$K \rightarrow 2\pi$ AND HADRONIC SUM RULES

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1. Introduction

It has been observed experimentally that $\Delta I=1/2$ and $\Delta I=3/2$ amplitudes of $K\rightarrow 2\pi$ decays differ by an order of magnitude. One finds[1) that

$$a^{3/2}(K^+ \rightarrow \pi^+\pi^o) = 1.82\times 10^{-8} \text{ GeV} ,$$
$$a^{1/2}(K^o \rightarrow \pi^+\pi^-) = 27.06\times 10^{-8} \text{ GeV} . \qquad (1.1)$$

This fact, known as the $\Delta I=1/2$ rule, is very difficult to explain within the standard model of weak interactions.

Another interesting point of K^o decays is CP violation. K^o appears in nature in two states, K_L and K_S, which are quantum-mechanical mixtures of K^o and $\overline{K}^o$:

$$|K_S\rangle = p|K^o\rangle - q|\overline{K}^o\rangle ,$$
$$|K_L\rangle = p|K^o\rangle + q|\overline{K}^o\rangle . \qquad (1.2)$$

The parameter $\varepsilon = (p-q)/(p+q)$ measures CP violation, which is directly related to the K_L-K_S mass difference.

Another possible source of CP violation (the so-called direct CP violation) is the difference in phases of the weak amplitudes $a_o \equiv a(K^o \rightarrow \pi\pi(I=0))$ and $a_2 \equiv a(K^o \rightarrow \pi\pi(I=2))$. One defines the ε' parameter as

$$\varepsilon' = \frac{1}{\sqrt{2}} \, \mathrm{Im}\left(\frac{a_2}{a_o}\right) e^{i(\delta_2-\delta_o+\frac{\pi}{2})} . \qquad (1.3)$$

From the phase-shift analysis it turns out that ε and ε' are almost parallel in the complex plane. Therefore, the parameter ε' is always given as a ratio ε'/ε. An interesting point concerning ε' is that if the $\Delta I=1/2$ rule is to be explained by the standard model, then there is a definite prediction for the magnitude of ε' [2,3].

Our aim is to examine matrix elements of the operators relevant to $K\to\pi\pi$ decays by using the method of hadronic sum rules. By doing this, we shall be able to set constraints on the $\Delta I=1/2$ and $\Delta I=3/2$ amplitudes, and consequently on ε'/ε.

2. Remarks on the present theoretical description

In the following we assume that weak transitions are due to the effective Hamiltonian derived in the minimal SU(2)×U(1) model, using the short-distance expansion for the product of weak currents. The effective Hamiltonian is of the form[4,5]

$$H_W(\Delta S=1) = \sqrt{2}\, G_F \cos\theta_c \sin\theta_c \sum_{n=1}^{6} c_n \mathcal{O}_n , \qquad (2.1)$$

where $\mathcal{O}_n$ are local four-quark operators and c_n are Wilson coefficients. QCD enters into the calculation through leading log corrections to the Wilson coefficients. The explicit content of the operators $\mathcal{O}_n$ may be found in Ref. 5. The operator $\mathcal{O}_4$ is a purely $\Delta I=3/2$ operator, whereas the other operators are $\Delta I=1/2$. $\mathcal{O}_5$ and $\mathcal{O}_6$, the so-called "penguins", are purely QCD-induced operators which vanish in the flavor-symmetry limit[5]. In this limit, the Hamiltonian may be written in a simpler form:

$$H_W(\Delta S=1) = \sqrt{2}\, G_F \cos\theta_c \sin\theta_c \left[c_- \mathcal{O}_- + c_+ \mathcal{O}_+\right], \qquad (2.2)$$

where

$$\mathcal{O}_\pm = (\bar{s}_L\gamma_\mu u_L)(\bar{u}_L\gamma^\mu d_L) \pm (\bar{s}_L\gamma_\mu d_L)(\bar{u}_L\gamma^\mu u_L) \ . \qquad (2.3)$$

For the calculations which follow, it is convenient to introduce $\mathcal{O}_+^{1/2} = \mathcal{O}_+ - \frac{2}{3}\mathcal{O}_4$, which is a pure $\Delta I=1/2$ operator.

