Interaction of Strings

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ABSTRACT

We rewiew and discuss in explicite terms the possible approaches to the interacting string theory. The interactions are introduced from the viewpoint of conformal mappings. Mandelstams light cone formulation is first summarized and then a discussion is given of the mappings that lead to Wittens form of the interacting field theory. The construction of a gauge invariant vertex is described.

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We review the string interactions involved in constructing the gauge invariant string field theory. We concentrate mostly on Wittens approach. Most of the new work reported was done [12] in colaboration with D. Gross.

The starting point for a field theoretic formulation of dual models is the path integral representation introduced by Hsue, Sakita and Virasoro [1] and Fairlic and Nielsen [2] where

$$\int [dX(z)^{\mu}] \exp \left[\frac{-1}{2\pi\alpha'} \int_D d^2 z \partial_z X^{\mu} \partial_{\bar{z}} X_{\mu} \right] \tag{1}$$

is a gaussian functional integral. The domain D is the upper half complex z-plane $D = [z; Imz \geq 0]$ for the open string and the full complex plane for the closed string. An arbitrary n-particle S-matrix scattering amplitude is obtained by inserting the appropriate external sources: $V_{(p,z)} = e^{ipX(z)}$ representing the scalar tachyons, $\dot{X}_{\mu}(z)e^{ipX(z)}$ representing vector states and in general [3,4]

$$S_n = \int \langle V(p_1, z_1) V(p_2, z_2) \ldots V(p_n, z_n) \rangle \prod_i dz_i \tag{2}$$

After functional integration there results the Koba-Nielsen type formula

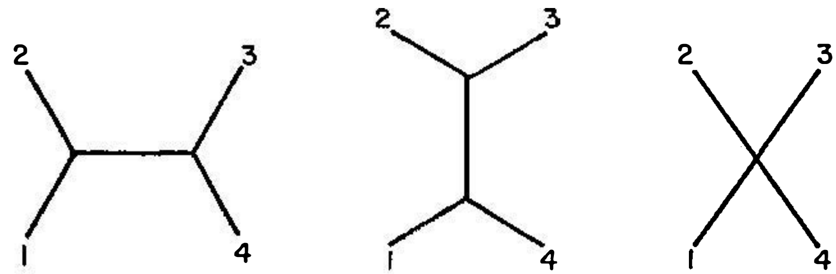
$$S_n(p_1, p_2, \ldots p_n) = \int \prod_{i=1}^{N} dz_i \prod_{i<j} |z_i - z_j|^{-2\pi\alpha' p_i p_j} \tag{3}$$

It is obtained from the Neumann function of the upper-half plane:

$$\begin{aligned} (\partial_x^2 + \partial_y^2) N(z, z') &= -2\pi\delta^{(2)}(z - z'), \\ N(z, z') &= \ln|z - z'||\bar{z} - z'|. \end{aligned} \tag{4}$$

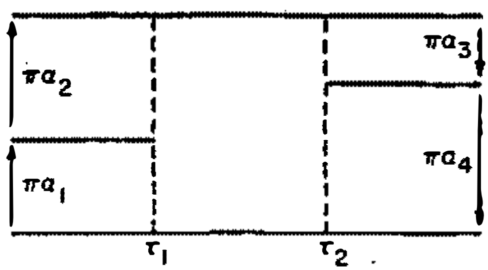
[in what follows we shall restrict our discussion to open strings only].

As it appears this representation has no resemblance to field theoretic representation of scattering amplitudes in terms of Feynman diagrams. For the four particle scattering process in field theory one has in general the diagrams



representing typicaly the s-channel, the t-channel and the quartic interaction contribution.

In the framework of the light-cone formulation [5,6] it was possible to reexpress the dual model as Feynman scattering processes of strings [7]. In this gauge the time propagation is along $= X^+ = X^0 + X^{25}$ and a special role is played by the p^+ momenta. The main point is that the uper-half z-plane can be mapped into a strip (ρ-plane)



where $\rho = \tau + i\sigma$ and the transvere string coordinates $X^i(\tau,\sigma)$ are the dynamical variables. The mapping is simply the Schwarz-Christoffel transformation.

$$\rho = \sum_{i=1}^{n} \alpha_i \ln(z - z_i) \tag{5}$$

The parameters α_i are conserved ($\sum \alpha_i = 0$) and they are identified with the momenta $\alpha_i = 2p_i^+$. Clearly for the 4-particle case in the picture one has a Feynman like process with interaction taking place at τ_1 (joining of strings 1 and 2) and at τ_2 (splitting of the intermediate string into strings 3 and 4).

