A new application of quasi power increasing sequences

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Abstract. *By applying the concept of a β- power increasing sequence, the author presents a generalization of a result of Leindler [8] dealing with $\mid \bar{N}, p_n \mid_k$ summability for the $\mid \bar{N}, p_n, \theta_n \mid_k$ summability factors. Some new results have also been obtained.*

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$. A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{1}$$

holds for all $n \geq m \geq 1$ (see [8]). It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the n-th (C,1) mean of the sequence (na_n). A series $\sum a_n$ is said to be summable $\mid C, 1 \mid_k$, $k \geq 1$, if (see [5],[7])

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n \mid^k < \infty. \tag{2}$$

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Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{4}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{5}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{6}$$

In the special case $p_n = 1$ for all values of n $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability.
Let (θ_n) be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty. \tag{7}$$

If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability. Also if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $\mid C, 1 \mid_k$ summability. Furthermore if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ (see [3]) summability.

2. Known results

Bor [4] has proved the following theorem for $\mid \bar{N}, p_n \mid_k$ summability factors.

Theorem A. *Let (X_n) be an almost increasing sequence and let the condition*

$$\sum_{n=1}^{m} \frac{1}{n} \mid s_n \mid^k = O(X_m) \tag{8}$$

be satisfied. If the sequences (β_n) and (λ_n) satisfy the conditions

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{9}$$

$$\beta_n \to 0, \tag{10}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{11}$$

$$\mid \lambda_n \mid X_n = O(1), \tag{12}$$

and furthermore if (p_n) *is a positive sequence such that*

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \mid s_n \mid^k = O(X_m), \tag{13}$$

then the series $\sum a_n\lambda_n$ *is summable* $\mid \bar{N}, p_n \mid_k, k \geq 1$.

Leindler [8] has proved Theorem A by using a quasi $\beta-$ power increasing sequence instead of an almost increasing sequence. His theorem is as follows:

Theorem B. *Let* (X_n) *be a quasi* $\beta-$ *power increasing sequence for some* $0 < \beta < 1$. *If all conditions from (8) to (13) are satisfied, then the series* $\sum a_n\lambda_n$ *is summable* $\mid \bar{N}, p_n \mid_k, k \geq 1$.

3. The main result

The aim of this paper is to generalize Theorem B for $\mid \bar{N}, p_n, \theta_n \mid_k$ summability. Now we shall prove the following theorem.

Theorem. *Let* $\left(\frac{\theta_n p_n}{P_n}\right)$ *be a non-increasing sequence. If all the conditions of Theorem B are satisfied with the condition (13) replaced by*

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \mid s_n \mid^k = O(X_m), \tag{14}$$

then the series $\sum a_n\lambda_n$ *is summable* $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$.

If we take $\theta_n = \frac{P_n}{p_n}$, then we get Theorem B. In this case condition (14) reduces to condition (13) and the condition $\left(\frac{\theta_n p_n}{P_n}\right)$ which is a non-increasing sequence is automatically satisfied.

We need the following lemma for the proof of our Theorem.

Lemma 1 ([8]). *Under the conditions on* (X_n), (β_n) *and* (λ_n) *as taken in the statement of the Theorem, the following conditions hold :*

$$nX_n\beta_n = O(1), \tag{15}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{16}$$

4. Proof of the Theorem

Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=1}^{v} a_r\lambda_r. \tag{17}$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v, \quad n \geq 1. \tag{18}$$

Using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v + \frac{p_n}{P_n} s_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \quad say. \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3. \tag{19}$$

Firstly by using Hölder's inequality, we have that

$$\begin{aligned} \sum_{n=2}^{m+1} \theta_n^{k-1} \mid T_{n,1} \mid^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} \mid \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v \mid^k \\ &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \mid \lambda_v \mid\mid \lambda_v \mid^{k-1} \mid s_v \mid^k \\ &\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} p_v \mid \lambda_v \mid\mid s_v \mid^k \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} p_v \mid \lambda_v \mid\mid s_v \mid^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} p_v \mid \lambda_v \mid\mid s_v \mid^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{1}{P_v} \\ &= O(1) \sum_{v=1}^{m} \mid \lambda_v \mid \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k \mid s_v \mid^k \\ &= O(1) \sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{r=1}^{v} \theta_r^{k-1} \left(\frac{p_r}{P_r}\right)^k \mid s_r \mid^k \\ &\quad + O(1) \mid \lambda_m \mid \sum_{v=1}^{m} \theta_v^{k-1} \left(\frac{p_v}{P_v}\right)^k \mid s_v \mid^k \\ &= O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m \\ &= O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty, \end{aligned}$$

in view of hypotheses of the Theorem and Lemma.
Also we get that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1} \mid T_{n,2} \mid^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} \mid \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k \\
&\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \mid s_v \mid^k \beta_v \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} P_v \mid s_v \mid^k \beta_v \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} P_v \beta_v \mid s_v \mid^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \beta_v \mid s_v \mid^k \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \\
&= O(1) \left(\frac{\theta_1 p_1}{P_1}\right)^{k-1} \sum_{v=1}^{m} \beta_v \mid s_v \mid^k \\
&= O(1) \sum_{v=1}^{m} v\beta_v \frac{\mid s_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{\mid s_r \mid^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} \frac{\mid s_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1) \mid \Delta\beta_v \mid - \mid \beta_v \mid \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid \beta_v \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of hypotheses of the Theorem and Lemma..
Finally, as in $T_{n,1}$ we have that

$$
\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,3} \mid^k &= \sum_{n=1}^{m} \theta_n^{k-1} \mid \frac{p_n}{P_n} s_n \lambda_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k \mid \lambda_n \mid\mid \lambda_n \mid^{k-1} \mid s_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \theta_n^{k-1} \left(\frac{p_n}{P_n}\right) \mid s_n \mid^k = O(1) \quad as \quad m \to \infty.
\end{aligned}
$$

Therefore we get that

$$\sum_{n=1}^{m} \theta_n^{k-1} \mid T_{n,r} \mid^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3.$$

This completes the proof of the Theorem. □

If we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a new result concerning the $\mid C, 1 \mid_k$ summability factors. Also, if we take $p_n = 1$ for all values of n, then we have a new result for $\mid C, 1, \theta_n \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then we have another new result for $\mid R, p_n \mid_k$ summability. Finally, if we take $p_n = \frac{1}{n+1}$, then we get a result for $\mid \bar{N}, \frac{1}{n+1}, \theta_n \mid_k$ summability.

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