By making use of the soft-pion limit (SPL), the $K\to 2\pi$ amplitudes are described via $K\to\pi$ transition amplitudes as follows:

$$a^{3/2}(K^+\to\pi^+\pi^0) = \kappa \frac{3}{\sqrt{2}\, f_\pi} (\sqrt{2}\, G_F \cos\theta_c \sin\theta_c) \cdot c_4 \langle\pi^+|\mathcal{O}_4|K^+\rangle ,$$

$$a^{1/2}(K^0 \to \pi^+\pi^-) = \kappa \frac{1}{f_\pi} (\sqrt{2}\, G_F \cos\theta_c \sin\theta_c) \cdot \{c_- \langle \pi^+ | \mathcal{O}_- | K^+ \rangle + c_+ \langle \pi^+ | \mathcal{O}_+^{1/2} | K^+ \rangle\} + a^{peng}(K^0 \to \pi^+\pi^-) \ . \tag{2.4}$$

Here $a^{peng}(K^0 \to \pi^+\pi^-)$ denotes the penguin amplitude which is somewhat more complicated due to the presence of the anomalous commutator term[6)]

$$a^{peng}(K^0 \to \pi^+\pi^-) = \kappa \frac{1}{f_\pi} (\sqrt{2}\, G_F \cos\theta_c \sin\theta_c) \cdot c_5 \{\langle \pi^+ | \mathcal{O}_5 | K^+ \rangle + \langle \pi^+ | \mathcal{O}_5^{(c)} | K^+ \rangle\} + (5 \to 6) \ . \tag{2.5}$$

The second term in the brackets in (2.5) is the matrix element of the anomalous commutator $\mathcal{O}_5^{(c)}$ which is given by

$$\mathcal{O}_5^{(c)} = -\frac{16}{9} \langle 0 | \bar{d}d | 0 \rangle \bar{s}(1 + \gamma_5) d \ . \tag{2.6}$$

The quantities c_n are QCD corrected Wilson coefficients with typical values[7)] $c_- = 2.8$, $c_+ = 0.6$, $c_4 = 0.4$, $c_5 = -0.15$, $c_6 = -0.05$.

The constant κ in (2.4) and (2.5) is a factor of continuation from the SPL to the physical amplitude, typically of the order of 1-2 [8)].

The basic problem is the calculation of the matrix elements $\langle \pi | \mathcal{O}_n | K \rangle$ in (2.4) and (2.5). The crudest way is the so-called vacuum-saturation approximation[9)] in which

$$\langle \pi | J_\mu J^\mu | K \rangle \approx \langle \pi | J_\mu | 0 \rangle \langle 0 | J_\mu | K \rangle \ .$$

A more refined approach is to insert other physical intermediate states in addition to the vacuum[10)]. One may also use quark models, such as the bag model[8)] and the harmonic-oscillator type of models[7)]. Recent estimates based on lattice QCD[11)] seem to support the vacuum-saturation estimate. The reported results, however, are only tentative and still subject to large systematic errors. In the next section we present the method based on hadronic sum rules[12)].

3. Constraints on matrix elements from hadronic sum rules

Consider the two-point function

$$\psi(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T(\mathcal{O}(x)\,\mathcal{O}^\dagger(0))|0\rangle \ , \tag{3.1}$$

with an absorptive part given by

$$\frac{1}{\pi}\,\mathrm{Im}\,\psi(t) = \frac{1}{2\pi}\sum_\Gamma \langle 0|\mathcal{O}|\Gamma\rangle\langle\Gamma|\mathcal{O}^\dagger|0\rangle(2\pi)^4\delta(p-\Sigma p_\Gamma). \tag{3.2}$$

Each state $|\Gamma\rangle$ gives a positive contribution! Inserting a particular one, for example $|K^+\pi^-\rangle$, we find that

$$\frac{1}{\pi}\,\mathrm{Im}\,\psi(t) \geq \frac{1}{16\pi^2}\,\frac{\lambda^{1/2}(t,m_K^2,m_\pi^2)}{t}\,|F(t)|^2\theta(t-(m_K+m_\pi)^2), \tag{3.3}$$

where $t = (p_\pi - p_K)^2$. The function $F(t)$ is an analytic function with a branch cut $(m_K+m_\pi)^2 \leq t < \infty$. Its value at $t = 0$ is our matrix element $F(0) = \langle\pi^+|\mathcal{O}|K^+\rangle$.

The function (3.1) obeys a dispersion relation up to an arbitrary polynomial. It follows from QCD that a polynomial is of the fourth order. To get rid of this arbitrariness, we take five derivatives of the function $\psi(q^2)$ and find that

$$\mathcal{F}(Q^2) \equiv -\frac{\partial^5\psi(q^2)}{(\partial Q^2)^5} = 5!\int_{t_o}^{\infty} dt\,\frac{1}{(t+Q^2)^6}\,\frac{1}{\pi}\,\mathrm{Im}\,\psi(t)$$

$$\geq \frac{15}{2\pi^2}\int_{t_o}^{\infty} dt\,\frac{1}{(t+Q^2)^6}\,(1-\frac{t_o}{t})^{1/2}$$