The functional integral can be formaly factorized into the propagation and interaction terms

$$\int d\tau_1\, d\tau_2 \int \prod_{i=1}^{6} dX_\mu^{(i)} D(-\infty; X^{(1)}) D(-\infty; X^{(2)}),$$
$$V(X^{(1)}, X^{(2)}, X^{(5)}) D(X^{(5)}, X^{(6)}) D(X^{(3)}, \infty) D(X^{(4)}, \infty). \tag{6}$$

with the propagator

$$D(x, x') = \int_0^\infty d\tau \int \mathcal{D}X e^{-\int^\tau d\tau\, d\sigma (\partial X)^2} \equiv \langle x | e^{\frac{-\tau}{\alpha}(L_0 - 1)} | x' \rangle \tag{7}$$

and a formal δ-function type interaction

$$V(1;2;3) = \prod_\sigma \delta\Big(X^{(3)} - \Theta_1 X^{(1)} - \Theta_2 X^{(2)}\Big) \tag{8}$$

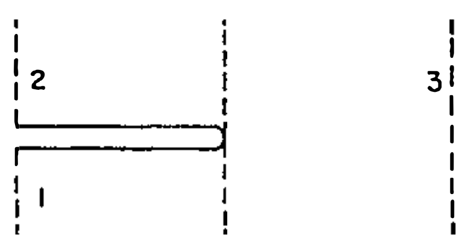
The positive and negative regions of the $T = \tau_2 - \tau_1$ integration correspond to the S and t-chanel contributions respectively. Of central importance is the explicite form of the three string interaction which represents the field theory vertex. It can be given in the creation- anihilation oscillator basis representing the discrete Fourier decomposition of the string coordinates

$$X^\mu(\sigma) = X_0^\mu + \sum_{n=1}^{\infty} (a_n^\mu - a_n^{\mu +}) \cos n\sigma \tag{9}$$

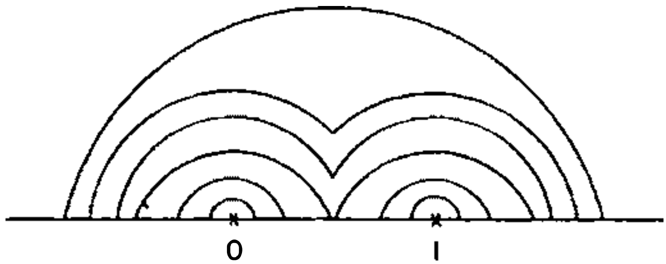
and momenta

$$P^\mu(\sigma) = P_0^\mu + \sum_n (a_n^\mu + a_n^\mu) \cos n\sigma \tag{10}$$

The δ-functions provide the set of constraines obeyed by the vertex, the explicite form is however obtained in terms of the conformal mapping and the associated Neumann function. One considers the three string problem with the mapping $\rho = \alpha_1 \ln(z-1) + \alpha_2 \ln z$ $(z_1 = 1, z_2 = 0, z_3 = 0)$ which corresponds to.



This string joining interaction of the light-cone gauge corresponds to a particular cutting of the uper- half z plane. A very useful procedure for describing the interaction in general was introduced in [8]. One thinks of the mapping function ρ as an electrostatic potential and draws equipotential lines. They describe string propagation. For the light cone case it is



The Neuman function for the open string scattering domain has the expansion

$$N(\rho,\rho') = -\delta_{rs} \sum_{n=1}^{\infty} \frac{2}{n} e^{-n|\xi_r-\xi_s|} \cos m\eta_r \cos n\eta_s, \\ + \sum 2N^{rs}_{mn} e^{m\xi_r+n\xi_s} \cos m\eta_r \cos n\eta_s. \tag{11}$$

where for the three strings $\rho = \alpha_r(\xi_r + i\eta_r)$ r=1,2,3.

