Generalization of some inequalities for the Gamma function

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Abstract. *An inequality involving the Euler gamma function is presented. This result generalizes several recently published results by Alsina and Tomás, Sándor, Bougoffa, and the author.*

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1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, x > 0.$$

C. Alsina and M. S. Tomás in [1], using a geometrical method, proved the following double inequality:

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1, \quad x \in [0,1], n \in \mathbb{N}. \tag{1}$$

Using the series representation of $\psi(x)$ J. Sándor [3] extended that result to obtain the following double inequality:

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1, \quad x \in [0,1], a \geq 1. \tag{2}$$

In [2] L. Bougoffa proved that if $1 + ax > 0, 1 + bx > 0$ then for all $a \geq b > 0$ the function

$$f(x) = \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b}, \quad x \geq 0 \tag{3}$$

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is decreasing on $[0, \infty)$ so for $x \in [0, 1]$ the following double inequality is true:

$$\frac{\Gamma(1+b)^a}{\Gamma(1+a)^b} \leq \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b} \leq 1. \tag{4}$$

Note: He also proved that under the conditions $1 + ax > 0, 1 + bx > 0$ the function given in (3) is also decreasing on $[0, \infty]$ if $0 > a \geq b$ and increasing on $[0, \infty]$ if $a > 0$ and $b < 0$.

In [4] the following result is presented.

Theorem 1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

where $x \in [0, 1], a \geq b > 0$, c and d are real numbers such that $bc \geq ad > 0$ and $\psi(b + ax) > 0$. Then f is increasing function on $[0, 1]$ and the following double inequality holds:

$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d} \leq \frac{\Gamma(a+b)^c}{\Gamma(a+b)^d}. \tag{5}$$

Equivalently under the conditions of the Theorem 1 the function

$$f_1(x) = \frac{1}{f(x)} = \frac{\Gamma(b+ax)^d}{\Gamma(a+bx)^c}$$

is decreasing on $[0, 1]$ so

$$\frac{\Gamma(a+b)^d}{\Gamma(a+b)^c} \leq \frac{\Gamma(b+ax)^d}{\Gamma(a+bx)^c} \leq \frac{\Gamma(b)^d}{\Gamma(a)^c}. \tag{6}$$

The idea of this paper is to consider the function

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f}, \quad x \geq 0$$

and to establish some generalizations of the above results.

2. Main results

In order to establish the proofs of the theorems, we need the following lemmas:

Lemma 1. *Let $0 < x \leq y$. Then*

$$\psi(x) \leq \psi(y). \tag{7}$$

Proof. In [5], page 21, we have the following:

$$\psi(x) - \psi(y) = (x - y) \cdot \sum_{n=0}^{\infty} \frac{1}{(x+n)(y+n)}.$$

Hence, if $x \le y$ then clearly $\psi(x) - \psi(y) \le 0$. □

Lemma 2. *Let a, b, c, d, e be real numbers such that $a + bx > 0$, $d + ex > 0$ and $a + bx \le d + ex$. Then*

$$\psi(a + bx) - \psi(d + ex) \le 0. \tag{8}$$

Proof. Based on Lemma 1. □

Lemma 3. *Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \le d + ex$ and $ef \ge bc > 0$. If*

(i) $\psi(a + bx) > 0$ or

(ii) $\psi(d + ex) > 0$

then

$$bc\psi(a + bx) - ef\psi(d + ex) \le 0. \tag{9}$$

Proof. (i) Let $\psi(a+bx) > 0$. From Lemma 2 we have $\psi(d+ex) \ge \psi(a+bx) > 0$. Multiplying both sides of inequality $ef \ge bc$ with $\psi(d + ex)$ we obtain.

$$ef\psi(d + ex) \ge bc\psi(d + ex) \ge bc\psi(a + bx),$$

or

$$bc\psi(a + bx) - ef\psi(d + ex) \le 0.$$

(ii) If $\psi(d + ex) > 0$, considering (8) we see that there are two possibilities for $\psi(a + bx)$.

Case 1. $\psi(a + bx) \le 0$, *Case 2.* $\psi(a + bx) > 0$.

So we have:

Case 1. $bc\psi(a + bx) \le 0$ and $ef\psi(d + ex) > 0$ so clearly (9) holds.