$$\cdot\,(1-\frac{t_1}{t})^{1/2}\,|F(t)|^2 \ , \tag{3.4}$$

where $Q^2 = -q^2$, $t_o = (m+m')^2$ and $t_1 = (m-m')^2$. If $\mathcal{F}(Q^2)$ is known, there is an upper bound on $F(0)$! By making use of the Peierls inequality and the Jensen formula for analytic functions, we obtain an upper bound of the form

$$\mathcal{F}(Q^2) \geq |F(0)|^2 \exp\int_{t_o}^{\infty} d\mu(t)\ln\rho(t,Q^2) \ , \tag{3.5}$$

where $d\mu(t)$ is a normalized measure

$$d\mu(t) = \frac{1}{\pi}\left(\frac{t_0}{t-t_0}\right)^{1/2}\frac{dt}{t}$$

such that $\int_{t_0}^{\infty} d\mu(t)=1$, and

$$\rho(t,Q^2) \equiv \frac{15}{2\pi^2}\pi t\left(\frac{t-t_0}{t_0}\right)^{1/2}\left(1-\frac{t_0}{t}\right)^{1/2}\left(1-\frac{t_1}{t}\right)^{1/2}\frac{1}{(t+Q^2)^6}\ . \tag{3.6}$$

The bound (3.5) may be considerably improved if we have some information about $F'(t)$ and $F''(t)$ at $t=0$. Such information can be provided by the K-π phase-shift analysis[13]. For details, see Ref. 14.

The next step is the calculation of $\mathcal{F}(Q^2)$ in (3.4) defined as a fifth derivative of the two-point function (3.2). This has to be done in both the low-Q^2 and high-Q^2 regions,separately.

Obviously, the high-Q^2 behavior of $\mathcal{F}(Q^2)$ is governed by perturbative QCD, whereas the low-Q^2 behavior is essentially nonperturbative. In QCD, the leading contributions come from the diagrams shown in Fig. 1a. Low-Q^2 calculations consist in summing up a class of hadronic contributions depicted in Fig. 1b. Details of the calculation may be found in Refs. 14 and 15.

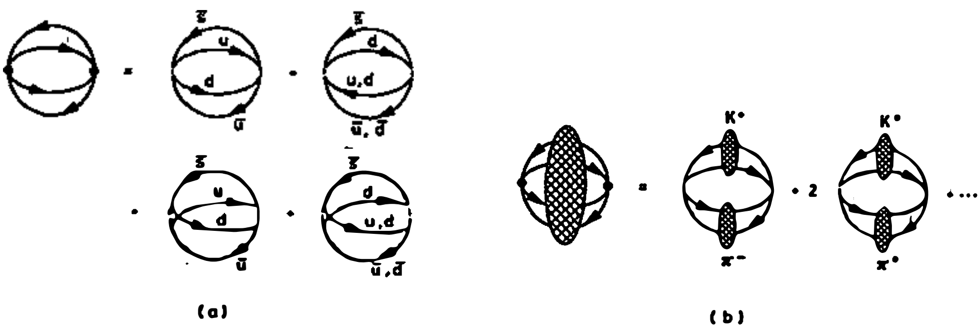

Fig. 1. Leading contributions to the function $\psi(q^2)$ for the case $\mathcal{O}_4$.

4. Results and discussion

The calculation of $\mathcal{F}(Q^2)$ eventually requires a numerical evaluation of the Feynman type of integrals. As an example, for penguins, $\mathcal{F}_5(Q^2)$ assumes a very simple form when only the dominant hadronic contribution is retained:

$$\mathcal{F}_5(Q^2) = \left(\frac{f_\pi m_\pi^2}{m_u+m_d}\,\frac{f_K m_K^2}{m_s+m_u}\right)^2 \frac{36}{16\pi^2}\int_0^1 dx\,\frac{x^5(1-x)^5}{[Q^2x(1-x)+m^2x+m_K^2(1-x)]^5} \quad (4.1)$$

This form is particularly convenient to relate it to the vacuum-saturation value of the matrix element

$$\langle\pi^+|\mathcal{O}_5|K^+\rangle_{vac} = -\left(1-\frac{1}{N_c^2}\right)\frac{f_\pi m_\pi^2}{m_u+m_d}\,\frac{f_K m_K^2}{m_s+m_u} = \frac{1-N_c^2}{N_c^2}\mathcal{M}_{vac}\ . \quad (4.2)$$

As regards upper bounds on matrix elements, the numerical results for $\mathcal{O}_4$ and $\mathcal{O}_5$ are presented in Fig. 2. The curves for $\mathcal{O}_+$ and $\mathcal{O}_-$ are very similar to the curve for $\mathcal{O}_4$. The upper bounds are given at the minimum of the curves. We summarize the results in Table 1, where we also give the results from the quark-model calculation[7]. We find that the quark-model results, compared with our bounds, are generally overestimated for the operators with left-left helicities. From the bounds given in Table 1, we may calculate the bounds on decay amplitudes using (2.4). If we forget about penguins for a moment, we obtain