From this Neumann function one constructs the closed boundary Neumann function and then after reduction of external legs one obtains the interaction vertex. It reads

$$|V\rangle = \exp\left[\frac{1}{2} a^r_{-n} \bar{N}^{rs}_{nm} a^s_{-m} + p^n_0 \bar{N}^{rs}_{0m} a^s_{-m}\right] |0_{123}\rangle \tag{12}$$

where

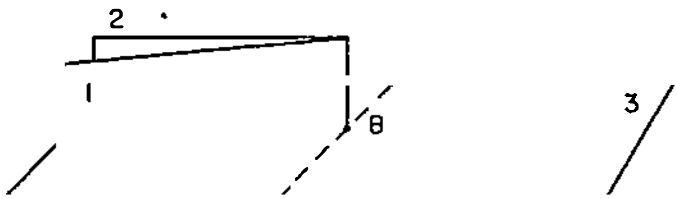
$$\bar{N}^{rs}_{mn} = \frac{-mn}{m\alpha_s + n\alpha_r} \alpha_1\alpha_2\alpha_3 \bar{N}^r_m \bar{N}^s_n \tag{13}$$

The coeficients

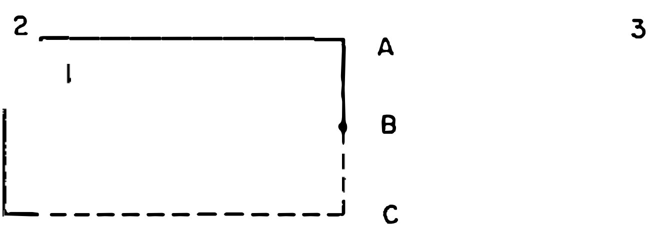
$$\bar{N}^r_m = \frac{1}{\alpha_r} f_m\left(\frac{-\alpha_{r+1}}{\alpha_r}\right) e^{m\tau_0/\alpha_r} \tag{14}$$

are obtained from the expansion of the inverted equation $z = z(\rho)$ in terms of $e^{(\xi_r+i\eta_r)}$. Once this form is known it was possible to establish that this vertex solves the δ-function overlap equations [9,8]. With the explicite form of the vertex it was then possible to write down the light cone interacting field theory [7,8]. Acctualy an additional quartic interactions was also needed to complete the theory. Extensions to closed strings and superstrings were also following [10].

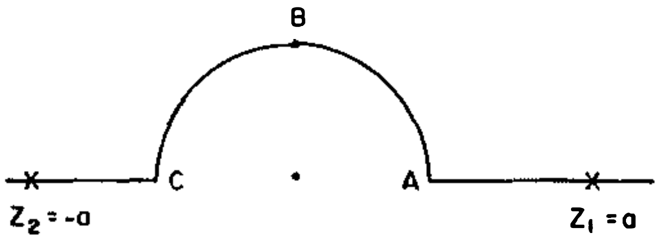
Let us now describe the conformal transformations that give the vertex and the scattering geometry suggested by Witten [11] in his approach to string field theory. The basic point is to write a conformal mapping that would transform the dual model uper half plane into a scattering process of strings of equal lenght. The process is given in fig. for three strings: half's of string 1 and string 2 anihilate and the other half's rearange into string 3:



In the complex ρ plane we can represent this process through a double sheeted structure. It is obtained by cutting the real three dimensional interaction proces along the interaction half-line to obtain:



Strings 1 and 2 now overlap each other and the rearangement is taking place along ABC. The point is not only of constructing a mapping that transforms the upper half plane into the ρ-plane curve outlined but we also need a procedure for identifing the half strings AB and AC at the interaction.To achieve this let us consider a modified upper half plane as in fig. The three strings are at $Z_1 = a$, $Z_2 = -a$ and $Z_3 = \infty$. The upper half plane is deformed into the picture



Here one has cutt out a unit circle from 1 to -1, this will corespond to the line ABC. A mapping that connects the Z to ρ plane is the following.

$$\rho = \ln(Z^2 - a^2) + \ln(Z) - \ln\left(Z^2 - \frac{1}{a^2}\right) \tag{15}$$

One can check that it goes as follows: at $Z = a$ there is an $i\pi$ jump for string 1 and then at $Z = 1$ or $\rho = i\pi + \tau_0$ with the interaction time

$$\tau_0 = \ln(a^2) \tag{16}$$

there is a turn by $\frac{\pi}{2}$, likewise at Z=-1 and then at $Z = -a$ one again has a jump of $i\pi$ corresponding to the string 2. The fact that the immage of the circle corresponds to a straight interaction line ABC follows from the fact that for $Z = e^{i\varphi}$.

$$\rho = i\varphi + \ln\frac{e^{2i\varphi} - a^2}{e^{2i\varphi} - a^{-2}} = \ln a^2 + i\theta(\varphi) \tag{17}$$

