2-a^2)} \exp^{\frac{a}{b}t} \big(b \sinh t - a \cosh t\big), \tag{5.7}$$

and the second curve by

$$x(t) = \frac{1}{a} \exp^{\frac{a}{b}t}, \qquad y(t) = \frac{a^2}{b(b^2-a^2)} \exp^{\frac{a}{b}t} \big(b \sinh t - a \cosh t\big),$$

$$z(t) = \frac{a^2}{b(b^2-a^2)} \exp^{\frac{a}{b}t} \big(b \cosh t - a \sinh t\big). \tag{5.8}$$

These curves lie on the two cones of revolution (in the sense of G_3^1)

$$y^2 - z^2 = \frac{a^6}{b^2(b^2-a^2)} x^2,$$

and

$$z^2 - y^2 = \frac{a^6}{b^2(b^2-a^2)} x^2,$$

respectively. Note that for $a < b$ the first curve is space-like and the second time-like, and for $a > b$ vice versa.

Moreover, the obtained curves are isogonal trajectories of cone generators. We prove this statement for the curve given by (5.7) and for the corresponding cone of revolution. First of all, we find that the tangent vector of (5.7) has the form

$$(x'(t), y'(t), z'(t)) = \Big(\frac{1}{b} \exp^{\frac{a}{b}t}, \frac{a^2}{b^2} \exp^{\frac{a}{b}t} \sinh t, \frac{a^2}{b^2} \exp^{\frac{a}{b}t} \cosh t\Big),$$

and its improper point F_t (at infinity) has the following homogeneous coordinates

$$F_t\Big(0 : 1 : \frac{a^2}{b} \sinh t : \frac{a^2}{b} \cosh t\Big).$$

The improper point F_g of a generator g is

$$F_g\Big(0 : 1 : \frac{a^3}{b(b^2-a^2)}(b \cosh t - a \sinh t) : \frac{a^3}{b(b^2-a^2)}(b \sinh t - a \cosh t)\Big).$$

In the pseudo-Galilean space the angle between a tangent t and a generator g is defined as the pseudo-Euclidian distance of the points F_t and F_g in the absolute plane ω. Hence we have

$$\varphi = \frac{a^2}{\sqrt{|a^2-b^2|}} = const$$

and our statement is proved.

In a similar way, for the curve given by (5.8) we get the same result.

Since $\frac{\tau}{\kappa} = const.$, the curves are general helices and moreover they are also conical helices.

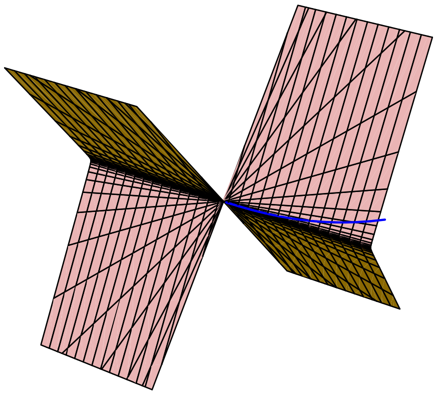

Figure 1. *Conical helix*

Remark 5.1. *For $a = b$ the system (5.3) has solution which is not an admissible curve, because this condition implies equality $\frac{dy}{dt} = \pm\frac{dz}{dt}$, that is not allowed by definition of an admissible curve.*

B) $\mathcal{K} = const. \neq 0$, $\mathcal{T} = 0$

It follows

$$\kappa = \frac{1}{as+c}, \qquad \tau = 0.$$

The system (5.2) have the same form as in Galilean case and it follows

$$y''' = \frac{\kappa'}{\kappa} \cdot y'', \qquad z''' = \frac{\kappa'}{\kappa} \cdot z''.$$

Finally, we have

$$x = s, \qquad y(s) = \frac{1}{a^2}(as+b)\Big[\ln(as+b) - 1\Big], \qquad z = 0$$

and this is an isotropic logarithmic spiral in G_3^1.

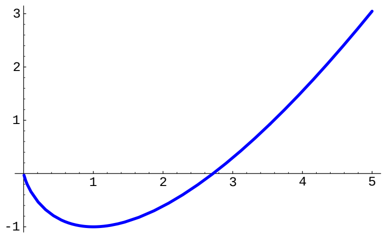

Figure 2. *Isotropic logarithmic spiral*

C) $\mathcal{K} = 0$, $\mathcal{T} = const. \neq 0$

These curves are characterized by

$$\kappa = const. \neq 0, \qquad \tau = const. \neq 0,$$

and therefore $\frac{\tau}{\kappa} = const.$ holds.

According to [4] these are circular helices.

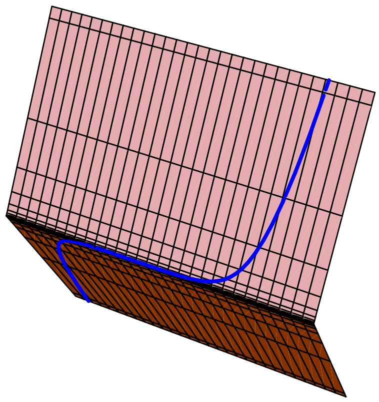

Figure 3. *Circular helix*

D) $\mathcal{K} = 0$, $\mathcal{T} = 0$

The natural equations of these curves are given by

$$\kappa = const. \neq 0, \qquad \tau = 0.$$

According to [1] these curves are the isotropic circles of the pseudo-Galilean space, i.e. curves in G_3^1 are isometric to the parabolas

$$y = \frac{\kappa}{2}x^2, \quad z = 0.$$

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