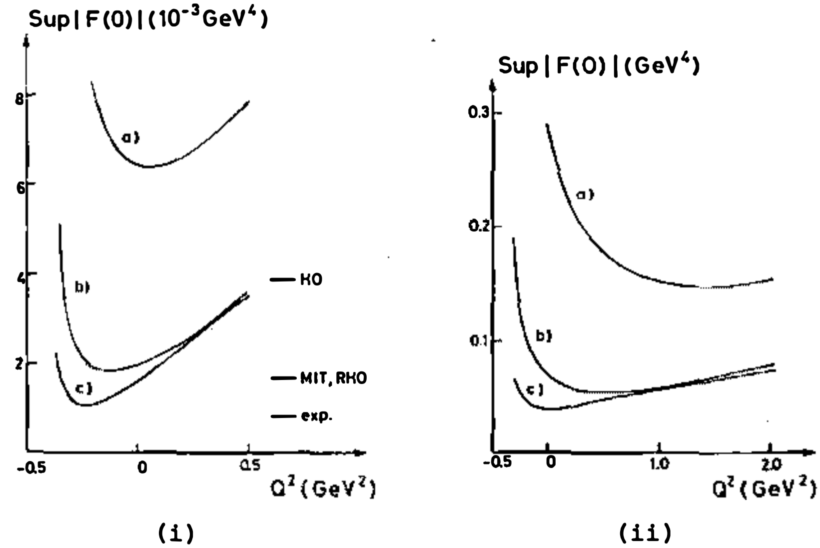

Fig. 2. Upper bounds for (i) $F(0) = \langle\pi^+|\mathcal{O}_4|K^+\rangle$ and (ii) $F(0) = \langle\pi^+|\mathcal{O}_5|K^+\rangle$. Curves (b) and (c) correspond to the improved bounds when the information about $F'(0)$ and $F''(0)$ is provided.

TABLE 1

Upper bounds on K-π matrix elements in 10^{-3} GeV^4 with corresponding results in the bag model, the harmonic oscillator (HO) and the relativized harmonic oscillator (RHO).

Operator	Our upper bound $\|F(0)\| \leq$	Bag	HO	RHO
$\mathcal{O}_4$	1.1	-1.7	3.9	1.7
$\mathcal{O}_-$	0.67	-0.84	1.97	0.84
$\mathcal{O}_+^{1/2}$	0.22	-0.28	0.66	0.28
$\mathcal{O}_5$	38	26.7	-	-

$$a_{3/2}(K^+\to\pi^+\pi^0) \leq 2.36\times10^{-8}\ \text{GeV}\ ,$$
$$a_{1/2}(K^0\to\pi^+\pi^-) \leq 5.16\times10^{-8}\ \text{GeV}\ . \qquad (4.3)$$

Compared with the experimental values in (1.1), eq. (4.3) shows that the $\Delta I=1/2$ amplitude is far too small and requires that about 80% of its value should be explained by penguins! Unfortunately, we find that this is not possible, although it seems from Table 1 that penguins give a large contribution. This is because there is an anomalous term, the contribution of which is precisely equal to the vacuum-saturation estimate but with an opposite sign! This yields the following constraints on penguins, with the anomalous term included:

$$0.1\ \mathcal{M}_{vac} \leq \langle\pi^+|\mathcal{O}_5+\mathcal{O}_5^{(c)}|K^+\rangle \leq \mathcal{M}_{vac}\ , \qquad (4.4)$$

where $\mathcal{M}_{vac}$ is given in (4.2). As far as we keep the sign stable, the limits in (4.4) lead to 1.6×10^{-8} GeV$\leq -a^{peng}(K^0\to\pi^+\pi^-)$ $\leq 16.1\times10^{-8}$ GeV, which means that the penguins act destructively to the $\Delta I=1/2$ amplitude.

Finally, our lower bound in (4.4) may be used to set a lower bound on the parameter ε' since it is directly related to the size of penguins. Following the procedure of Gilman and Hagelin[3,16], we find that all their bounds on ε'/ε should be reduced by a factor of 10. Moreover, since we have shown that penguins contribute to $\Delta I=1/2$ suppression, we determine the

the sign of ε' to be the opposite of ε following the same line of arguments as in Ref. 3. Our result seems to be confirmed by recently reported preliminary Fermilab measurements[17].

To conclude, we may say that the $\Delta I=1/2$ rule still remains an open problem. Probably, its solution lies outside of the scope of the standard model. The penguin-type operators, which obviously do not solve the $\Delta I=1/2$ rule in the standard model, are also induced by supersymmetry and possibly these additional effective operators yield the desired effect.

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