Concerning the parameter a in the mapping one notes the following. It's most symmetric value, as will become clear, is $a = \sqrt{3}$. However for general a we only have a shift in the interaction time and the presence of this freedom signals a symmetry of the vertex. Namely one can show that changes of a are induced by the following transformations.

$$Z' = Z + \epsilon\frac{Z - Z^3}{1 + Z^2} \tag{18}$$

Acctualy this is only a particular example of an infinite set of invariance transformations, they are generated by the largest anomaly free subgroup of the Virasoro algebra

$$K_n = L_n - (-)^n L_{-n} \tag{19}$$

This large group replaces the SU(1,1) subgroup of the dual model.

To achive the second important property which is the identification of the segment AB (this is on string 1) with the segment CB (on the string 2) we proceed as follows. One uses and imposes a symmetry of inversion

$$Z \to -1/Z \tag{20}$$

in the Z-plane, which corresponds to identification of the half curve $(0, \pi/2)$ with $(\pi, \pi/2)$. Imposition of invariance under eq.(20) now supplies the identification of the half-segments interaction time. The fact that this is achieved by a symetry transformation will allow a construction of the Neumann function. Namely in addition to the old image charge effect at $\bar{Z}'$ we will now have new image charges. We can describes the situation in simpler and much more symetric terms if we move from the upper half plane to the circle.

Consider a transformation of the Z-plane into a unit circle

$$z = \frac{Z + i}{Z - i} \tag{21}$$

The sources in eq.(15) then correspond to three real strings located at

$$z_1 = e^{i\pi/6},\ z_2 = -e^{-i\pi/6},\ z_3 = e^{i\pi/2} \tag{22}$$

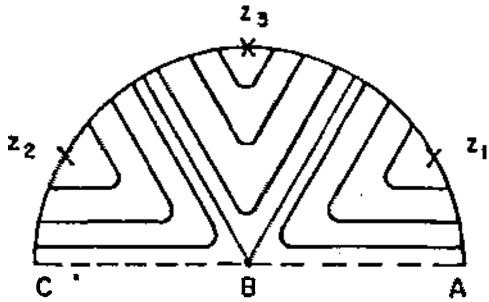
and three reflected strings

$$z_1 = -z_1,\ z_2 = -z_2,\ z_3 = -z_3 \tag{23}$$

since the inversions eq.(20) now read as reflections $z \to -z$. The mapping now corresponds to that of six half-strings with alternating lengths $(\alpha_1 = 1, \alpha_3 = -1, \alpha_2 = 1, \alpha_{-1} = -1, \alpha_{-3} = +1, \alpha_{-2} = -1)$ and it equals

$$\rho = \sum_{i=1}^{3} (\ln(z - z_i) - \ln(z - z_{-i})) \tag{24}$$

The six string process described by this conformall mapping is nicely described by the electrostatic analogue diagram which has the following form (we only draw the upper half circle):



This makes clear the half string rearrangement process. We also see that demanding reflection symmetry leads us to the three string rearrangement process, which we were interested in. For the corresponding Neumann function which usually had the image charge at $1/\bar{z}'$, now due to identification of the lower with the uper half of the disk we add the charges at $-z'$ and -$1/\bar{z}'$, giving

$$N(z,z') = \log|z^2 - z'^2||z^2 - 1/\bar{z}'^2| \tag{25}$$

We summarize below the steps involved in the construction of the vertex operator of the gauge invariant field theory [12]. The above conformal mapping eq.(24) which gives the rearangement of equal lenght strings is simply

$$\rho = \ln\frac{(z^3 - i)}{(z^3 + i)} - \frac{i\pi}{2} \tag{26}$$

In what follows we use z to represent the complex variable on the disk. On each string the conformal mapping is inverted to read

$$z = z(\varsigma) = z_a\left(\frac{1 + ie^{\varsigma}}{1 - ie^{\varsigma}}\right)^{\frac{1}{3}}; a = \pm1, \pm2, \pm3 \tag{27}$$

The Fourier coeficients that enter the vertex are given by

$$\left(\frac{1 + ie^{\varsigma}}{1 - ie^{\varsigma}}\right)^{\frac{1}{3}} = \sum_{n=2k} A_{2k}e^{n\varsigma} + i\sum_{n=2k+1} A_{2k+1}e^{n\varsigma} \tag{28}$$