Case 2. The possibility $\psi(a + bx) > 0$ was proved in (i). □

Lemma 4. *Let a, b, c, d, e, f be real numbers such that $a + bx > 0, d + ex > 0, a + bx \le d + ex$ and $bc \ge ef > 0$. If*

(i) $\psi(d + ex) < 0$ or

(ii) $\psi(a + bx) < 0$

then

$$bc\psi(a + bx) - ef\psi(d + ex) \le 0. \tag{10}$$

Proof. (i) Let $\psi(d+ex) < 0$. From Lemma 2 we have $\psi(a+bx) \le \psi(d+ex) < 0$. Multiplying both sides of inequality $bc \ge ef$ with $\psi(a + bx)$ we obtain.

$$bc\psi(a + bx) \le ef\psi(a + bx) \le ef\psi(d + ex),$$

or

$$bc\psi(a + bx) - ef\psi(d + ex) \le 0.$$

ii) If $\psi(a + bx) < 0$, considering (8) we see that there are two possibilities for $\psi(d + ex)$.

Case 1. $\psi(d + ex) \ge 0$, *Case 2.* $\psi(d + ex) < 0$.

Now we proceed in the same way as in Lemma 3 □

Theorem 2. *Let f_1 be a function defined by*

$$f_1(x) = \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f}, \quad x \geq 0 \tag{11}$$

where a, b, c, d, e, f are real numbers such that: $a + bx > 0, d + ex > 0, a + bx \leq d + ex, ef \geq bc > 0$. If $\psi(a + bx) > 0$ or $\psi(d + ex) > 0$ then the function f_1 is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the following double inequality holds:

$$\frac{\Gamma(a+b)^c}{\Gamma(d+e)^f} \leq \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f} \leq \frac{\Gamma(a)^c}{\Gamma(d)^f}. \tag{12}$$

Proof. Let g_1 be a function defined by $g_1(x) = \log f_1(x)$ for $x \in [0, \infty)$. Then:

$$g_1(x) = c \log \Gamma(a + bx) - f \log \Gamma(d + ex).$$

So

$$g_1^{'}(x) = bc\frac{\Gamma^{'}(a+bx)}{\Gamma(a+bx)} - ef\frac{\Gamma^{'}(d+ex)}{\Gamma(d+ex)} = bc\psi(a+bx) - ef\psi(d+ex).$$

Using (9), we have $g_1^{'}(x) \leq 0$. It means that g_1 is decreasing on $[0, 1]$. This implies that f_1 is decreasing on $[0, 1]$.

Hence for $x \in [0, 1]$ we have $f_1(1) \leq f_1(x) \leq f_1(0)$ or

$$\frac{\Gamma(a+b)^c}{\Gamma(d+e)^f} \leq \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f} \leq \frac{\Gamma(a)^c}{\Gamma(d)^f}.$$

This concludes the proof of Theorem 2. □

In a similar way, using Lemma 4, it is easy to prove the following theorem.

Theorem 3. *Let f_1 be a function defined by*

$$f_1(x) = \frac{\Gamma(a+bx)^c}{\Gamma(d+ex)^f}, \quad x \geq 0, \tag{13}$$

where a, b, c, d, e, f are real numbers such that: $a + bx > 0, d + ex > 0, a + bx \leq d + ex, bc \geq ef > 0$. If $\psi(d + ex) < 0$ or $\psi(a + bx) < 0$ then the function f_1 is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the inequality (12) holds.

It is easy to verify by Theorem 2 and Theorem 3 that the following remarks hold.

Remark 1. *Considering (12) with $a = 1, b = 1, c = n, n \in \mathbb{N}, d = 1, e = n, n \in \mathbb{N}, f = 1$ we obtain inequality (1).*

Remark 2. *Considering (12) with $a = 1, b = 1, c = a, a \geq 1, d = 1, e = a, f = 1$ we obtain inequality (2).*

Remark 3. *If in (12) we take $a = 1, c = a, d = 1, e = a, f = b$, with $c \geq f > 0$ we obtain inequality (4).*

Remark 4. *If in (12) we take $a = b, b = a, c = d, d = a, e = b, f = c$ $ef \geq bc > 0$, with $a \geq b > 0$ and $\psi(b + ax) > 0$, we obtain inequality (6).*

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References

[1] C. ALSINA, M. S. TOMÁS, *A geometrical proof of a new inequality for the gamma function*, J. Ineq. Pure Appl. Math. **6**(2005), Article 48. http://jipam.vu.edu.au/article.php?sid=517

[2] L. BOUGOFFA, *Some inequalities involving the Gamma Function*, J. Ineq. Pure Appl. Math. **7**(2006), Article 179. http://jipam.vu.edu.au/article.php?sid=796

[3] J. SÁNDOR, *A note on certain inequalities for the gamma function*, J. Ineq. Pure Appl. Math. **6**(2005), Article 61. http://jipam.vu.edu.au/article.php?sid=534

[4] A. SH. SHABANI, *Some inequalities for the Gamma Function*, J. Ineq. Pure Appl. Math. **8**(2007), Article 49. http://jipam.vu.edu.au/article.php?sid=852

[5] M. A. CHAUDHRY, S. M. ZUBAIR, *A class of incomplete Gamma function with applications*, CRC Press, 2002.