Similarly for $\omega = 2$ the coeficients are

$$\left(\frac{1+ie^{\varsigma}}{1-ie^{\varsigma}}\right)^{\frac{2}{3}} = \sum_{n=2k} B_{2k}e^{n\varsigma} + i\sum_{n=2k+1} B_{2k+1}e^{n\varsigma} \tag{29}$$

All these coeficients can be found explicitely, they also satisfy simple recurssion relations which read

$$\begin{aligned} A_{n+1} &= \frac{-\frac{2}{3}A_n + (n-1)A_{n-1}}{n+1}, \\ B_{n+1} &= \frac{-\frac{4}{3}B_n + (n-1)B_{n-1}}{n+1}. \end{aligned} \tag{30}$$

The Neumann function is constructed from the function

$$M_0(\rho,\rho') = (\partial_\tau + \partial'_\tau)N(\rho,\rho') \tag{31}$$

It is determined for six strings to read

$$M_0(\rho,\rho') = 2\sum_{a=\pm1}^{3} \alpha_a Re(\partial_\tau \ln(z-z_a))Re(\partial_{\tau'} \ln(z-z_a)) \tag{32}$$

This follows from the Laplace equation and boundary conditions. The method is formulated by Mandelstam [7].

Using the explicit form of the mapping we have

$$\frac{\partial \rho}{\partial z} = -6i\frac{z^2}{z^6+1} \tag{33}$$

and

$$\begin{aligned} M_0(\rho,\rho') = \frac{1}{2\cdot 6}\Bigg[&Re\left(z+\frac{1}{z}\right)Re\left(\frac{z'^2}{i}+\frac{i}{z'^2}\right)+ \\ &Re\left(\frac{z}{i}+\frac{i}{z}\right)Re\left(z'^2+\frac{1}{z'^2}\right)+ \\ &2Re(z^3)+(z\leftrightarrow z'). \end{aligned} \tag{34}$$

The terms $Re(z^3)$ and $Re(z'^3)$ can be droped, they contribute to the zero made coeficients only but become irrelevant due to momentum conservation.

For the three String vertex we use the Neumann function that is symetric under reflections ($z \to -z$), this means an identification of strings at z_a and z_{-a}. Consequently the Neumann function for the three string problerm is constructed as

$$N^{rs}_{nm} = \frac{M^{rs}_{nm}}{\alpha_r n + \alpha_s m} + \frac{M^{r-s}_{nm}}{\alpha_r n + \alpha_{-s} m}; r,s = 1,2,3 \tag{35}$$

The structure of the Neumann coeficients agrees with the general srtucture following from the δ-function overlap equations. The precise correspondence was established in [12] the form of the vertex which we have constructed is as follows. The general structure is as in eq.(12):

$$|V\rangle = \exp\{\frac{1}{2}\alpha^r_{-m}\mathcal{N}^{rs}_{nm}\alpha^s_{-m} + p^r_0\mathcal{N}^{rs}_{0m}\alpha^s_{-m} + \frac{1}{2}\mathcal{N}_{00}(\sum_{r=1}^{3} p_r^2)\}|O_{123}\rangle \tag{36}$$

with new coeficients $\mathcal{N}^{rs}_{nm}$. The form in the string index space $(r, s = 1, 2, 3)$ is very simple and is given by the matrix representation

$$\mathcal{N} = -\frac{1}{6}\left[(\tilde{C} + \mathcal{U} + \bar{\mathcal{U}})1 + (\tilde{C} - \mathcal{U} + \frac{\bar{\mathcal{U}}}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + i\frac{\sqrt{3}}{2}(\mathcal{U} - \bar{\mathcal{U}})\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\right] \tag{37}$$

$\tilde{C}$, U, $\bar{U}$ are matrices in the n, m space and ere given by:

$$\tilde{C}_{nm} = 2\frac{(-)^n}{n}\delta_{n,m} \tag{38}$$

$$(\mathcal{U} + \bar{\mathcal{U}})_{nm} = -2(-)^n\{\frac{A_nB_m + B_nA_m}{n+m} + \frac{A_nB_m - B_nA_m}{n-m}\}$$

$$(\mathcal{U} - \bar{\mathcal{U}})_{nm} = -2i\{\frac{A_nB_m - B_nA_m}{n+m} + \frac{A_nB_m + B_nA_m}{n-m}\}$$

These are valid when $n = 0$ reducing to

$$\mathcal{U}_{0m} = -\frac{1}{m}[A_{m=2k} - iA_{m=2k+1}] \tag{39}$$

Also

$$\mathcal{U}_{00} = -\frac{1}{2}\ln\frac{3^3}{2^4} \tag{40}$$

The above list is valid even for $n = m$ with $(\mathcal{U} - \bar{\mathcal{U}})_{nn} = 0$ and a nontrivial limit to be involved in $(\mathcal{U} + \bar{\mathcal{U}})_{nn}$. This can also be found and the explicit result is given in [17]. The above is then a complete description of the gauge invariant vertex.

The Neumann function vertex described above corresponds to the intinitesimal time $(\Delta\tau \to 0)$ limit of the transition amplitude (fig.[5]). As such it is a representation of a local δ-function type interaction which is

$$\mathcal{V} = \prod_{0\leq\sigma\leq\frac{\pi}{2}}\prod_{r=1}^{3}\delta(X_r(\sigma) - X_{r+1}(\pi - \sigma)) \tag{41}$$

This can be established by showing that $|V\rangle$ obeys the δ-function overlap equations associated with (41). Those can be worked out in a rather elegant form (this is due to a high degree of symmetry involved. One constructs the linear combinations

$$Q_3(\sigma) = \frac{1}{\sqrt{3}}(X_1(\sigma) + X_2(\sigma) + X_3(\sigma))$$
$$Q(\sigma) = \frac{1}{\sqrt{3}}(X_1 + e^{i\frac{\pi}{3}}X_2 + e^{-i\frac{\pi}{3}}X_3) \tag{42}$$

and the overlaps read:

$$Q_3(\sigma) - Q_3(\pi - \sigma) = 0 \tag{43a}$$

$$Q(\sigma) = \begin{cases} e^{i\frac{\pi}{3}}Q(\sigma) & \text{for } 0 \le \sigma \le \frac{\pi}{2} \\ e^{-i\frac{\pi}{3}}Q(\sigma) & \text{for } \frac{\pi}{2} \le \sigma \le \pi \end{cases} \tag{43b}$$

The demonstration that $|V\rangle$ obeys these δ-function overlap equations is given in ref.[12]. It conects the second with the first quantized Polyakov type theory.

The vertex operator enjoys a large degree of symmetry. (this feature distinguishes it very much from a light cone type vertex). Namely consider the σ-reparametrization subgroup of the Virasoro algebra

$$R_n = L_n - L_{-n} \tag{44}$$

The correspondding generators given by

$$K_n = e^{i\frac{\pi}{2}L_0}R_n e^{-i\frac{\pi}{2}L_0} \ . \tag{45}$$

one symmetries of the vertex. They naively leave the overlaps invariant, with the ghosts they can be shown to anihilate the vertex

$$(\hat{K}_n^1 + \hat{K}_n^2 + \hat{K}_n^3)|V_3\rangle = 0 \tag{46}$$

and represent a symmetry. They together with the BRST operator Q plays a central role in the gauge invariance of Witten's action:

$$S = \int (\Psi Q \Psi + \frac{2}{3}\Psi * \Psi * \Psi) \tag{47}$$

In the whole approach the ghost degrees of freedom play a central role. The ghosts appear already in the first quantized theory where Polyakov generalized [14] the original functional integral eq.(1) to

$$\int \mathcal{D}X_\mu \mathcal{D}\bar{C}\mathcal{D}C \exp[-S] \tag{48}$$

with the action

$$S = \int d^2z(\partial X^\mu \partial X^\mu + i\bar{C}_a \mathcal{D}^{ab} C_b) \tag{49}$$

Here

$$\mathcal{D}_{ab} = \begin{pmatrix} \partial_1 & \partial_0 \\ -\partial_0 & \partial_1 \end{pmatrix}, \quad C = \begin{pmatrix} C_0 \\ C_1 \end{pmatrix} \tag{50}$$

The full Virasoro generators $L_n = L_n^x + L_n^c$ do not have a central charge and the BRST operator

$$Q = \sum C_n L_{-n} \tag{51}$$

is nilpotent $Q^2 = 0$ in $D = 26$.

The string field Ψ is a functional of ghosts coordinates also $\Psi = \Psi[X^\mu, \bar{C}, C]$ and so are the vertex operators. The notion that this set of coordinates $X^\mu, \bar{C}, C$ with the BRST charge has a differential geometric structure [16] is of fundamental importance. The free actions constructed are shown to be gauge invariant and after gauge fixing equivalent to the light cone gauge string theory [17].

Generalizations to include interactions tried using the known light cone vertex. Apart from a 26 dimensional coordinate the additional part was to work out the ghost vertex [18]. In the first quantized language the arbitrary string lenghts necessary in the light cone approach do not present problems but they lack interpretation at the field theoretic level.

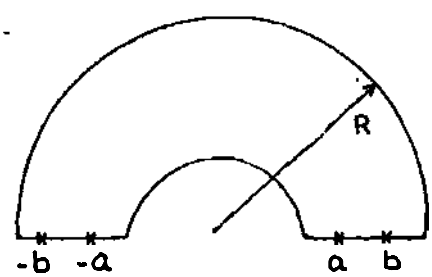
Wittens generalization is notable in several respects. first of all it has a differential geometric structure, in addition to Q taken as a derivative Witten has defined an integration $\int$ and a multiplication $*$. The integration corresponds to folding of a single string

$$\int \Psi = \int \mathcal{D}X \prod_\sigma \delta(X^\mu(\sigma) - X^\mu(\pi - \sigma))\Psi[X] \tag{52}$$

and the multiplication $*$ is closely related to the vertex $|V\rangle$ that we considered above. In all these, the ghost contribution has to be included, Witten has suggested using a bosonized ghost field $\phi(\sigma)$. For this the vertex is the same as for the coordinates, one only has the additional ghost number factors $e^{\pm i\frac{3}{2}\phi(\frac{\pi}{2})}$ at the midpoints. However it is also possible to work out a vertex in terms of fermionic ghosts.

With the explicite operator vertex that we have described Wittens theory has a concrete representation. The equations of motion can be written down and in principle one could look for classical solutions. On shell the theory gives the dual model amplitudes. First of all any three point amplitude can be seen to agree (a method for a full proof can be given). Concerning higher N-point scattering amplitudes one can say the following. Using the vertex operator one writtes down standard Feynman type diagrams. Now since we have the fact that the field theoretic δ-function vertex is represented by the Neumann function form on an infinitesimal surface it follows (gluing) that the whole contribution is given by the Neumann function of the Riemannian surface representing the Feynman diagram.One needs to prove that the measure is the first quantized Faddeev-Popov determinant and then one can consider the whole process using the simple Polyakov integral [13].

Let us just describe shortly the nature of the four particle scattering mapping that would be induced by our basic three point mapping (eq.(15)). For this case in addition to the semicircle (-1,+1) we shall add another semicircle of radius R closing the contour. This represents the second interaction line where one has a splitting of a single string in two. The Z plane is therefore



The radius R is clearly related to the interaction time which is the distance between the interaction lines. Strings 1 and 2 are located as before at $Z_1 = a$ $Z_2 = -a$ and strings 3 and 4 are at $Z_3 = -b$ and $Z_4 = b$. The basic mapping is now

$$\ln(Z^2 - a^2) - \ln(Z^2 - b^2) \tag{53}$$

but since one is now to imposse two symmetry operations

$$Z \to -\frac{1}{Z}, \quad Z \to -R\frac{1}{Z} \tag{54}$$

there will be an infinite series of additional image strings. In this way the symmetry operations eq.(54) lead us to ($\omega \equiv \frac{1}{R^2}$):

$$\begin{aligned} &\ln(1 - \frac{a^2}{Z^2}) - \ln(1 - \frac{1}{a^2Z^2}) \\ &+ \sum_{n=1}^{\infty} \left[\left(\ln\left(1 - \omega^n \frac{a^2}{z^2}\right) + \ln\left(1 - \omega^n \frac{Z^2}{a^2}\right) \right) \right. \\ &- \left(\ln\left(1 - \omega^n \frac{1}{a^2Z^2}\right) + \ln(1 - \omega^n a^2 Z^2) \right) \end{aligned} \tag{55}$$

This clearly is a sum of two eliptic functions since

$$\begin{aligned} \ln \Psi(x,\omega) = \ln(1-x) - \frac{1}{2}\ln x + \frac{\ln^2(x)}{2\ln\omega} + \\ \sum_{n=1}^{\infty} [\ln(1-\omega^n x) + \ln(1 - \omega^n \frac{1}{x}) - 2\ln(1-\omega^n)] \end{aligned} \tag{56}$$

The above four point eliptic function mapping must be related to the particular mapping used by Giddings [13].

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